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THE PARTIALLY OBSERVED STOCHASTIC MINIMUM PRINCIPLE*

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Abstract. Using stochastic flows and the generalized differentiation formula of Bismut and Kunita, the change in cost due to a strong variation of an optimal control is explicitly calculated. Differentiating this expression gives a minimum principle in both the partially observed and stochastic open loop situations. In the latter case the equation satisfied by the adjoint process is obtained by applying a martingale representation result.

Key words. stochastic control, minimum principle, adjoint process, stochastic flow

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1. Introduction. Various proofs have been given of the minimum principle satisfied by an optimal control in a partially observed stochastic control problem. See, for example, the papers by Bensoussan [1], Elliott [8], Haussmann [11], and the recent paper [14] by Haussmann in which the adjoint process is identified. The simple case of a partially observed Markov chain is discussed in the University of Maryland lecture notes [9] of Elliott.

In this article we show that the minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. The results of Bismut [5], [6] and Kunita [16] on stochastic flows enable us to compute in an easy and explicit way the change in the cost due to a "strong variation" of an optimal control. The only technical difficulty is the justification of the differentiation. As we wished to exhibit the simplification obtained by using the ideas of stochastic flows, the result is not proved under the weakest possible hypotheses. In § 6, stochastic open loop controls are considered and a similar minimum principle with an explicit adjoint process is derived in § 7. If the optimal control is Markov, the equation satisfied by the adjoint process is obtained in § 8 using the martingale representation result of [10]. This simplifies the proof of Haussmann [12]. Finally in § 9 it is pointed out how Bensoussan's minimum principle [2] follows from our result if the drift coefficient is differentiable in the control variable.

2. Dynamics. Suppose the state of the system is described by a stochastic differential equation

$$d\xi_t = f(t, \xi_t, u)\, dt + g(t, \xi_t)\, dw_t, \qquad (2.1)$$
$$\xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leqq t \leqq T.$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. We shall make the following assumptions:

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(A_1) x_0 is given; if x_0 is a random variable and P_0 its distribution, the situation when $\int |x|^q P_0(dx) < \infty$ for some $q > n+1$ can be treated, as in [14], by including an extra integration with respect to P_0.

(A_2) $f:[0, T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t, x), continuously differentiable in x and for some constant K_1, $(1+|x|)^{-1}|f(t, x, u)| + |f_x(t, x, u)| \leqq K_1$.

(A_3) $g:[0, T] \times R^d \to R^d \otimes R^n$ is a matrix-valued function, Borel measurable, continuously differentiable in x, and for some constant K_2, $|g(t, x)| + |g_x(t, x)| \leqq K_2$.

The observation process is given by

$$dy_t = h(\xi_t)\, dt + dv_t, \qquad y_t \in R^m, \quad y_0 = 0, \quad 0 \leqq t \leqq T. \tag{2.2}$$

In the above equations $w = (w^1, \cdots, w^n)$ and $v = (v^1, \cdots, v^d)$ are independent Brownian motions. We also assume the following:

(A_4) $h: R^d \to R^m$ is Borel measurable, continuously differentiable in x, and for some constant K_3, $|h(t, x)| + |h_x(t, x)| \leqq K_3$.

Remark 2.1. These hypotheses can be weakened. For example, in (A_4), h can be allowed linear growth in x. Because g is bounded, a delicate argument then implies the exponential Z of (2.3) is in some L^p space, $1 < p < \infty$. (See, for example, Theorem 2.2 of [13].) However, when h is bounded, Z is in all the L^p spaces (see Lemma 2.3). Also, if we require f to have linear growth in u, then the set of control values U can be unbounded as in [14]. Our objective, however, is not the greatest generality but is to demonstrate the simplicity of the techniques of stochastic flows.

Let $\hat{P}$ denote Wiener measure on $C([0, T], R^n)$ and μ denote Wiener measure on $C([0, T], R^m)$. Consider the space $\Omega = C([0, T], R^n) \times C([0, T], R^m)$ with coordinate functions (w_t, y_t) and define Wiener measure P on Ω by

$$P(dw, dy) = \hat{P}(dw)\mu(dy).$$

DEFINITION 2.2. Write $Y = \{Y_t\}$ for the right continuous complete filtration on $C([0, T], R^m)$ generated by $Y_t^0 = \sigma\{y_s : s \leqq t\}$. The set of admissible control functions $\underline{U}$ will be the Y-predictable functions on $[0, T] \times C([0, T], R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$ write $\xi^u_{s,t}(x)$ for the strong solution of (2.1) corresponding to control u, and with $\xi^u_{s,s}(x) = x$. Write

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))'\, dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2\, dr\right) \tag{2.3}$$

and define a new probability measure P^u on Ω by $dP^u/dP = Z^u_{0,T}(x_0)$. Then under P^u, $(\xi^u_{0,t}(x_0), y_t)$ is a solution of (2.1) and (2.2), that is, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion v such that y_t satisfies (2.2). A version of Z defined for every trajectory y of the observation process is obtained by integrating by parts the stochastic integral in (2.3).

LEMMA 2.3. *Under hypothesis* (A_4) *for* $t \leqq T$,

$$E[(Z^u_{0,t}(x_0))^p] < \infty \quad \text{for all } u \in \underline{U} \text{ and all } p, \quad 1 \leqq p < \infty.$$

Proof.

$$Z^u_{0,t}(x_0) = 1 + \int_0^t Z^u_{0,r}(x_0) h(\xi^u_{0,r}(x_0))'\, dy_r.$$

Therefore, for any p there is a constant C_p such that

$$E[(Z^u_{0,t}(x_0))^p] \leqq C_p\left[1+E\left(\int_0^t (Z^u_{0,r}(x_0))^2 h(\xi^u_{0,r}(x_0))^2\,dr\right)^{p/2}\right].$$

The result follows by Gronwall's inequality.

Cost 2.4. We shall suppose the cost is purely terminal and given by some bounded, continuously differentiable function

$$c(\xi^u_{0,T}(x_0)),$$

which has bounded derivatives. Then the expected cost, if control $u \in \underline{U}$ is used, is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

In terms of P, under which y_t is always a Brownian motion, this is

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.4}$$

3. Stochastic flows. For $u \in \underline{U}$ write

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)\,dr + \int_s^t g(r, \xi^u_{s,r}(x))\,dw_r \tag{3.1}$$

for the solution of (2.1) over the time interval $[s, t]$ with initial condition $\xi^u_{s,s}(x) = x$. In the sequel we wish to discuss the behavior of (3.1) for each trajectory y of the observation process. We have already noted that there is a version of Z defined for every y. The results of Bismut [5] and Kunita [16] extend easily and show the map

$$\xi^u_{s,t}: R^d \to R^d$$

is, almost surely, for each $y \in C([0, T], R^m)$ a diffeomorphism. Bismut [5] initially gives proofs when the coefficients f and g are bounded, but points out that a stopping time argument extends the results to when, for example, the coefficients have linear growth.

Write $\|\xi^u(x_0)\|_t = \sup_{0\leqq x\leqq t} |\xi^u_{0,s}(x_0)|$. Then, as in Lemma 2.1 of [13], for any p, $1 \leqq p < \infty$, using Gronwall's and Jensen's inequalities,

$$\|\xi^u(x_0)\|^p_T \leqq C\left(1+|x_0|^p + \left|\int_0^t g(r, \xi^u_{0,r}(x_0))\,dw_r\right|^p\right)$$

almost surely, for some constant C.

Therefore, using Burkholder's inequality and hypothesis (A_3), $\|\xi^u(x_0)\|_T$ is in L^p for all p, $1 \leqq p < \infty$.

Suppose $u^* \in \underline{U}$ is an optimal control; then $J(u^*) \leqq J(u)$ for any other $u \in \underline{U}$. Write $\xi^*_{s,t}(\cdot)$ for $\xi^{u^*}_{s,t}(\cdot)$. The derivative $\partial\xi^*_{s,t}(x)/\partial x$ is the matrix solution C_t of the equation for $s \leqq t$,

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t\,dt + \sum_{i=1}^n g^{(i)}_x(t, \xi^*_{s,t}(x))C_t\,dw^i_t \quad \text{with } C_s = I. \tag{3.2}$$

Here I is the $n \times n$ identity matrix and $g^{(i)}$ is the ith column of g. From hypotheses (A_2) and (A_3), f_x and g_x are bounded. When we write $\|C\|_t = \sup_{0\leqq s\leqq t}|C_s|$, an application of Gronwall's, Jensen's, and Burkholder's inequalities again implies $\|C\|_T$ is in

L^p for all p, $1 \leqq p < \infty$. Consider the related matrix-valued stochastic differential equation

$$D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r)' \, dr - \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x))' \, dw^i_r + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x))')^2 \, dr. \tag{3.3}$$

Then it can be checked that $D_t C_t = I$ for $t \geqq s$, so that D_t is the inverse of the Jacobian, that is, $D_t = (\partial \xi^*_{s,t}(x)/\partial x)^{-1}$. Again, because f_x and g_x are bounded we have that $\|D\|_t$ is in every L^p, $1 \leqq p < \infty$.

For a d-dimensional semimartingale z_t Bismut [5] shows that $\xi^*_{s,t}(z_t)$ is well-defined and gives the semimartingale representation of this process. In fact if $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i \, dw^i_r$ is a d-dimensional semimartingale, Bismut's formula states that

$$\begin{aligned} \xi^*_{s,t}(z_t) = z_s &+ \int_s^t \Bigg(f(r, \xi^*_{s,r}(z_r), u^*_r) + \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r) \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \\ &+ \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \xi^*_{s,r}(z_r)}{\partial x^2}(H_i, H_i) \Bigg) dr \\ &+ \int_s^t \frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \, dA_r + \sum_{i=1}^n \int_s^t \left(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \right) dw^i_r. \end{aligned} \tag{3.4}$$

DEFINITION 3.1. We shall consider perturbations of the optimal control u^* of the following kind. For $s \in [0, T)$, $h > 0$ such that $0 \leqq s < s + h \leqq T$, for any other admissible control $\tilde{u} \in \underline{U}$ and $A \in Y_s$ define a strong variation of u^* by

$$u(t, w) = \begin{cases} u^*(t, w) & \text{if } (t, w) \notin [s, s+h] \times A, \\ \tilde{u}(t, w) & \text{if } (t, w) \in [s, s+h] \times A. \end{cases}$$

Applying (3.4) as in Theorem 5.1 of [7], we have the following result.

THEOREM 3.2. *For the perturbation u of the optimal control u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r)) \, dr. \tag{3.5}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

Proof. Note that the equation defining z_t involves only an integral in time; there is no martingale term, so to apply (3.4) we have $H_i = 0$ for all i. Therefore, from (3.4)

$$\begin{aligned} \xi^*_{s,t}(z_t) = x &+ \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r) \, dr \\ &+ \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right) \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r)) \, dr \\ &+ \int_s^t g(r, \xi^*_{s,r}(z_r)) \, dw_r. \end{aligned}$$

However, the solution of (3.1) is unique so

$$\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x).$$

Remark 3.3. Note that the perturbation $u(t)$ equals $u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$\xi^*_{s,t}(z_t) = \xi^*_{s,t}(z_{s+h}) = \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)).$$

4. Augmented flows. Consider the augmented flow that includes as an extra coordinate the stochastic exponential $Z^*_{s,t}$ with a "variable" initial condition $z \in R$ for $Z^*_{s,s}(\cdot)$. That is, consider the $(d+1)$-dimensional system given by

$$\xi^*_{s,t}(x) = x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)\, dr + \int_s^t g(r, \xi^*_{s,r}(x))\, dw_r,$$

$$Z^*_{s,t}(x, z) = z + \int_s^t Z^*_{s,r}(x, z) h(\xi^*_{s,r}(x))'\, dy_r.$$

Therefore, from the first equation in the proof of Lemma 2.3 we have

$$\begin{aligned} Z^*_{s,t}(x, z) &= z Z^*_{s,t}(x) \\ &= z \exp\left(\int_s^t h(\xi^*_{s,r}(x))'\, dy_r - \frac{1}{2}\int_s^t h(\xi^*_{s,r}(x))^2\, dr\right) \end{aligned}$$

and we see there is a version of the enlarged system defined for each trajectory y by integrating by parts the stochastic integral. The augmented map $(x, z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x, z))$ is then almost surely a diffeomorphism of R^{d+1}. Note that $\partial\xi^*_{s,t}(x)/\partial z = 0$, $\partial f/\partial z = 0$ and $\partial g/\partial z = 0$. The Jacobian of this augmented map is, therefore, represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \partial\xi^*_{s,t}(x)/\partial x & 0 \\ \partial Z^*_{s,t}(x, z)/\partial x & \partial Z^*_{s,t}(x, z)/\partial z \end{pmatrix},$$

and for $1 \leqq i \leqq d$ as in (3.2)

$$\begin{aligned} \frac{\partial Z^*_{s,t}(x, z)}{\partial x_i} = \sum_{j=1}^m \int_s^t \Bigg(Z^*_{s,r}(x, z) \frac{\partial h^j(\xi^*_{s,r}(x))}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}(x)}{\partial x_i} \\ + h^j(\xi^*_{s,r}(x)) \frac{dZ^*_{s,r}(x, z)}{\partial x_i} \Bigg) dy^j_r. \end{aligned} \tag{4.1}$$

(Here the double index k is summed from 1 to n.)

We shall be interested in the solution of this differential system (4.1) only in the situation when $z = 1$, so we shall write $Z^*_{s,t}(x)$ for $Z^*_{s,t}(x, 1)$. The following result is motivated by formally differentiating the exponential formula for $Z^*_{s,t}(x)$.

LEMMA 4.1.

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = Z^*_{s,t}(x)\left(\int_s^t h_x(\xi^*_{s,r}(x)) \cdot \frac{\partial \xi^*_{s,r}(x)}{\partial x} \cdot dv_r\right)$$

where $v = (v^1, \cdots, v^n)$ is the Brownian motion in the observation process.

Proof. From (4.1) we see $\partial Z^*_{s,t}(x)/\partial x$ is the solution of the stochastic differential equation

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = \int_s^t \left(\frac{\partial Z^*_{s,r}(x)}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}(x)}{\partial x}\right) dy_r. \tag{4.2}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\left(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} \cdot dv_r\right)$$

where

$$dy_r = h(\xi^*_{s,t}(x))\, dt + dv_t.$$

Because

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r$$

the product rule gives

$$\begin{aligned} L_{s,t}(x) &= \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x}\, dv_r + \int_s^t \left(\int_s^r h_x \cdot \frac{\partial \xi^*_{s,\sigma}}{\partial x} \cdot dv_\sigma \right) Z^*_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r \\ &\quad + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) \cdot h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x}\, dr \\ &= \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r + \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} \cdot dy_r. \end{aligned}$$

Therefore, $L_{s,t}(x)$ is also a solution of (4.2), so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^*_{s,t}(x)}{\partial x}.$$

Remark 4.2. As noted at the beginning of this section we can consider the augmented flow

$$(x, z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x, z)) \quad \text{for } x \in R^d, z \in R,$$

and we are only interested in the situation when $z = 1$, so we write $Z^*_{s,t}(x)$.

LEMMA 4.3. *$Z^*_{s,t}(z_t) = Z^u_{s,t}(x)$ where z_t is the semimartingale defined in* (3.6).

Proof. $Z^u_{s,t}(x)$ is the process uniquely defined by

$$Z^u_{s,t}(x) = 1 + \int_s^t Z^u_{s,r}(x) h'(\xi^u_{s,r}(x))\, dy_r. \tag{4.3}$$

Consider an augmented $(d+1)$-dimensional version of (3.5) defining a semimartingale $\bar{z}_t = (z_t, 1)$, so the additional component is always identically one. Then applying (3.4) to the new component of the augmented process, we have

$$\begin{aligned} Z^*_{s,r}(z_r) &= 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^*_{s,r}(z_r))\, dy_r \\ &= 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^u_{s,r}(x))\, dy_r \end{aligned}$$

by Theorem 3.2. However, (4.3) has a unique solution so $Z^*_{s,t}(z_t) = Z^u_{s,t}(x)$.

Remark 4.4. Note that for $t > s + h$

$$Z^*_{s,t}(z_t) = Z^*_{s,t}(z_{s+h}).$$

5. The minimum principle. Control u will be the perturbation of the optimal control u^* as in Definition 3.1. We shall write $x = \xi^*_{0,s}(x_0)$. Then the minimum cost is

$$\begin{aligned} J(u^*) &= E[Z^*_{0,T}(x_0) c(\xi^*_{0,T}(x_0))] \\ &= E[Z^*_{0,s}(x_0) Z^*_{s,T}(x) c(\xi^*_{s,T}(x))]. \end{aligned}$$

The cost corresponding to the perturbed control u is

$$\begin{aligned} J(u) &= E[Z^*_{0,s}(x_0) Z^u_{s,T}(x) c(\xi^u_{s,T}(x))] \\ &= E[Z^*_{0,s}(x_0) Z^*_{s,T}(z_{s+h}) c(\xi^*_{s,T}(z_{s+h}))] \end{aligned}$$

by Theorem 3.2 and Lemma 4.3. Now $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are almost surely differentiable with continuous derivatives and z_t, given by (3.5), is absolutely continuous. Therefore,

$$J(u)-J(u^*)=E[Z^*_{0,s}(x_0)(Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))-Z^*_{s,T}(x)c(\xi^*_{s,T}(x)))]$$
$$=E\left[\int_s^{s+h}\Gamma(s,z_r)(f(r,\xi^*_{s,r}(z_r),u^*_r)-f(r,\xi^*_{s,r}(x),u^*_r))\,dr\right]$$

where by Lemma 4.1

$$\Gamma(s,z_r)=Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\left\{c_\xi(\xi^*_{s,T}(z_r))\frac{\partial\xi^*_{s,T}(z_r)}{\partial x}\right.$$
$$\left.+c(\xi^*_{s,T}(z_r))\left(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\frac{\partial\xi^*_{s,\sigma}}{\partial x}(z_r)\,dv_\sigma\right)\right\}\left(\frac{\partial\xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}.$$

Note that this expression gives an explicit formula for the change in the cost resulting from a variation in the optimal control. The only remaining problem is to justify differentiating the right-hand side.

From Lemma 2.3, Z is in every L^p space, $1\leqq p<\infty$, and from the remarks at the beginning of § 3, $C_T=\partial\xi^*_{s,T}/\partial x$ and $D_T=(\partial\xi^*_{s,T}/\partial x)^{-1}$ are in every L^p space, $1\leqq p<\infty$. Consequently, Γ is in every L^p space, $1\leqq p<\infty$.

Therefore,

$$J(u)-J(u^*)=\int_s^{s+h}E[(\Gamma(s,z_r)-\Gamma(s,x))(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[(\Gamma(s,x)-\Gamma(r,x))(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[\Gamma(r,x)(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r)$$
$$-f(r,\xi^*_{s,r}(x),u_r)+f(r,\xi^*_{s,r}(x),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[\Gamma(r,x)(f(r,\xi^*_{0,r}(x_0),u_r)-f(r,\xi^*_{0,r}(x_0),u^*_r))]\,dr$$
$$=I_1(h)+I_2(h)+I_3(h)+I_4(h),\quad\text{say}.$$

Now,

$$|I_1(h)|\leqq K_1\int_s^{s+h}E[|\Gamma(s,z_r)-\Gamma(s,x)|(1+\|\xi^u(x_0)\|_{s+h})]\,dr$$
$$\leqq K_1h\sup_{s\leqq r\leqq s+h}E[|\Gamma(s,z_r)-\Gamma(s,x)|(1+\|\xi^u(x_0)\|_{s+h})],$$
$$|I_2(h)|\leqq K_2\int_s^{s+h}E[|\Gamma(s,x)-\Gamma(r,s)|(1+\|\xi^u(x_0)\|_{s+h})]\,dr$$
$$\leqq K_2h\sup_{s\leqq r\leqq s+h}E[|\Gamma(s,x)-\Gamma(r,x)|(1+\|\xi^u(x_0)\|_{s+h})],$$
$$|I_3(h)|\leqq K_3\int_s^{s+h}E[|\Gamma(r,x)|\,\|\xi^*_{s,r}(z_r)-\xi^*_{s,r}(x)\|]\,dr$$
$$\leqq K_3h\sup_{s\leqq r\leqq s+h}E[|\Gamma(r,x)|\,\|\xi^u_{s,\cdot}(x)-\xi^*_{s,\cdot}(x)\|_{s+h}].$$

The differences $|\Gamma(s, z_r)-\Gamma(s, x)|$, $|\Gamma(s, x)-\Gamma(r, x)|$ and $\|\xi^u_{s,\cdot}(x)-\xi^*_{s,\cdot}(x)\|_{s+h}$ are all uniformly bounded in some L^p, $p \geqq 1$, and

$$\lim_{r\to s} |\Gamma(s, z_r)-\Gamma(s, x)| = 0 \quad \text{a.s.},$$

$$\lim_{r\to s} |\Gamma(s, x)-\Gamma(r, x)| = 0 \quad \text{a.s.},$$

$$\lim_{h\to 0} \|\xi^u_{s,\cdot}(x)-\xi^*_{s,\cdot}(x)\|_{s+h} = 0.$$

Therefore,

$$\lim_{r\to s} \|\Gamma(s, z_r)-\Gamma(s, x)\|_p = 0,$$

$$\lim_{r\to s} \|\Gamma(s, x)-\Gamma(r, x)\|_p = 0, \quad \text{and}$$

$$\lim_{h\to 0} \|(\|\xi^u_{s,\cdot}(x)-\xi^*_{s,\cdot}(x)\|_{s+h})\|_p = 0 \quad \text{for some } p.$$

Consequently, $\lim_{h\to 0} h^{-1} I_k(h) = 0$, for $k = 1, 2, 3$.

The only remaining problem concerns the differentiability of

$$I_4(h) = \int_s^{s+h} E[\Gamma(r, x)(f(r, \xi^*_{0,r}(x_0), u_r)-f(r, \xi^*_{0,r}(x_0), u^*_r))]\, dr.$$

The integrand is almost surely in $L^1([0, T])$ so $\lim_{h\to 0} h^{-1} I_4(h)$ exists for almost every $s \in [0, T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the control u. Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

(1) Because the trajectories are, almost surely, continuous, Y_ρ is countably generated by sets $\{A_{i\rho}\}$, $i = 1, 2, \cdots$ for any rational number $\rho \in [0, T]$. Consequently, Y_t is countably generated by the sets $\{A_{i\rho}\}$, $\rho \leqq t$.

(2) Let G_t denote the set of measurable functions from (Ω, Y_t) to $U \subset R^k$. (If $u \in \underline{U}$ then $u(t, w) \in G_t$.) Using the L^1-norm, as in [8], there is a countable dense subset $H_\rho = \{u_{j\rho}\}$ of G_ρ, for rational $\rho \in [0, T]$. If $H_t = \bigcup_{\rho \leqq t} H_\rho$ then H_t is a countable dense subset of G_t. If $u_{j\rho} \in H_\rho$ then, as a function constant in time, $u_{j\rho}$ can be considered as an admissible control over the time interval $[t, T]$ for $t \geqq \rho$.

(3) The countable family of perturbations is obtained by considering sets $A_{i\rho} \in Y_t$, functions $u_{j\rho} \in H_t$, where $\rho \leqq t$, and defining as in (3.1) the following:

$$u^*_{j\rho}(s, w) = \begin{cases} u^*(s, w) & \text{if } (s, w) \notin [t, T] \times A_{i\rho}, \\ u_{j\rho}(s, w) & \text{if } (s, w) \in [t, T] \times A_{i\rho}. \end{cases}$$

Then for each i, j, ρ

$$\lim_{h\to 0} h^{-1} \int_s^{s+h} E[\Gamma(r, x)(f(r, \xi^*_{0,r}(x_0), u^*_{j\rho})-f(r, \xi^*_{0,r}(x_0), u^*))]\, dr \tag{5.1}$$

exists and equals

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho})-f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}]$$

for almost all $s \in [0, T]$. Therefore, considering this perturbation we have

$$\lim_{h\to 0} h^{-1}(J(u^*_{j\rho})-J(u^*)) = E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho})-f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}]$$

$$\geqq 0 \quad \text{for almost all } s \in [0, T].$$

Consequently there is a set $S \subset [0, T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (5.1) exists for all i, j, ρ, and gives

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}] \geqq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underset{\sim}{U}$, we can deduce that if $s \notin S$, then

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u) - f(s, \xi^*_{0,s}(x_0), u^*))I_A] \geqq 0 \quad \text{for any } A \in Y_s. \tag{5.2}$$

Write

$$p_s(x) = E^*\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}(x)}{\partial x} + c(\xi^*_{0,T}(x_0))\left(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\frac{\partial \xi^*_{s,\sigma}(x)}{\partial x}\, dv_\sigma\right) \middle| \, Y_s \vee \{x\}\right]$$

where, as before, $x = \xi^*_{0,s}(x_0)$ and E^* denotes expectation under $P^* = P^{u^*}$. Then $p_s(x)$ is the co-state variable and we have in (5.2) proved the following "conditional" minimum principle.

THEOREM 5.1. *If $u^* \in \underset{\sim}{U}$ is an optimal control there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$*

$$E^*[p_s(x)f(s, x, u^*) \mid Y_s] \geqq E^*[p_s(x)f(s, x, u) \mid Y_s] \quad a.s.$$

That is, the optimal control u^* almost surely minimizes the conditional Hamiltonian and the adjoint variable is $p_s(x)$.

6. Stochastic open loop controls. We shall again suppose the state of the system is described by a stochastic differential equation

$$d\xi_t = f(t, \xi_t, u)\, dt + g(t, \xi_t)\, dw_t, \qquad \xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leqq t \leqq T \tag{6.1}$$

where x_0, f, and g satisfy the same assumptions A_1, A_2, and A_3 as in § 2.

Suppose $w = (w^1, \cdots, w^n)$ is an n-dimensional Brownian motion on a probability space (Ω, F, P), with a right continuous complete filtration $\{F_t\}$, $0 \leqq t \leqq T$. Rather than controls depending on some observation process y we now consider controls that depend on the "noise process" w. These are sometimes called "stochastic open loop" controls [4].

DEFINITION 6.1. The set of admissible controls $\underset{\sim}{V}$ will be the F_t-predictable functions on $[0, T] \times \Omega$ with values in a compact subset V of some Euclidean space R^k.

Remark 6.2. For each $u \in \underset{\sim}{V}$ there is, therefore, a strong solution of (6.1) and we shall write $\xi^u_{s,t}(x)$ for the solution trajectory given by

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)\, dr + \int_s^t g(r, \xi^u_{s,r}(x))\, dw_r. \tag{6.2}$$

Again, because u is a (predictable) parameter the results of [2], [5], or [16] extend to this situation, so the derivative $\partial \xi^u_{s,t}/\partial x(x) = C^u_{s,t}$ exists and is the solution of

$$C^u_{s,t} = I + \int_s^t f_\xi(r, \xi^u_{s,r}(x), u_r)C^u_{s,r}\, dr + \sum_{k=1}^n \int_s^t g^{(k)}_\xi(r, \xi^u_{s,r}(x))C^u_{s,r}\, dw^k_r. \tag{6.3}$$

Suppose $D^u_{s,t}$ is the matrix-valued process defined by

$$\begin{aligned} D^u_{s,r} = I - \int_s^t D^u_{s,t}\left(f_\xi(r, \xi^u_{s,r}(x), u_r) - \sum_{k=1}^n g^{(k)}_\xi(r, \xi^u_{s,r}(x))^2\right) dr \\ - \sum_{k=1}^n \int_s^t D^u_{s,r}\, g^{(k)}_\xi(r, \xi^u_{s,r}(x))\, dw^k_r. \end{aligned} \tag{6.4}$$

Using the Itô rule as in § 3 we see that $d(D^u_{s,t}C^u_{s,t})=0$ and $D^u_{s,s}C^u_{s,s}=I$, so

$$D^u_{s,t}=(C^u_{s,t})^{-1}.$$

As before, if

$$\|\xi^u(x_0)\|_t=\sup_{0\leqq s\leqq t}|\xi^u_{0,s}(x_0)|,$$

$$\|C^u\|_T=\sup_{0\leqq s\leqq T}|C^u_{0,s}|, \qquad \|D^u\|_T=\sup_{0\leqq s\leqq T}|D^u_{0,s}|,$$

then applications of Gronwall's, Jensen's, and Burkholder's inequalities imply that

$$\|\xi^u(x_0)\|_k, \quad \|C^u\|_T, \quad \text{and} \quad \|D^u\|_T$$

are in L^p for all p, $1\leqq p<\infty$.

Cost 6.3. As in § 2, we shall suppose the cost is purely terminal and given by a bounded C^2 function

$$c(\xi^u_{0,T}(x_0)).$$

Furthermore, we shall assume

$$|c(x)|+|c_x(x)|+|c_{xx}(x)|\leqq K_3(1+|x|^q)$$

for some $q<\infty$.

The expected cost if a control $u\in \underline{V}$ is used, therefore, is

$$J(u)=E[c(\xi^u_{0,T}(x_0))].$$

Suppose there is an optimal control $u^*\in \underline{V}$ so that

$$J(u^*)\leqq J(u) \quad \text{for all } u\in \underline{V}.$$

Notation 6.4. If u^* is an optimal control, write ξ^* for ξ^{u^*}, C^* for C^{u^*}, etc.

DEFINITION 6.5. Consider perturbations of u^* of the following kind. For $s\in[0,T]$, $h>0$ such that $0\leqq s<s+h\leqq T$ and $A\in F_s$ define, for any other $\tilde{u}\in \underline{V}$, a strong variation of u^* by

$$u(t,w)=\begin{cases} u^*(t,w) & \text{if } (t,w)\notin[s,s+h]\times A,\\ \tilde{u}(t,w) & \text{if } (t,w)\in[s,s+h]\times A.\end{cases}$$

The following result is established exactly as Theorem 3.2.

THEOREM 6.6. *For any perturbation u of* u^* *consider the process*

$$z_r=x+\int_s^t\left(\frac{\partial\xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))\,dr. \tag{6.5}$$

Then the process $\xi^*_{s,t}(z_t)$ *is indistinguishable from* $\xi^u_{s,t}(x)$.

Note if $t>s+h$, $\xi^*_{s,t}(z_t)=\xi^*_{s,t}(z_{s+h})=\xi^*_{s+h,t}(\xi^u_{s,s+h}(x))$.

7. An open loop minimum principle. Now

$$\begin{aligned}J(u^*)&=E[c(\xi^*_{0,T}(x_0))]\\&=E[c(\xi^*_{s,T}(x))]\end{aligned}$$

where $x=\xi^*_{0,s}(x_0)$.

Similarly,

$$\begin{aligned}J(u)&=E[c(\xi^u_{0,T}(x_0))]\\&=E[c(\xi^u_{s,T}(x))]\\&=E[c(\xi^*_{s,T}(z_{s+h}))].\end{aligned}$$

Therefore,

$$J(u)-J(u^*)=E[c(\xi^*_{s,T}(z_{s+h}))-c(\xi^*_{s,T}(x))].$$

Because $\xi^*_{s,T}(\cdot)$ is differentiable this is

$$=E\left[\int_s^{s+h} c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}}{\partial x}(z_r)\cdot\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}(f(r,\xi^*_{s,r}(z_r),u_r) -f(r,\xi^*_{s,r}(z_r),u^*_r))\,dr\right]. \tag{7.1}$$

As in § 5, this gives an explicit formula for the change in the cost resulting from a "strong variation" in the optimal stochastic open loop control. It involves a time integration over $[s, s+h]$ and, again, the only remaining problem is to justify the differentiation of the right-hand side of (7.1).

Write

$$\Gamma(s,r,z_r)=c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}}{\partial x}(z_r)\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}$$

and

$$\begin{aligned} p_s(x)&=E\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}}{\partial x}(x)\,|\,F_s\right]\\ &=E[\Gamma(s,s,x)\,|\,F_s], \end{aligned} \tag{7.2}$$

where, as above, $x=\xi^*_{0,s}(x_0)$.

Then arguments similar to those of § 5—but in fact simpler because Z is not involved—enable us to show that there is a set $S\subset[0,T]$ of zero Lebesque measure such that if $s\notin S$,

$$E[\Gamma(s,s,x)(f(s,\xi^*_{0,s}(x_0),u)-f(s,\xi^*_{0,s}(x_0),u^*))I_A]\geqq 0$$

for any $u\in V$ and $A\in F_s$.

That is, in terms of the adjoint variable $p_s(x)$ we have the following minimum principle for stochastic open loop controls.

THEOREM 7.1. *If $u^*\in V$ is an optimal stochastic open loop control there is a set $S\subset[0,T]$ of zero Lebesgue measure such that if $s\notin S$*

$$p_s(x)f(s,x,u^*)\leqq p_s(x)f(s,x,u)\quad a.s.$$

for all $u\in V$. That is, the optimal control u^* almost surely minimizes the Hamiltonian with adjoint variable $p_s(x)$.

Remark 7.2. Under certain conditions the minimum cost attainable under the stochastic open loop controls is equal to the minimum cost attainable under the Markov feedback controls of the form $u(s,\xi^u_{0,s}(x_0))$. See for example [3], [12]. If u_M is a Markov control, with a corresponding, possibly weak, solution trajectory ξ^{u_M}, then u_M can be considered as a stochastic open loop control $u_M(w)$ by putting

$$u_M(w)=u_M(s,\xi^{u_M}_{0,s}(x_0,w)).$$

This means the control in effect "follows" its original trajectory ξ^{u_M} rather than any new trajectory. That is, the control is similar to the adjoint strategies considered by Krylov [15]. The significance of this is that when we consider variations in the state trajectory ξ, and derivatives of the map $x\to\xi_{s,t}(x)$, the control does not react, and so we do not introduce derivatives in the u variable.

If the optimal control u^* is the Markov, then the process ξ^* is Markov and

$$
\begin{aligned}
p_s(x) &= E[\Gamma(s, s, x) \mid F_s] \\
&= E[\Gamma(s, s, x) \mid x].
\end{aligned}
\tag{7.3}
$$

8. The adjoint process. Suppose the optimal stochastic open loop control u^* is Markov. The Jacobian $\partial\xi^*_{s,T}/\partial x$ exists, as does $(\partial\xi^*_{s,T}/\partial x)^{-1}$ and higher derivatives.

THEOREM 8.1. *Suppose the optimal control u^* is Markov. Then*

$$
\begin{aligned}
p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0))C_{0,t}] &- \int_0^s p_r(\xi^*_{0,r}(x_0)) f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)\, dr \\
&+ \int_0^s p_x(r, \xi^*_{0,r}(x_0)) g(r, \xi^*_{0,r}(x_0))\, dw_r \\
&- \int_0^s p_x(r, \xi^*_{0,r}(x_0)) g(r, \xi^*_{0,r}(x_0)) g_\xi(r, \xi^*_{0,r}(x_0))\, dr.
\end{aligned}
$$

Proof. Write $f_\xi(r)$ for $f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)$ and $g(r)$ for $g(r, \xi^*_{0,r}(x_0))$, etc. By uniqueness of the solutions to (6.1)

$$
\xi^*_{0,T}(x_0) = \xi^*_{s,T}(\xi^*_{0,s}(x_0))
\tag{8.1}
$$

so, differentiating,

$$
C_{0,T} = C_{s,T} C_{0,s}
\tag{8.2}
$$

where $C_{0,T} = C^*_{0,T}$, etc. (without the *).

From (7.2) and (7.3)

$$
p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0)) C_{s,T} \mid F_s],
$$

so from (8.2)

$$
p_s(x) C_{0,s} = E[c_\xi(\xi^*_{0,T}(x_0)) C_{0,T} \mid F_s],
\tag{8.3}
$$

and this is a $(P, \{F_t\})$ martingale. Write $x = \xi^*_{0,s}(x_0)$, $C = C_{0,s}$. From the martingale representation result [10], the integrand in the representation of $p_s(x)C$ as a stochastic integral is obtained by the Itô rule, noting that only the stochastic integral terms will appear. These involve the derivatives in x and C. In fact, by considering the system $\bar{\xi}_{0,t}$ with components $\xi^*_{0,t}$ and $C_{0,t}$ and any real C^2 function Φ, the martingale

$$
\begin{aligned}
M_s &= E[\Phi(\bar{\xi}_{0,T} \mid F_s] = E[\Phi(\bar{\xi}_{0,T}) \mid x, C] = V(s, x, C) \\
&= V(0, x_0, I) + \int_0^s V_x(r, \xi^*_{0,r}(x_0), C_{0,r}) g(r)\, dw_r \\
&\quad + \sum_{k=1}^n \int_0^s V_C(r, \xi^*_{0,r}(x_0), C_{0,r}) g^{(k)}_\xi(r) C_{0,r}\, dw^k_r.
\end{aligned}
$$

Therefore, for the vector martingale (8.3)

$$
\begin{aligned}
p_s(x) C = E[c_\xi(\xi_{0,T}(x_0)) C_{0,T}] &+ \int_0^s p_x(r, \xi^*_{0,r}(x_0)) g(r)\, dw_r C_{0,r} \\
&+ \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0)) g^{(k)}_\xi(r) C_{0,r}\, dw^k_r.
\end{aligned}
\tag{8.4}
$$

Recall that $D_{0,s}=C^{-1}$, so forming the product of (6.4) and (8.4) by using the Itô rule, we have

$$
\begin{aligned}
p_s(x) &= (p_x(x)C)D_{0,s} \\
&= E[c_\xi(\xi^*_{0,T}(x_0))C_{0,T}] - \int_0^s p_r(\xi^*_{0,r}(x_0))f_\xi(r)\,dr \\
&\quad - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)\,dw_r^k + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2\,dr \\
&\quad + \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)\,dw_r + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)\,dw_r^k \\
&\quad - \sum_{k=1}^n \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)\,dr - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2\,dr \\
&= E[c_\xi(\xi^*_{0,T}(x_0))C_{0,T}] - \int_0^s p_r(\xi^*_{0,T}(x_0))f_\xi(r)\,dr \\
&\quad + \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)\,dw_r - \sum_{k=1}^n \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)\,dr,
\end{aligned}
$$

thus establishing the result.

This verifies by a simple, direct method the formula of Haussmann [12] without any requirement that the diffusion coefficient matrix gg^* is nonsingular. However we do not identify $p_s(x)$ with the gradient of the minimum cost process; this follows from arguments as in [12].

9. Conclusion. Using the theory of stochastic flows the effect of a perturbation of an optimal control is explicitly calculated in both the partially observed and stochastic open loop cases. The only difficulty is to justify the differentiation. The adjoint variable $p_s(x)$ is explicitly identified.

THEOREM 9.1. *If f is differentiable in the control variable u, and if the random variable $x=\xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under the measure P^*, then the inequality of Theorem* 5.1 *implies*

$$\sum_{j=1}^k (u_j(x)-u_j^*(s)) \int_{R^d} \Gamma(s,x)\frac{\partial f}{\partial u_j}(s,x,u^*)q_s(x)\,dx \geqq 0.$$

This is the result of Bensoussan's paper [1].

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