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Expander Families as Information Patterns for Distributed Control of Vehicle Platoons ⋆

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Abstract: A state space formulation of the distributed control of a 1-D vehicle platoon is considered. The objective is to understand the effects of the underlying information exchange pattern between the vehicles on the control performance of the platoon. The symmetric control case is considered where each vehicle gives equal weight to all the information available to it in determining its control law. It is shown that expander families of graphs when used as information patterns result in stability margins decaying to zero at rate at most $O(1/N)$; an improvement over the previously known $O(1/N^2)$ decay with nearest neighbor type information patterns.

Keywords: Intelligent vehicle highway systems (IVHS), Expander graphs, Vehicle platoons

1. INTRODUCTION

Distributed control of large vehicle platoons has been an active area of research in the control community. In particular, the one dimensional variant of the problem has received special interest due to potential applications in increasing throughput of Intelligent Vehicle/Highway Systems(IVHS) (see Varaiya (1993)). The idea is that some lead vehicles in the platoon are given desired trajectory information by a supervisory layer in the IVHS and the whole platoon follows this desired trajectory while maintaining prescribed safe inter vehicle spacing. The constraint is that each vehicle in the platoon can use only the local information available to it (for instance the sensed distance from predecessor and follower) in determining its (local) control action.

Several researchers have analyzed this scenario and we discuss only some of the results here. A double integrator model for individual vehicle dynamics is a common abstraction. Using identical double integrator dynamics for all vehicles in the platoon, Seiler et al. (2004) and Barooah and Hespanha (2005) analyze the problem in the frequency domain. They consider the cases where the local information available to a vehicle is limited to the (sensed) relative distance from its predecessor and from both predecessor and follower. For both cases they conclude that the H_∞ norm of the transfer function from disturbances acting on the vehicles to the spacing errors and of that between the spacing errors grow without bounds in the size of the platoon. In a similar setting, it is shown in Middleton and Braslavsky (2010) that this is the case with a general LTI model for the individual dynamics and local information restricted to a fixed neighborhood of the individual.

The state space formulation of the problem was studied in Hao et al. (2010). A more general notion of information pattern is introduced with each vehicle receiving information from its neighbors on a D-dimensional lattice. By defining the stability margin in terms of the least damped eigenvalue of the appropriate closed loop matrix, it is shown that the stability margin decays to zero as $O(1/N^{2/D})$, where N is the number of vehicles in the platoon. For the case where only a few vehicles (independent of size of platoon) are provided desired trajectory information, this translates to the nearest neighbor type information pattern with D being 1 yielding a decay of $O(1/N^2)$. This result assumed what has come to be known as *symmetric control*, where an agent weighs the information from all its neighbors on the information graph equally. In a more recent result Hao and Barooah (2012), the authors argue that by introducing asymmetry in weighing the information between the predecessor and the follower, the stability margin can be bounded away from zero uniformly in N.

In Bamieh et al. (2012), the authors analyze the infinite dimensional limiting case of the problem making use of tools from infinite dimensional systems theory. They study the 'rigidity of the formation' or coherence of the whole platoon by defining it in terms of the variance of spacing errors when each vehicle is subjected to additive white Gaussian disturbances. They conclude that in the one dimensional setting nearest neighbor type information patterns inevitably lead to loss of coherence.

The general trend of the results is a conclusion about a certain inadequacy in control performance when the information available to the individual is constrained to be of the nearest neighbor type. While some (Middleton and Braslavsky (2010), Hao et al. (2010)) have mentioned the possibility of using more general information patterns, it does not seem that much progress has been made in this direction. With technological advances, precise and dynamic measurements of absolute positions of vehicles (eg. GPS) and their communication over inter-vehicle communication networks (see Part 4, Chapter 13 Samad and Annaswamy (2011)) is practically possible. This gives the control engineer the flexibility to choose from a wider class of information patterns. However, such a choice must be made being mindful of the demand imposed on the communication network.

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The objective of this paper is to analyze the role of the underlying information pattern on the control performance of the 1-D vehicle platooning problem. In section 2 we formulate the problem in a state space setting similar to Hao et al. (2010) with symmetric weighing of all the information while allowing for an arbitrary graph serving as the information pattern. In section 3 we derive a lower bound on the achievable performance (stability margin) in terms of parameters of the information pattern. In section 4 we introduce expander families or expander graphs and argue that they strike the balance by providing better control performance with tolerable communication overheads when used as information patterns. Conclusions and directions for future work are discussed in section 5.

NOTATION

The spectrum of matrix A is denoted by $\sigma(A)$ and its i^{th} row and j^{th} column element is denoted by $A(i,j)$. $A > 0$ (≥ 0) denotes A is symmetric positive (semi-) definite. The i^{th} element of vector p is denoted by $p(i)$, the vector $[1,\ldots,1]^T \in \mathbb{R}^N$ is denoted by $\mathbf{1}$ and the $N \times N$ identity matrix is denoted by I_N. The Kronecker delta function is denoted by $\delta(i,j)$. Given a graph $G = (V,E)$, $\mathcal{N}(i)$ denotes the adjacency list of vertex i, $deg(i) = |\mathcal{N}(i)|$ is the degree of the i^{th} vertex, $A_G \in \mathbb{R}^{|V|\times|V|}$ denotes its (normalized) adjacency matrix given by

$$A_G(i,j) = \begin{cases} \dfrac{1}{deg(i)} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

and the (normalized) Laplacian $L_G = I - A_G$. A *family of graphs* is an infinite sequence of graphs $\{G_n\}_{n\in\mathbb{N}}$, $G_n = (V_n,E_n)$, with increasing number of vertices such that $n \to \infty \Rightarrow |V_n| \to \infty$.

2. PROBLEM FORMULATION

Our goal is to understand the effect of the underlying information pattern on the performance of the symmetric distributed control algorithm for the 1-D vehicle platoon. To this end, we assume identical vehicle dynamics and identical controller parameters for every vehicle. Next, we quantify what we mean by control performance by defining an appropriate stability margin. This section closely follows the set up in Hao et al. (2010).

2.1 System Model

We consider N vehicles indexed by $i \in \{1,\ldots,N\}$, each governed by double-integrator dynamics $\ddot{x}_i = u_i$, where x_i and u_i are real valued functions of time (argument suppressed) denoting the position and control input of the i^{th} vehicle. The first vehicle in the formation, $i = 1$, is given the desired/reference trajectory information $x_{1,d}$. Reference inter vehicle distances $\Delta_{i,j}$ between vehicle i and its nearest neighbors $j \in \{i-1,i+1\}$ are specified for $i = 2,\ldots,N-1$. Since the formation is one dimensional, such reference spacing between nearest neighbors in effect specifies inter vehicle distance between any two vehicles i and j, denoted by $\Delta_{i,j}$. Further, specifying $x_{1,d}$ specifies the reference trajectory $x_{i,d} = x_{1,d} + \Delta_{1,j}$ for all vehicles $i = 2,\ldots,N$. Note that vehicle 1 alone is given the reference trajectory input $x_{1,d}$. Such information is assumed to be given externally. For example, in the case of an IVHS, the supervisory traffic control system can command the reference trajectory to the platoon leader.

Each vehicle is assumed to have sensing capabilities and can measure relative distances and velocities to its nearest neighbors i.e. can measure $(x_i - x_j)$ and $(\dot{x}_i - \dot{x}_j)$ for $j \in \{i-1,i+1\}$ (the first and the last vehicle can measure relative distance and velocity only from vehicles 2 and $N-1$ respectively). Apart from sensing, each vehicle is also capable of inter-vehicle communication and can send and receive relative position and velocity information instantaneously to and from other vehicles. We construct an abstraction of all such inter-vehicle sensing and communication and represent it by an undirected graph. Each vehicle is represented by a vertex and an edge is drawn between two vertices if the corresponding vehicles have access to their relative spacing information. We call this graph the *information graph* and denote the adjacency list of vertex i by $\mathcal{N}(i)$.

The control objective is to enable tracking of $x_{1,d}$ by vehicle 1 while the others follow maintaining the prescribed $\Delta_{i,j}$ spacings. In particular, we want to achieve this by means of a local feedback law for every vehicle based on the local information available to it. Consider the following feedback law

$$u_i = \frac{1}{deg(i)} \sum_{j\in\mathcal{N}(i)} [-k(x_i - x_j - \Delta_{i,j}) - b(\dot{x}_i - \dot{x}_j)] \qquad (1)$$
$$+ \delta(1,i)[-k(x_1 - x_{1,d}) - b(\dot{x}_1 - \dot{x}_{1,d})].$$

Noting that the relative measurements can be written as $x_i - x_j - \Delta_{i,j} = (x_i - x_{i,d}) - (x_j - x_{j,d})$, we define $z = [x_1^T - x_{1,d}^T, \dot{x}_1^T - \dot{x}_{1,d}^T, \ldots, x_N^T - x_{N,d}^T, \dot{x}_N^T - \dot{x}_{N,d}^T]^T$. Substituting u_i from (1) to the double-integrator $\ddot{x}_i = u_i$ and assuming $\ddot{x}_{1,d} \equiv 0$, we obtain the following closed loop system

$$\dot{z} = (I_N \otimes A_1 + (L + D_{ext}) \otimes A_2)z \qquad (2)$$

$$\text{where } A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ -k & -b \end{bmatrix},$$

$L \in \mathbb{R}^{N\times N}$ is the Laplacian of the information graph and D_{ext} is the diagonal matrix with $D_{ext}(i,i) = \delta(1,i)$. We denote the closed loop matrix by $A_{cl} = I_N \otimes A_1 + (L + D_{ext}) \otimes A_2$.

2.2 Stability Margin

We consider the real part of the least damped eigenvalue of A_{cl} as a measure of stability of (2). The following was proved in Hao et al. (2010).

Theorem 1. The spectrum of A_{cl} is

$$\sigma(A_{cl}) = \bigcup_{\gamma\in\sigma(L+D_{ext})} \{\sigma(A_1 + \gamma A_2)\}$$
$$= \bigcup_{\gamma\in\sigma(L+D_{ext})} \left\{ -\frac{\gamma b}{2}\left(1 \pm \sqrt{1 - \frac{4k}{\gamma b^2}}\right)\right\} \qquad (3)$$

The proof relies on Schur's triangulation theorem and proceeds along the lines of using the unitary matrix U that puts $U^{-1}(L + D_{ext})U$ into an upper triangular matrix, in the similarity transformation $(U^{-1} \otimes I_2)A(U \otimes I_2)$ which yields an upper block diagonal matrix with $(A_1 + \gamma A_2)$ on the block diagonal for every $\gamma \in \sigma(L + D_{ext})$.

As a consequence of the Perron Frobenius theorem (see Horn and Johnson (1990)), it is known that the Laplacian of a connected graph has a simple eigenvalue at zero and the corresponding eigenvector is $\mathbf{1}$. Further, all other eigenvalues of the

Laplacian of a connected undirected graph are strictly positive and hence $L \geq 0$.

Lemma 2. For a connected undirected graph with Laplacian L, $L + D_{ext} > 0$ and hence $0 \notin \sigma(L + D_{ext})$.

Proof. $L + D_{ext} = L + e_1 e_1^T$ is positive semidefinite. For any $x \in \mathbb{R}^N$,

$$x^T [L + e_1 e_1^T] x = 0 \Leftrightarrow Lx = 0 \text{ and } x(1) = 0.$$

Since the graph is connected, $Lx = 0 \Leftrightarrow x \in col(\mathbf{1})$, where $col(\mathbf{1})$ denotes column span of $\mathbf{1}$. Now $x^T Lx = 0 \Leftrightarrow Lx = 0 \Leftrightarrow x \in col(\mathbf{1})$ which contradicts $x(1) = 0$ unless $x = 0$. We have $x^T (L + D_{ext}) x = 0 \Rightarrow x = 0$ implying $L + D_{ext} > 0$ and $0 \notin \sigma(L + D_{ext})$.

From Theorem 1, the eigenvalues of A_{cl} are the roots of the polynomial $\prod_{\gamma \in \sigma(L+D_{ext})} (s^2 + b\gamma s + k\gamma) = 0$. Since $k, b > 0$ and from Lemma 2.1 all the elements in $\sigma(L + D_{ext})$ are positive, the real parts of all eigenvalues of A_{cl} are strictly negative (i.e. A_{cl} is Hurwitz) and can be explicitly written as (3). Define

$$\gamma_{min} = \min \sigma(L + D_{ext}). \tag{4}$$

We will be concerned about the asymptotic behavior of γ_{min} within a family of information graphs. Observe that as $N \longrightarrow \infty$, the only possibility for an eigenvalue of A_{cl} to converge to zero is if $\gamma_{min} \rightarrow 0$ (since $\gamma_{min} = 0 \Leftrightarrow 0 \in \sigma(L + D_{ext}) \Leftrightarrow 0 \in \sigma(A_{cl})$). If $\gamma_{min} \longrightarrow 0$, we can pick a sufficiently large N_0 such that $1 - \frac{4k}{\gamma_{min} b^2} < 0$ for all $N > N_0$ and the real part of the corresponding eigenvalue of A_{cl} is given by $\gamma_{min} b/2$. And if $\gamma_{min} \longrightarrow 0$, the real parts of the least damped eigenvalues of A_{cl} approach zero in the limit. With this interpretation, we call γ_{min} the *stability margin* of (2).

3. STABILITY MARGIN AND THE INFORMATION PATTERN

In order to understand the effect of the information pattern on the stability margin we try to relate the spectral properties of the information graph to the latter. In particular, we try to relate γ_{min} and $\lambda_{min} = \min\{\sigma(L) \setminus \{0\}\}$. What we have at hand is essentially the problem of relating an eigenvalue of the sum of two matrices to the eigenvalues of the summands which is known to be difficult. It should be noted that we need something tighter than some known inequalities regarding addition of rank one positive semidefinite matrices to another semi-definite matrix; for instance Theorem 4.3.4, pp. 182, Horn and Johnson (1990) in this case gives $0 \leq \gamma_{min} \leq \lambda_{min}$. We use the special structure of the matrices involved in proving Theorem 4 which provides bounds for the stability margin γ_{min} in terms of λ_{min}. First a technical lemma.

Lemma 3. Let $p(s)$ be a monic polynomial of degree $N > 1$ with real roots $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_N > 0$. Then $p(s)$ is strictly convex for N even and strictly concave for N odd in the interval $(-\infty, \alpha_N)$.

The proof is based on the sign definiteness of the second derivative of $p(s)$ for $s \in (-\infty, \alpha_N)$.

Theorem 4. Let L be the normalized Laplacian of a connected undirected graph $G = (V, E)$ and D_{ext} be a diagonal matrix with $D_{ext}(i,i) = \delta(1,i)$. Then γ_{min} defined in (4), and the second smallest eigenvalue of L, λ_{min}, satisfy

$$\frac{\lambda_{min}}{4N} < \gamma_{min} \tag{5}$$

where $N = |V|$.

Proof. The idea is to perform some similarity transformations on $L + D_{ext}$ to get it in a form where its characteristic polynomial can be expressed in terms of the eigenvalues of L. Then we use a first order Taylor approximation of the characteristic polynomial at zero to obtain the desired lower bound.

From the spectral theorem, there exists an orthogonal matrix P such that

$$P^T LP = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{min} & \\ & & & & 0 \end{bmatrix} \doteq \Lambda$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} = \lambda_{min} > 0$ are the eigenvalues of L. Let the set of vectors $\{e_i\}_{i=1}^N$ denote the natural basis of $\mathbb{R}^N$. Let the i^{th} column of P be denoted by p_i and that of P^T by r_i. Since $P^T P = PP^T = I$, we have $p_i^T p_j = r_i^T r_j = \delta(i,j)$. Thus $P^T LPe_N = 0 \Rightarrow Lp_N = 0 \Leftrightarrow p_N = \frac{1}{\sqrt{N}}\mathbf{1}$.

We prove a more general result by allowing D_{ext} to be a rank one matrix with any one of the diagonal elements being one and the rest of the entries being zero. We can write D_{ext} as $e_k e_k^T$ for some k; $k = 1$ corresponds to the D_{ext} in the statement of the theorem. Next, we perform the similarity transformation $P^T (L + D_{ext}) P = \Lambda + r_k r_k^T$,

$$P^T (L + D_{ext}) P = \Lambda + \begin{bmatrix} | & | & & | \\ r_k(1) r_k & r_k(2) r_k & \cdots & r_k(N) r_k \\ | & | & & | \end{bmatrix}.$$

Since $p_N = \frac{1}{\sqrt{N}}\mathbf{1}$, $r_k(N) = \frac{1}{\sqrt{N}}$ for any k. One can perform elementary column operations $j^{th}\ column \rightarrow j^{th}\ column - \frac{r_k(j)}{r_k(N)}$ $N^{th}\ column$ for all $j \in \{1, \cdots, N-1\}$. Such an elementary operation can be expressed as right multiplication by a matrix $\tilde{T}$ given by

$$\tilde{T} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ -\frac{r_k(1)}{r_k(N)} & \cdots & -\frac{r_k(N-1)}{r_k(N)} & 1 \end{bmatrix}.$$

Applying these column operations on $P^T (L + D_{ext}) P$,

$$P^T (L + D_{ext}) P\tilde{T} = \begin{bmatrix} | & & | & | \\ \lambda_1 e_1 & \cdots & \lambda_{min} e_{N-1} & r_k(N) r_k \\ | & & | & | \end{bmatrix}.$$

Let $\tilde{\Lambda}$ be the $(N-1) \times (N-1)$ matrix obtained by removing the last row and last column of Λ and $\tilde{r}_k$ be the $(N-1)$ dimensional column vector obtained by removing the last element of r_k. Then

$$\tilde{T} = \begin{bmatrix} I_{N-1} & 0 \\ -\frac{1}{r_k(N)}\tilde{r}_k^T & 1 \end{bmatrix}, \quad \tilde{T}^{-1} = \begin{bmatrix} I_{N-1} & 0 \\ \frac{1}{r_k(N)}\tilde{r}_k^T & 1 \end{bmatrix}$$

and we can rewrite

$$P^T (L + D_{ext}) P\tilde{T} = \begin{bmatrix} \tilde{\Lambda} & r_k(N)\tilde{r}_k \\ 0 & r_k^2(N) \end{bmatrix}.$$

Left multiplying by $\tilde{T}^{-1}$,

$$\tilde{T}^{-1}P^T(L+D_{ext})P\tilde{T} = \begin{bmatrix} \tilde{\Lambda} & r_k(N)\tilde{r}_k \\ \frac{1}{r_k(N)}\tilde{r}_k^T\tilde{\Lambda} & 1 \end{bmatrix} \doteq \hat{L}.$$

Since $\hat{L} = (P\tilde{T})^{-1}(L+D_{ext})(P\tilde{T})$, the spectrum of $\hat{L}$ is the same as the spectrum of $L+D_{ext}$.

We now find the characteristic polynomial of $L+D_{ext}$ by finding that of $\hat{L}$. For some $\hat{x} \in \mathbb{R}^N$, $\hat{x} \neq 0$, $(sI-\hat{L})\hat{x} = 0 \Rightarrow$

$$(s-\lambda_j)\hat{x}(j) = r_k(N)r_k(j)\hat{x}(N) \text{ for all } j \in \{1,\cdots,N-1\}$$
$$\text{and } (s-1)\hat{x}(N) - \sum_{j=1}^{N-1}\frac{r_k(j)}{r_k(N)}\lambda_j\hat{x}(j) = 0.$$

Substituting for $\hat{x}(j)$ for all $j \in \{1,\cdots,N-1\}$ in terms of $\hat{x}(N)(\neq 0)$, we obtain the characteristic polynomial for $L+D_{ext}$

$$\chi(s) = \left[\prod_{j=1}^{N-1}(s-\lambda_j)\right]\left[(s-1) - \sum_{i=1}^{N-1}\frac{r_k^2(j)}{s-\lambda_j}\right].$$

Since $L+D_{ext}$ is symmetric positive definite, the roots of $\chi(s) = 0$ are real and positive; the smallest being γ_{min}. From Lemma 3, for even (odd) $N > 1$, $\chi(s)$ is strictly convex (concave) in the interval $(-\infty,\gamma_{min})$. For N even, $\chi(s) > 0$ for $s \in (-\infty,\gamma_{min})$ and the tangent at $s = 0$ is below the curve due to strict convexity implying that in a plot of s vs. $\chi(s)$, the intercept of tangent with the horizontal axis is a strict lower bound for γ_{min}. A similar argument can be made for the case where N is odd. The derivative of $\chi(s)$ at $s = 0$ is

$$\left.\frac{d}{ds}\chi(s)\right|_{s=0} = (-1)^{N-1}\left(\prod_{j=1}^{N-1}\lambda_j\right)\left[\frac{1}{N}\sum_{j=1}^{N-1}\frac{1}{\lambda_j} + 1 + \sum_{j=1}^{N-1}\frac{r_k^2(j)}{\lambda_j}\right]$$

and the value of $\chi(s)$ at $s = 0$ is $\chi(0) = \frac{1}{N}(-1)^N\left(\prod_{j=1}^{N-1}\lambda_j\right)$ (we have used $\sum_{j=1}^{N-1} r_k^2(j) + r_k^2(N) = 1$ and $r_k(N) = \frac{1}{\sqrt{N}}$ for simplifications). The equation of the tangent to $\chi(s)$ at $s = 0$ is given by $\chi_T(s) = \chi(0) + s\chi'(0)$. If N is even (odd), the intercept with the vertical axis, $\chi(0)$, is positive (negative) and the slope $\chi'(0)$ is negative (positive) resulting in a positive horizontal axis intercept. Solving for $\chi_T(s^*) = 0$ gives s^*, a lower bound to γ_{min}.

$$\begin{aligned} 0 < s^* = \frac{\chi(0)}{\chi'(0)} &< \gamma_{min} \\ \Rightarrow \frac{1}{\gamma_{min}} &< N\left[\frac{1}{N}\sum_{j=1}^{N-1}\frac{1}{\lambda_j} + 1 + \sum_{j=1}^{N-1}\frac{r_k^2(j)}{\lambda_j}\right] \\ &\leq \frac{N}{\lambda_{min}} + N + \frac{N}{\lambda_{min}}\sum_{j=1}^{N-1}r_k^2(j) \\ &\leq \frac{N}{\lambda_{min}}(2+\lambda_{min}). \end{aligned} \tag{6}$$

The lower bound in (5) follows by noting that $\lambda_i \leq 2$ for all i.

Remark. The lowerbound in (5) holds even with $m > 1$ (independent of N) 'lead' vehicles each given reference trajectory information externally which translates to D_{ext} having m ones on its diagonal and rest of the elements zero. This is a direct consequence of splitting D_{ext} as a sum of a rank one matrix $e_k e_k^T$ and a rank $(m-1)$ matrix $D_{ext} - e_k e_k^T$ for some k such that $D_{ext}(k,k) = 1$ and applying Weyl's inequality to the sum $(L + e_k e_k^T) + (D_{ext} - e_k e_k^T)$ where the smallest eigenvalue of $L + e_k e_k^T$ satisfies the bounds in (5).

For a nearest neighbor type information graph, it has been proved in Hao et al. (2010) that with a control law similar to (1), the stability margin γ_{min} decays to zero as $O(1/N^2)$. We would like to improve upon the asymptotic rate by changing the information graph. It is clear from the bounds in (5) that the asymptotic behavior of γ_{min} is closely related to the asymptotic behavior of the second smallest eigenvalue of the Laplacian λ_{min}.

In terms of asymptotic behavior, λ_{min} can either be bounded away from zero or can approach zero with some rate. If λ_{min} is bounded away from zero for arbitrarily large N for a family of graphs, we can conclude from (5), that $\gamma_{min} > O(1/N)$ with such a family chosen as the information graph. Asymptotically, a $O(1/N)$ decay is strictly slower than the $O(1/N^2)$ resulting from the nearest neighbor type information graphs and hence is better as the stability margin approaches zero slower. A question arises regarding which families of graphs satisfy the property that λ_{min} is bounded away from zero as $N \to \infty$?

4. EXPANDER FAMILIES AS INFORMATION PATTERNS

We now define an expander family of graphs and briefly present some results from graph theory relevant to the analysis of expander families. We then discuss the use of members of expander families as information graphs in the vehicle platoon problem.

Definition 5. The *edge expansion* of a d-regular undirected graph $G = (V,E)$ is given by

$$h(G) = \min_{S\subset V, 0<|S|\leq|V|/2}\frac{|E(S,V\setminus S)|}{d\,|S|}$$

where $E(S,V\setminus S)$ is the set of edges which are incident on a vertex in S and $V\setminus S$.

Thus edge expansion is the minimum across all nontrivial cuts $(S,V\setminus S)$ of the ratio of the number of edges across a cut and the number of vertices in the smaller set in the cut. Intuitively, a graph with 'large' edge expansion can be interpreted as a 'better connected' graph as one has to remove a 'large' number of edges to disconnect a 'large' enough component.

Definition 6. A family of d-regular graphs $\{G_n\}_{n\in\mathbb{N}}$ is an *expander family* if for some $\varepsilon > 0$, $h(G_n) > \varepsilon$ for all n.

Let us denote the adjacency matrix of G_n by A_n, $|V_n| = N$ and the eigenvalues of A_n by $1 = \alpha_1 \geq \alpha_2 \cdots \geq \alpha_N$. The following inequality is attributed to Cheeger (see pp. 454, Theorem 2.4 Hoory et al. (2006))

$$\frac{1-\alpha_2}{2} \leq h(G) \leq \sqrt{2\cdot(1-\alpha_2)}. \tag{7}$$

The eigenvalues of the corresponding Laplacian L_n and A_n get related as $\lambda_{N-i+1} = 1-\alpha_i$ with $\alpha_1 = 1$ corresponding to the zero eigenvalue of L_n and $\lambda_{min} = 1-\alpha_2$. Substituting in (7) we have

$$\frac{\lambda_{min}}{2} \leq h(G) \leq \sqrt{2\cdot\lambda_{min}}.$$

This leads to an equivalent characterization of expander families as a family of graphs such that for some $\varepsilon > 0$, $\lambda_{min} > \varepsilon$

for all n. Note that in the discussion at the end of the previous section, this was the sought after property for outperforming nearest neighbor type information patterns.

Explicit constructions of expander families based on results from different areas of mathematics have been known since 1970s. It is known that a random d-regular graph in the limit is a good expander with high probability. A technique called *zig-zag product*, which is related to the replacement product of two graphs, has also been discovered which can be applied to construct expander families. In general, it is rather easy to construct expander graphs or modify graphs to become expanders (see Hoory et al. (2006) and Krebs and Shaheen (2011)). Finding the appropriate expander family to serve as the information graph in the platoon problem will require further work and will involve several implementation related considerations such as physical or hop distance in the communication network between two neighbors on the information graphs etc. We do not attempt to answer these questions in detail here. Instead, we present an example applicable when the number of vehicles in the platoon is a prime.

Fact 7. (pp. 453, §2.2, Hoory et al. (2006)) Let $\{p_i\}_{i\in\mathbb{N}}$ be an infinite sequence of increasing primes. The 3-regular family of graphs $\{S_{p_i}\}_{i\in\mathbb{N}}$, $S_{p_i} = (V_i, E_i)$, with $V_i = \mathbb{Z}_{p_i}$ and for every $a \in V_i$, $(a, a+1), (a, a-1)$ and $(a, a^{-1}) \in E$ is an expander family (all operations are mod p_i).

The motivation for picking this family of expanders is that the edges $(a, a-1)$ and $(a, a+1)$ have a physical meaning as the sensed distance between a vehicle and its immediate successor and predecessor. In figure 1 we plot the sparsity pattern of a member of this family using the Matlab function $spy(\cdot)$ where the off diagonal elements are seen to be non zero.

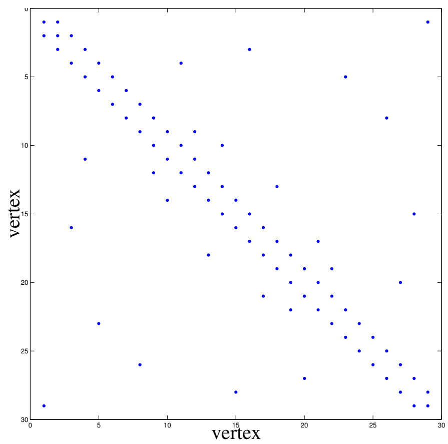

Fig. 1. Sparsity pattern of S_{29}

Several known explicit constructions of expander families yield multi-graphs i.e. graphs with multiple self loops and multiple edges. It is worth mentioning at this point that all the analysis of section 3 holds for information graphs that are possibly multi-graphs. One has to account for these by modifying the normalized adjacency matrix entries as $A(i,j) = \frac{|j\in\mathcal{N}(i)|}{deg(i)}$, where $|j \in \mathcal{N}(i)|$ is the multiplicity of j in the adjacency list of i.

We would like to make another feature of expander families explicit. While expander families have good connectivity properties, due to the restriction of d-regularity in the definition the number of edges are of order $O(N)$. In contrast, a complete graph has very good connectivity but at the 'cost' of having a larger number of edges of the order $O(N^2)$. From an implementation point of view, not only does using expanders as information graphs improve control performance in terms of stability margins, it also limits the amount of inter vehicle communication required. For instance if a multi-hop wireless network were used for such communication, the number of messages at any given point of time would be at most $O(N)$ giving flexibility in designing such protocols.

This brings us to another issue in the choice of the expander family. After making a choice for the information graph based on such spectral considerations, a question arises as to which vertex of the graph be assigned to which vehicle. Since any assignment does not change the spectral properties of the information graph one can try to do this assignment so as to reduce the inter-vehicle communication. If the communication is over a multi-hop network it would be prudent to reduce the hop count of the longest edge on the graph. This means finding the permutation of the vertex assignment which results in the least bandwidth adjacency matrix. This is the graph bandwidth minimization problem which is known to be NP hard Feige (2000). Several approximation algorithms are known and we show the results of the Cuthill-McKee algorithm implemented by the Matlab function $symrcm(\cdot)$ used to reduce the bandwidth of S_{541} in Fig. 2.

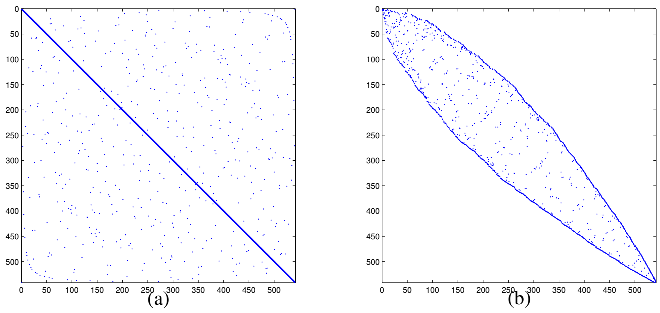

Fig. 2. Sparsity plots of (a) S_{541} and (b) bandwidth reduced S_{541}

5. DISCUSSION AND CONCLUSIONS

We begin with some numerical illustrations. Fig. 3 shows an experimental verification of the bound in (5) with the plot on log scale along the vertical axis of γ_{min} being strictly above λ_{min}/N against the first 150 primes as the number of vehicles along the horizontal axis with the expander family S_n as the information pattern. The nontrivial gap between the curves suggests that the lowerbound is not tight and that our conclusion of an at most $O(1/N)$ decay rate for the stability margin may be pessimistic i.e. the decay rate may be slower. In Fig. 4 we compare the stability margin with expander as the information pattern against the nearest neighbor type information pattern along the same axis. Notice that the decay rate in the case of the former is slower than the latter, providing validation to our argument of using an expander as the information pattern. The main goal of this paper is to bring forth the possibility of more general, albeit simple, information patterns in the vehicle platoon problem and present an analysis where different kind of patterns can be compared. We have also tried to address the issue of reducing the load on the communication network while improving control performance. It is clear that optimizing a cost

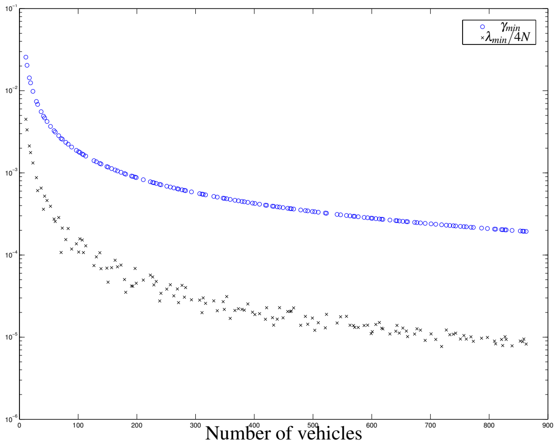

Fig. 3. Plot of γ_{min} and $\lambda_{min}/4N$ on log scale vs. number of vehicles

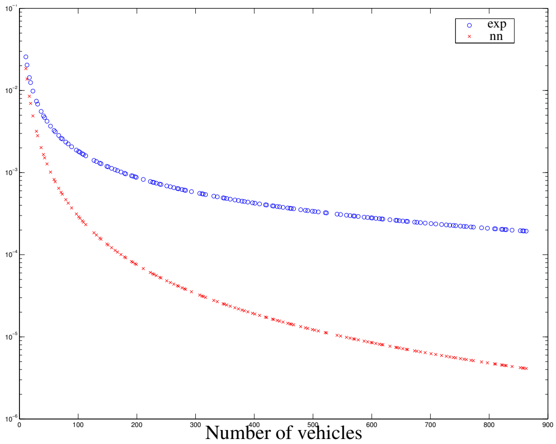

Fig. 4. Plot of stability margins on log scale with expanders (exp) and nearest neighbor (nn) as information patterns vs. number of vehicles

of communication over the set of all possible information patterns is NP hard; instead we argue that expander families, due to their sparsity, lower the demand on the communication network while their spectral properties help improve control performance. However, we have ignored the issue of implementing non-nearest neighbor type information patterns, such as the $\{S_{p_i}\}$ family, here. How can the relative distance and velocity information between far away vehicles in the platoon be communicated? Current vehicle-to-vehicle wireless communication technologies make the implementation of longer inter-vehicle communication implementation straightforward. A promising implementation approach is to use substantial quantization to send the necessary information, thus implementing quantized controls.

Several open questions remain. The result in this paper is only valid for the symmetric control case. In Hao and Barooah (2012), it is shown that by introducing asymmetry in the weights in (1), the stability margin can be bounded away from zero for arbitrarily large N for nearest neighbor type information patterns. The asymmetric case has to be analyzed for more general information patterns.

The relation between information pattern and other important metrics of control performance such as string stability has to be investigated. For instance, consider the linear consensus protocol $\dot{x} = -Lx + w$, where w is unit variance white Gaussian noise, and the associated macroscopic metric *deviation from average* $V_{dav} = \lim_{t\to\infty} \mathbb{E}[\mathbf{e}^T(t)\mathbf{e}(t)]$ proposed in Bamieh et al. (2012), where $\mathbf{e}(t) = x(t) - \frac{1}{N}\sum_{i=1}^{N} x_i(t)\mathbf{1}$. Let $J = \frac{1}{N}\mathbf{1}\mathbf{1}^T$, then it can be verified that $V_{dav} = tr\int_0^\infty (e^{-Lt}(I_N - J))^T e^{-Lt}(I_N - J)dt$ evaluates to $\sum_{j=1}^{N-1}\frac{1}{2\lambda_j} \Rightarrow \frac{V_{dav}}{N} \leq \frac{1}{2\,\lambda_{min}}$. If an expander is used as the information graph, $\frac{V_{dav}}{N}$ is bounded above by a constant independent of N while it has been shown to grow as $O(N)$ for nearest neighbor type information graph in Bamieh et al. (2012) for the consensus problem. Similar macroscopic measures scale as $O(N^3)$ for the 1-D platoon control problem with nearest neighbor type information graphs and analogous results for more general information patterns are open.

Another important question is the choice of the right expander family for the platoon control problem. We will address some of these questions in future work.

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