FLOW CONTROL IN TIME-VARYING, RANDOM SUPPLY CHAINS*

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1. Extended Abstract.

Supply chains consist of multiple activities, which cover the design, procurement, manufacturing, distribution, and consumption of goods. They repeatedly demonstrate the co-existence of operational optimization with operational vulnerability [3]. This was most recently and dramatically demonstrated in the aftermath of several events, such as the Hurricane Katrina or the tragic earthquake of March 13, 2011 off the northeastern coast of Japan [3]. As a consequence, the stakeholders were forced to recognize the *time-varying* and *random* nature of supply chains.

In this paper we propose a control algorithm for the flow of products in a time varying, random supply chain aimed at maximizing the profit of a firm. The algorithm we propose is dynamic, distributed, and results as a solution of a stochastic optimization problem. Additionally, the algorithm does not require knowledge of the probability distributions of the supply chain. This extended abstract is based on a technical report that can be found at [1].

1.1. Problem setup.

We assume that a set of firms $\mathcal{F}$ are involved in the production, storage and distribution of a homogeneous product. The firms are considering a set of manufacturing facilities $\mathcal{M}$, a set of warehouses $\mathcal{W}$ and serves a set of retail outlets/demand markets $\mathcal{R}$. We use a supply network model similar to the one introduced in [2], with the main difference that we assume a *time varying* and *stochastic supply chain network*, and for notational simplicity we consider only one firm. Let $\mathcal{N}$ be the set of all nodes in the network (with a typical node denoted by i), i.e., $\mathcal{N} = \{\mathcal{F} \cup \mathcal{M} \cup \mathcal{W} \cup \mathcal{R}\} \cup \{i' | i \in \mathcal{W}\}$, with cardinality $N = |\mathcal{N}|$. Note that a warehouse i is represented by two nodes in the network (by using i' as well) in order to clearly emphasize the flow of product passing through the warehouse, i.e., through the link (i, i'). We denote by $\mathcal{L}$ the set of links of the supply chain, i.e., $\mathcal{L} = \{(i, j), i, j \in \mathcal{N}\}$ through which products "flow" from node i to node j, where the flow of product is driven by the demand at the retailers/markets (we assume that links of the form (i, i') are also included in $\mathcal{L}$).

We make the assumption that the supply chain operates in slotted time, with slots normalized to integral units so that slot times occur at times $t \in \{0, 1, 2, \ldots\}$. We denote by $S(t)$ the supply chain network state during slot t, which reflects possible disruptions in manufacturing and transportation, power outages, technical malfunctions, etc.

ASSUMPTION 1.1. *The process $S(t)$ belongs to a finite set $\mathcal{S}$ and evolves according to an identically, independently distributed random process.*

We denote by $\mu_{i,j}(t)$ the amount of product flowing through the link (i, j) during time slot t. We let $d_i(t)$ denote the market demand at retailer i, during slot t and we assume it to be a random process. It is reasonable to assume that the quantity of product flowing between different entities is upper-bounded, and hence we make the following assumption.

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Assumption 1.2. *The flows $\mu_{i,j}(t)$ are positive for all time-slots t and there exist positive scalars μ_i^{max} such that*

$$\sum_b \mu_{i,b}(t) \leq \mu_i^{max},\ \forall i \in \mathcal{N}, \forall (i,b) \in \mathcal{L},\ \forall t. \tag{1.1}$$

The following definitions introduce the time averages of the product flows in the supply chain.

Definition 1.3. *The time average flows of product and their long-run time averages, respectively are given by*

$$\bar{\mu}_{i,j}(t) = \frac{1}{t}\sum_{\tau=0}^{t-1} E\{\mu_{i,j}(\tau)\},\ \bar{\mu}_{i,j} = \lim_{t\to\infty} \bar{\mu}_{i,j}(t),\ \forall (i,j) \in \mathcal{L}. \tag{1.2}$$

Additionally, we have the following assumption on the market demands.

Assumption 1.4. *The market demands $d_i(t)$ are independent and identically distributed random processes with mean given by*

$$\bar{d}_i = E\{d_i(t)\},\ \forall i \in \mathcal{R}. \tag{1.3}$$

Let us also define the aggregate vectors of product flows $\boldsymbol{\mu}(t) = (\mu_{i,j}(t), (i,j) \in \mathcal{L})$, and market demands $\boldsymbol{d}(t) = (d_i(t),\ i \in \mathcal{M})$.

1.2. Flow control algorithm and performance results. We assume that the revenue function of the firm depends on the quantity of products that reach the retailers/markets in the long-run and we denote it by

$$\boldsymbol{f}(\bar{\boldsymbol{\mu}}) = \sum_{i\in\mathcal{W}, j\in\mathcal{R}} \boldsymbol{f}_{i',j}(\bar{\mu}_{i',j}),$$

for $i \in \mathcal{W},\ j \in \mathcal{R}$ and $(i',j) \in \mathcal{L}$. We also consider a cost function associated with each link $(i,j) \in \mathcal{L}$ which we denote by $g_{i,j}(\bar{\mu}_{i,j})$. These cost functions depend on the flow of product on the links and are generated by activities such as acquiring raw materials, manufacturing, transportation or warehouse usage. The total cost function is given by

$$\boldsymbol{g}(\bar{\boldsymbol{\mu}}) = \sum_{i\in\mathcal{F}, j\in\mathcal{M}} \boldsymbol{g}_{i,j}(\bar{\mu}_{i,j}) + \sum_{i\in\mathcal{M}, j\in\mathcal{W}} \boldsymbol{g}_{i,j}(\bar{\mu}_{i,j}) + \sum_{i\in\mathcal{W}} \boldsymbol{g}_{i,i'}(\bar{\mu}_{i,i'}) + \sum_{i\in\mathcal{W}, j\in\mathcal{R}} \boldsymbol{g}_{i',j}(\bar{\mu}_{i',j}).$$

Assumption 1.5. *The functions $\boldsymbol{f}_{i,j}$ are non-negative, continuously differentiable and concave, while the functions $\boldsymbol{g}_{i,j}$ are non-negative, continuously differentiable and convex.*

We define the long-run profit function $\boldsymbol{h}$ as the difference between the revenue and the cost functions, i.e.,

$$\boldsymbol{h}(\bar{\boldsymbol{\mu}}) = \boldsymbol{f}(\bar{\boldsymbol{\mu}}) - \boldsymbol{g}(\bar{\boldsymbol{\mu}}).$$

We introduce the following stochastic optimization problem,

$$\begin{aligned} \max_{\bar{\boldsymbol{\mu}}} \quad & \boldsymbol{h}(\bar{\boldsymbol{\mu}}) \\ \text{subject to: } & \sum_{a\in\mathcal{F}} \bar{\mu}_{a,i} = \sum_{b\in\mathcal{W}} \bar{\mu}_{i,b}, \forall i \in \mathcal{M}, \sum_{a\in\mathcal{M}} \bar{\mu}_{a,i} = \bar{\mu}_{i,i'}, \forall i \in \mathcal{W}, \\ & \bar{\mu}_{i',i} = \sum_{b\in\mathcal{R}} \bar{\mu}_{i',b}, \forall i \in \mathcal{W}, \sum_{a\in\mathcal{W}} \bar{\mu}_{a',i} \leq \bar{d}_i, \forall i \in \mathcal{R}. \end{aligned} \tag{1.4}$$

where $\bar{\mu}_{i,j} = E\{\mu_{i,j}(t)\}$ for all $(i,j) \in \mathcal{L}$.

In what follows we describe a randomized flow control algorithm, which picks values for $\mu_{i,j}(t)$ at each time instant based only on local knowledge of the state $S(t)$ of the supply chain at time t. The main idea of the algorithm consists of replacing the equality constrains with two inequality constraints and associating to each of the inequality constraints a queue. For example

$$\sum_{a\in\mathcal{F}} \bar{\mu}_{a,i} = \sum_{b\in\mathcal{W}} \bar{\mu}_{i,b},\ \forall i \in \mathcal{M}$$

is replaced by

$$\sum_{a\in\mathcal{F}} \bar{\mu}_{a,i} \leq \sum_{b\in\mathcal{W}} \bar{\mu}_{i,b},\ \sum_{a\in\mathcal{F}} \bar{\mu}_{a,i} \geq \sum_{b\in\mathcal{W}} \bar{\mu}_{i,b},\ \forall i \in \mathcal{M}.$$

To the first and the second previous inequalities we associate the following two queues with dynamics, respectively

$$U_i^1(t+1) = \max\left\{U_i^1(t) - \sum_b \mu_{i,b}(t), 0\right\} + \sum_a \mu_{a,i}(t), \forall i \in \mathcal{M}, \tag{1.5}$$

$$U_i^2(t+1) = \max\left\{U_i^2(t) - \sum_a \mu_{a,i}(t), 0\right\} + \sum_b \mu_{i,b}(t), \forall i \in \mathcal{M}. \tag{1.6}$$

In addition to the queues introduced in (1.5) and (1.6), we define queues corresponding to the rest of the constraints, with dynamics given by

$$U_i^1(t+1) = \max\left\{U_i^1(t) - \mu_{i,i'}(t), 0\right\} + \sum_a \mu_{a,i}(t),\ U_i^2(t+1) = \max\left\{U_i^2(t) - \sum_a \mu_{a,i}(t), 0\right\} + \mu_{i,i'}(t), \forall i \in \mathcal{W},$$

$$U_{i'}^1(t+1) = \max\left\{U_{i'}^1(t) - \sum_b \mu_{i',b}(t), 0\right\} + \mu_{i,i'}(t), U_{i'}^2(t+1) = \max\left\{U_{i'}^2(t) - \mu_{i,i'}(t), 0\right\} + \sum_b \mu_{i',b}(t), \forall i \in \mathcal{W}$$

$$U_i(t+1) = \max\left\{U_i^1(t) - d_i(t), 0\right\} + \sum_a \mu_{a,i}(t), \forall i \in \mathcal{R}.$$

1.3. Flow control algorithm. Next, we describe an algorithm that not only stabilizes the queues but also gets arbitrarily close to the optimal solution of (1.4). The closeness is characterized by a positive scalar δ, used as a parameter in the algorithm.

- *Control of the raw material flow:* At each time t it chooses the amount $\mu_{1,b}$ of raw material sent to manufacturer b, where $\mu_{1,b}$ is the solution of the following optimization problem:

$$\min_{\mu_{1,b}} \sum_{b\in\mathcal{M}} \left(\delta \boldsymbol{g}_{1,b}(\mu_{1,b}) + \left[U_b^1(t) - U_b^2(t)\right]\mu_{1,b}\right)$$
$$\text{subject to:} \quad \sum_{b\in\mathcal{M}} \mu_{1,b} \leq \mu_1^{max}, \mu_{1,b} \geq 0, \forall b.$$

- *Control of the flow of product from the manufacturers to the warehouses:* The amount of product sent to each warehouse b at time slot t is given by $\mu_{i,b}$, obtained as solution of the following optimization problem:

$$\min_{\mu_{i,b}} \sum_b \delta \boldsymbol{g}_{i,b}(\mu_{i,b}) - \left(\left[U_i^1(t) - U_b^1(t)\right] + \left[U_b^2(t) - U_i^2(t)\right]\right)\mu_{i,b}$$
$$\text{subject to:} \quad \sum_{b\in\mathcal{W}} \mu_{i,b} \leq \mu_i^{max},\ \mu_{i,b} \geq 0, \forall b,$$

  for all $i \in \mathcal{M}$, $b \in \mathcal{W}$ and $(i,b) \in \mathcal{L}$ which are active at time t, as per the state of the supply chain given by $S(t)$.

- *Control of the flow of product within the warehouses:* The amount of product allowed in the warehouse at time slot t is given by $\mu_{i,i'}$, obtained as solution of the following optimization problem:

$$\begin{aligned}&\min_{\mu} \delta \boldsymbol{g}_{i,i'}(\mu) - \left(\left[U_i^1(t) - U_{i'}^1(t)\right] + \left[U_{i'}^2(t) - U_i^2(t)\right]\right)\mu \\ &\text{subject to:} \qquad 0 \le \mu \le \mu_i^{max}\end{aligned}$$

  for all $i \in \mathcal{W}$ and $(i,i') \in \mathcal{L}$ which are active at time t, as per the state of the supply chain given by $S(t)$.
- *Control of the flow of product from the warehouse to retailers:* The amount of product sent to the retailer b at time slot t is given by $\mu_{i',b}$, where $\mu_{i',b}$ are obtained as solution of the following optimization problem:

$$\begin{aligned}&\min_{\mu_{i',b}} \sum_{b \in \mathcal{R}} \delta \boldsymbol{g}_{i',b}(\mu_{i',b}) - \delta \boldsymbol{f}_{i',b}(\mu_{i',b}) - \left[\left(U_{i'}^1(t) - U_b^1(t)\right) - U_{i'}^2(t)\right]\mu_{i',b} \\ &\text{subject to:} \qquad \sum_{b \in \mathcal{R}} \mu_{i',b} \le \mu_{i'}^{max}, \mu_{i',b} \ge 0, \ \forall b,\end{aligned}$$

  for all $i \in \mathcal{W}$, $b \in \mathcal{R}$ and $(i',b) \in \mathcal{L}$ which are active at time t, as per the state of the supply chain given by $S(t)$.

1.4. Performance of the algorithm. Let $\boldsymbol{\mu}^*$ and $\boldsymbol{h}^*$ be the optimal solution and cost value, respectively of the optimization problem (1.4). The next Theorem describes the performance of the flow control algorithm.

THEOREM 1.6. *Let Assumptions 1.1 through 1.5 hold. For any positive parameter δ the flow control algorithm described in Section 1.3 stabilizes the (virtual) queues associated with the constraints of the optimization problem (1.4) and gives the following upper bounds:*

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \left(\sum_{j=1}^{2} \sum_{i \in \mathcal{W}} E\{U_i^j(\tau)\} + \sum_{i \in \mathcal{M}} E\{U_i^j(\tau) + U_{i'}^j(\tau)\} + \sum_{i \in \mathcal{R}} E\{U_i(\tau)\right) \le c_1 + c_2\delta \tag{1.7}$$

$$\liminf_{t\to\infty} \boldsymbol{h}(\bar{\boldsymbol{\mu}}(t)) \ge \boldsymbol{h}(\boldsymbol{\mu}^*) - \frac{c_3}{\delta}, \tag{1.8}$$

where c_1, c_2 and c_3 are positive scalars depending on the parameters of the problem.

Note that inequality (1.7) shows that under the flow control algorithm, the queues remain stable, i.e., the long-run time averages of the flows are feasible. In addition, inequality (1.8) shows that under the flow control algorithm we can get arbitrarily close to the optimal solution, by making δ arbitrarily large.

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