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Existence, Uniqueness, and Asymptotic Behavior of Solutions to a Class of Zakai Equations with Unbounded Coefficients

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***Abstract*—Conditions are given to guarantee the existence and uniqueness of solutions to the Zakai equation associated with the nonlinear filtering of diffusion processes. The conditions permit stronger than polynomial growth of the coefficients, and depend instead on the relative growth rates. The results are derived by adapting, through a sequence of exponential transformations, the classical existence and uniqueness theorems for parabolic PDE's due to Besala to the "robust" form of the Zakai equation. In this process we also obtain sharp estimates for the tail behavior of the conditional density. Examples, including observations through a polynomial sensor and estimation of the state of a "bilinear" system, are worked out in detail. Our results are compared to those of Fleming and Mitter, Pardoux, and Sussmann who, among others, have obtained existence and uniqueness theorems for a more limited class of problems by different methods.**

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I. Introduction and Summary of Results[1]

The general nonlinear filtering problem for diffusion processes involves computing the conditional distribution of a diffusion $x(t)$ given nonlinear observations of $x(t)$ in additive Gaussian noise. Conditional statistics which exist and are of interest may be computed from this distribution. If the conditional distribution is absolutely continuous with respect to Lebesgue measure, then it has a density which, at least formally, satisfies a stochastic partial differential equation—the Duncan–Mortensen–Zakai (DMZ) equation. Background information on this equation and other aspects of the nonlinear filtering problem may be found in the anthology [3]. We are particularly interested in conditions for the existence, uniqueness, and representation of solutions to the DMZ equation and in the tail behavior of the resulting solutions. These are important considerations for the numerical treatment or small parameter asymptotic analysis (see, e.g., [2]) of nonlinear filtering problems.

When the state process $x(t)$ evolves in a bounded domain in R^n, or when the state space is unbounded, but the coefficients of the DMZ equation are bounded and possibly degenerate, then a satisfactory existence and uniqueness theory is available [4]–[6]. When, however, $x(t)$ evolves in R^n and the coefficients of the DMZ equation are unbounded functions of x (but bounded in t), then existence and uniqueness results are only available for "mildly" unbounded coefficients [1], [7]–[10]. In this paper we prove that the nonlinear filtering problem is well-posed for a large class of systems with strongly unbounded coefficients (greater than polynomial growth in x), and we provide precise estimates for the tail behavior (as $|x| \to \infty$) of the conditional density. These results are stated in detail and proved in Section II. Among the examples covered by our conditions are systems whose coefficients have polynomial growth in $|x|$ and the "bilinear" filtering problem. These special cases are analyzed in Section III.

Our approach is to apply the methods of Besala [11] for classical parabolic PDE's to the "robust" form of the DMZ equation. Besala's theorems are based on a maximum principle and the use of weight functions—standard devices for the treatment of PDE's. Since the robust form of the DMZ equation may be regarded as a parabolic PDE, it is entirely appropriate that it be treated by classical methods.

To set the problem, we consider the pair of Itô stochastic differential equations[2]

$$\begin{aligned} dx(t) &= f(x(t))\,dt + g(x(t))\,d\alpha(t) \\ dy(t) &= h(x(t))\,dt + d\beta(t) \\ x(0) &= x_0, \quad y(0) = 0, \qquad 0 \leqslant t \leqslant T < \infty. \end{aligned} \tag{1.1}$$

Here α, β are standard R-valued Wiener processes, mutually independent, and independent of x_o which is a random variable with density $p_o(x)$. The functions f, g, h are smooth ($f \in C^1(R)$, $g, h \in C^2(R)$) and may grow rapidly as $|x| \to \infty$. The filtering problem for (1.1) is to estimate $x(t)$ given the σ algebra $Y_t = \sigma\{y(s), 0 \leqslant s \leqslant t\}$.

Formally, the conditional density of $x(t)$ given Y_t is the normalization of $U(t,x) \geqslant 0$ which satisfies the DMZ equation[3]

$$\begin{aligned} dU(t,x) &= \left[a(x)U_{xx}(t,x) + b(x)U_x(t,x) \right. \\ &\quad \left. + c(x)U(t,x)\right] dt \\ &\quad + h(x)U(t,x)\,dy(t) \\ U(0,x) &= p_o(x), \qquad 0 \leqslant t \leqslant T \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} a(x) &= \frac{1}{2}g^2(x), \\ b(x) &= 2g(x)g_x(x) - f(x) \\ c(x) &= g_x^2(x) + g(x)g_{xx}(x) - f_x(x) - \frac{1}{2}h^2(x). \end{aligned} \tag{1.3}$$

We have used the Fisk–Stratonovich version of the stochastic calculus in writing (1.2). We shall study this equation indirectly by studying its associated "robust" form. Introducing

$$V(t,x) = \exp\left[-h(x)y(t)\right]U(t,x) \tag{1.4}$$

we find (formally) that V satisfies an "ordinary" parabolic PDE, pathwise in $y(t)$, $0 \leqslant t \leqslant T$

$$\begin{aligned} V_t(t,x) &= A(x)V_{xx}(t,x) + B(t,x)V_x(t,x) \\ &\quad + C(t,x)V(t,x) \\ V(0,x) &= p_o(x), \qquad 0 \leqslant t \leqslant T \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} A(x) &= a(x), \\ B(t,x) &= b(x) + 2a(x)h_x(x)y(t) \\ C(t,x) &= c(x) + b(x)h_x(x)y(t) \\ &\quad + a(x)\left[h_{xx}(x)y(t) + h_x^2(x)y^2(t)\right]. \end{aligned} \tag{1.6}$$

Equation (1.5) is the "pathwise-robust" form of the filtering problem. We will call it the "robust" DMZ equation. (See [12] for a discussion of this equation.) It is the starting point for our study.

For each "given" path $y(t)$, $0 \leqslant t \leqslant T$, (1.5) may be regarded as a classical PDE. Since the process y is equivalent to a Brownian motion under an invertible change of measure, we can assume that the paths $y(t)$, $0 \leqslant t \leqslant T$ are Hölder continuous [13]; therefore, the coefficients in (1.6) are jointly locally Hölder continuous in (x,t) whenever

[1]Some of the results of this paper [when $g(x) = 1$ in (1.1)] were announced in [1], [2].

[2]Only the scalar case is treated here. In [18], a class of vector problems ($x \in R^n$, $y \in R^m$) is treated by similar methods. The general vector case appears to be open.

[3]The paper [12] by Davis and Marcus in [3] has an especially clear derivation.

assumption A1) of Section II holds. [Thus, we could let f, g, h in (1.1) depend on t as long as $f, f_x, g, g_x, g_{xx}, h, h_x, h_{xx}, h_t$ were jointly locally Hölder continuous in (x, t).] Because the transformation (1.4) is invertible and (1.2), (1.5) are linear, existence and uniqueness results for (1.5) translate directly into corresponding results for (1.2).

Under various conditions on the relative growth of f, g, h (specifically, on f/g, $(f/g^2)_x$, f_x, h^2, gh_x, h_x, h_{xx}, etc. (see A1)–A3), B1)–B6) in Section II), we show that (1.5) has a fundamental solution (Theorem 1), that if the initial density $p_o(x)$ falls off sufficiently rapidly as $|x| \to \infty$, then (1.5) has a unique solution which falls off rapidly (Theorem 2), and in this case, that the tail behavior may be computed exactly (Theorem 3). These results are applied to the case when f, g, h are polynomials, and to the bilinear case when $g(x) = x$ (Theorem 4) in Section III. The conditions imposed on f, g exclude the occurrence of "explosions" in the x process in an entirely natural way.

Existence and uniqueness of solutions to (1.5) in the case $g(x)$ bounded (essentially $g(x) = 1$), $f \in C^3(R^n)$, f and ∇f bounded, $h(x)$ of polynomial growth, and $p_o(x)$ rapidly decaying have been derived by Fleming and Mitter [8] using a nonlinear transformation leading to an associated control problem. In [9] Sussmann treats the case $f = 0$, $g = 1$, $h(x) = x^3$ using measure theoretic arguments. He also obtains growth estimates on the conditional density. Pardoux [10] has also treated nonlinear filtering problems with mildly unbounded coefficients (f, h have linear growth, g bounded) starting with methods somewhat like those used here. The final form of his results is, however, very different from ours. His earlier paper [4] treats the case f, g, h bounded using arguments based on coercivity. It also contains many other interesting ideas. Michel [6] has analyzed regularity properties of solutions to Zakai equations with smooth f, g, h. Her results address bounded f, g, h, however, and focus on existence of conditional densities. Her methods are completely different from ours.

II. Conditions for Existence and Uniqueness, and Growth Estimates

Our assumptions on the coefficients in (1.6) are stated in terms of the original functions f, g, h. To state these succinctly, we will use the following relative order notation.

Definition: Let F, G: $R \to R$ and

$$L = \lim_{|x| \to \infty} \sup |F(x)/G(x)| \in [0, \infty].$$

Then $F = O(G)$ if $L < \infty$ and $F = o(G)$ if $L = O$.

The coefficients of the diffusion x are assumed to satisfy the following:

A1) $f \in C^1(R)$, $g \in C^2(R)$, f_x, g_{xx} are locally Hölder continuous;

A2)[4] $g(x) \geq \lambda > 0$, $\forall x \in R$ and some λ;

A3) $-\int_0^x (f/g^2)(\xi)\, d\xi \geq M$, $\forall x \in R$ and some M;

A4) $(f/g^2)_x = o(f^2/g^4)$, $f_x = o(f^2/g^2)$; and

A5) The martingale problem for (f, g) is well-posed.

The last condition implies that the stochastic differential equation for x has a unique weak solution for all $t \geq 0$. A sufficient condition for this is the existence of a Lyapunov function for the backwards Kolmogorov equation associated with the process x [14]. If the integral in A3) diverges to $+\infty$ as $|x| \to \infty$, then its exponential could serve as the Lyapunov function. If the martingale problem is not well posed, then the process x may have "explosions" (escape times which are finite with probability one). In this case the conditional distribution of $x(t)$ given Y_t may have singular components which are not computed by the DMZ equation.

The observation function h is assumed to satisfy the following.

B1) $h \in C^2(R)$, h_{xx} is locally Hölder continuous;

B2) either $g^2 h_{xx}, (g^2 h_x)_x = o(h^2)$, or $g^2 h_{xx}, (g^2 h_x)_x = o(g^2 h_x^2)$

B3) either $gh_x = 0(h)$ or $gh_x = o(f/g)$;

B4) either $(g^2)_{xx} = o(h^2)$ or $(g^2)_{xx} = o(f^2/g^2)$;

B5) one of the two mutually exclusive cases holds:

a) either $h = 0(f/g)$ or $h = 0(gh_x)$; or

b) both $f/g = o(h)$ and $gh_x = o(h)$; in addition, $gh_x, g_x h = o(h^2)$

B6) in case B5a),

$$\lim_{|x| \to +\infty} \max\left\{|h(x)|, -\int_0^x (f/g^2)(\xi)\, d\xi\right\} = +\infty;$$

and in case B5b),

$$\lim_{|x| \to +\infty} \left|\int_0^x (h/g)(\xi)\, d\xi\right| = +\infty.$$

Remark 1: The growth conditions are relatively easy to understand in the case when f, g, h are polynomials, especially $f(x) = f_0 x^j$, $g(x) = g_0(1 + x^2)^k$, $h(x) = h_0 x^l$. This case is discussed in detail in Section III. While many systems with polynomial coefficients are covered by A1)–B6), the conditions do not permit

a) too rapid growth of h when g is superlinear ($k > 1/2$); or

b) unstable f with g sublinear.

Nor are cases like $f = 0$, $g = 1$, $h(x) = x \sin x$ permitted (B6) is violated), or cases in which $h(x)$ is highly oscillatory.

Remark 2: The conditions A1)–B6) are not necessary; different choices of the weight functions used in the proofs would lead to different growth restrictions. In fact, one could consider optimizing the choice of the weight functions.

In the analysis of the robust equation (1.5) we use certain exponential transformations which have parameters that are functionals of the path $y(t)$, $t \geq 0$. To express this dependence, we will use a sequence of stopping times $(t_k)_{k=0}^{\infty}$, with $t_0 = 0$ and $t_k \to +\infty$ as $k \to \infty$ which depend on the path $y(t)$, $t \geq 0$. (These are defined in the proof of Theorem 1.) If $V(t, x)$ is the solution of (1.5), then we

[4]See, however, the second example in Section III.

define

$$u^k(t,x) = V(t,x)\exp\left(\psi^k(x) - \gamma^k t\right),\quad t \in (t_k, t_{k+1}),\ x \in R \tag{2.1}$$

where

$$\psi^k(x) = \alpha\phi_1(x) + \beta_1^k\phi_2(x) + \beta_2\left[1 + \phi_2^2(x)\right]^{1/2}$$

$$\phi_1(x) = \begin{cases} -\int_0^x (f/g^2)(\xi)\,d\xi & \text{in case B5a)} \\ 0 & \text{in case B5b)} \end{cases}$$

$$\phi_2(x) = \begin{cases} h(x) & \text{in case B5a)} \\ \int_0^x (h/g)(\xi)\,d\xi & \text{in case B5b).} \end{cases} \tag{2.2}$$

The parameters α, β_2, $\{\beta_1^k, \gamma^k, t_k\}_{k=0}^{\infty}$ will be functionals of the path $y(t)$, $t \geq 0$. (These are also defined in the proof of Theorem 1.)

The function $u^k(t,x)$ satisfies the transformed robust equation

$$u_t^k(t,x) = a(x)u_{xx}^k(t,x) + b^k(t,x)u_x^k(t,x) + c^k(t,x)u^k(t,x),\qquad (t,x) \in (t_k, t_{k+1})\times R,\quad k = 0,1,2,\cdots$$

$$u^k(t_k,x) = \begin{cases} p_o(x)\exp\left[\psi^0(x)\right], & k = 0 \\ u^{k-1}(t_k,x)\exp\left[\psi^k(x) - \psi^{k-1}(x) + (\gamma^{k-1} - \gamma^k)t_k\right], & k \geq 1 \end{cases} \tag{2.3}$$

where for $(t,x) \in (t_k, t_{k+1})\times R$ and omitting the arguments

$$a(x) = \frac{1}{2}g^2$$

$$b^k(t,x) = -2a\psi_x^k + 2a_x - f + 2ah_x y$$

$$c^k(t,x) = a\left[\left(\psi_x^k\right)^2 - \psi_{xx}^k\right] - \psi_x^k[2ah_x y + 2a_x - f] + a\left(h_x^2 y^2 + h_{xx}y\right) + h_x y(2a_x - f) + \left(a_{xx} - f_x - \frac{1}{2}h^2\right) - \gamma^k. \tag{2.4}$$

Assumption A2) guarantees that each equation in (2.3) and the original (1.5) are nondegenerate parabolic equations. Assumptions A3) and B6) together with the constraints $\alpha > 0$, $\beta_2 > 0$, $\beta_2 > |\beta_1^k|$, imply that the weight functions $\psi^k(x)$ diverge to $+\infty$ as $|x| \to \infty$. The remaining growth conditions serve to identify the dominant terms (as $|x| \to \infty$) in the potential $c^k(t,x)$ in (2.3) and in the potential of the adjoint of (2.3). Assumption B3) permits us to select the functions ψ^k and the constants γ^k so that these potentials are nonpositive. This in turn permits the use of a maximum principle.

Under these assumptions we shall show that the robust equation (1.5) has a fundamental solution which may be used to construct a unique solution to the DMZ equation within a certain class of functions. To describe this class, we define the constants

$$\eta_i = \lim_{|x|\to\infty} \sup|g(\phi_i)_x|/\left[h^2 + f^2/g^2\right]^{1/2}$$

$$\nu_i = \lim_{|x|\to\infty} \inf|g(\phi_i)_x|/\left[h^2 + f^2/g^2\right]^{1/2},\qquad i = 1,2. \tag{2.5}$$

The assumptions imply η_1, $\nu_1 \in [0,1]$ and η_2, $\nu_2 \in [0,\infty)$ when B5a) holds, while $\eta_1 = 0 = \nu_1$, $\eta_2 = 1 = \nu_2$ when B5b) holds. The assumption that either B5a) or b) holds implies $(\nu_1 + \nu_2) > 0$. This will prove to be essential in establishing a lower bound on the unnormalized conditional density (Theorem 3).

Finally, we remind the reader that a *fundamental solution* of (1.5) is a real-valued function $\Gamma(t,x;s,z)$ defined for $0 \leq s < t \leq T$, $x, z \in R$, which satisfies the following conditions:

a) As a function of (t,x), Γ has continuous derivatives Γ_t, Γ_x, Γ_{xx} and satisfies (1.5) in $(s,T)\times R$;

b) If $\rho(x)$ is continuous and has compact support, then

$$\lim_{\substack{t\downarrow s\\ x\to z}} \int_{-\infty}^{\infty} \Gamma(t,x;s,\xi)\rho(\xi)\,d\xi = \rho(z).$$

Theorem 1: If A1)–A5), B1)–B6) hold, then for each Hölder continuous path $\{y(t),\ 0 \leq t < \infty\}$ of the observation process there exist constants α, $\{\beta_1^k, \gamma^k\}_{k=0}^{\infty}$, β_2 and an unbounded monotone increasing sequence of times $\{t_k\}_{k=0}^{\infty}$, $t_0 = 0$, (which may depend on the path) such that for each $k \geq 0$ there exists a fundamental solution $\tilde{\Gamma}_k(t,x;s,z)$ of (2.3). Then

$$\Gamma_k(t,x;s,z) = \tilde{\Gamma}_k(t,x;s,z)\exp\left[\psi^k(z) - \psi^k(x) + \gamma^k(t-s)\right] \tag{2.6}$$

is a fundamental solution of the robust DMZ equation (1.5) on (t_k, t_{k+1}). Moreover, $\tilde{\Gamma}_k(t,x;s,z)$ satisfies the inequalities

$$0 \leq \tilde{\Gamma}_k(t,x;s,z) \leq c_k/(t-s)^{1/2} \tag{2.7}$$

for some constant c_k and $x, z \in R$, $t \in (t_k, t_{k+1})$ and

$$\int_{-\infty}^{\infty} \tilde{\Gamma}_k(t,x;s,z)\,dz \leq 1 \tag{2.8a}$$

$$\int_{-\infty}^{\infty} \tilde{\Gamma}_k(t,x;s,z)\,dx \leq 1. \tag{2.8b}$$

Theorem 2: Suppose A1)–A5), B1)–B6) hold. Let $p_o(x)$ be continuous $p_o(x) \geq 0$ and assume that there exist constants $\theta_i > 0$, $i = 1,2$, such that $0 < \theta_1\eta_1 + \theta_2\eta_2 < 1$, and

$$p_o(x)\exp\left[\theta_1\phi_1(x) + \theta_2|\phi_2(x)|\right] \leq M,\qquad \forall x \in R \tag{2.9}$$

for some $M < \infty$. Then for any constants $\tilde{\theta}_i$, $0 < \tilde{\theta}_i < \theta_i$,

$i=1, 2$, there exists a unique solution to the DMZ equation (1.2) within the class of functions satisfying

$$\lim_{|x|\to\infty} \sup U(t,x)\exp\left[\tilde{\theta}_1\phi_1(x)+\tilde{\theta}_2|\phi_2(x)|\right]=0, \quad \forall t\geqslant 0. \tag{2.10}$$

This solution satisfies $U(t,x)=U^k(t,x)$, $t\in(t_k,t_{k+1})$

$$U^k(t,x)=e^{h(x)y(t)}\int_{-\infty}^{\infty}\Gamma_k(t,x;z,t_k)U^{k-1}(t_k,z)\,dz$$
$$U^0(0,x)=p_o(x) \qquad k=1,2,\cdots \tag{2.11}$$

where Γ_k is defined by (2.6).

Theorem 3: Suppose A1)–A5), B1)–B6) hold, and assume that when case B5) a) holds with $\nu_1>0, \nu_2>0$, that $-f(x)\,sgn(x)$ and $h_x(x)\,sgn(xh(x))$ are nonnegative for $|x|$ sufficiently large.[5] Let $p_o(x)$ satisfy the conditions in Theorem 2, and suppose further that there exist $M_0>0$, $K_0>0$ such that

$$M_0\exp\left[-K_0\phi(x)\right]\leqslant p_o(x) \qquad \forall x\in R \tag{2.12}$$

where

$$\phi(x)=\phi_1(x)+|\phi_2(x)|. \tag{2.13}$$

Then for any $T<\infty$, there exist positive constants M_1, M_2, K_1, K_2, which may depend on the path $\{y(t), 0\leqslant t\leqslant T\}$ such that the solution of the DMZ equation given by (2.11) satisfies

$$M_1\exp\left[-K_1\phi(x)\right]\leqslant U(t,x)\leqslant M_2\exp\left[-K_2\phi(x)\right] \quad \forall(t,x)\in[0,T]\times R. \tag{2.14}$$

The proofs of Theorems 1 and 2 are based on the results of Besala [11]. These provide a very general existence theory for the Cauchy problem for parabolic equations on a half-space with unbounded coefficients. The key result which we shall use is the following.

Lemma 1: (Besala [11]) *Let $a(t,x)$, $b(t,x)$, $c(t,x)$ (real valued) together with a_x, a_{xx}, b_x be locally Hölder continuous in $\mathfrak{D}=(t_0,t_1)\times R$. Assume that*

a) $a(t,x)\geqslant\lambda>0$, $\forall(t,x)\in\mathfrak{D}$, *for some constant* λ

b) $c(t,x)\leqslant 0$, $\forall(t,x)\in\mathfrak{D}$

c) $(c-b_x+a_{xx})(t,x)\leqslant 0$, $\forall(t,x)\in\mathfrak{D}$.

Then the Cauchy problem

$$u_t(t,x)=a(t,x)u_{xx}+b(t,x)u_x+c(t,x)u$$
$$u(0,x)=u_o(x), \qquad (t,x)\in\mathfrak{D} \tag{2.15}$$

has a fundamental solution $\Gamma(t,x;s,z)$ which satisfies

$$0\leqslant\Gamma(t,x;s,z)\leqslant c/(t-s)^{1/2} \tag{2.16}$$

for some positive constant c and

$$\int_{-\infty}^{\infty}\Gamma(t,x;s,z)\,dz\leqslant 1$$
$$\int_{-\infty}^{\infty}\Gamma(t,x;s,z)\,dx\leqslant 1. \tag{2.17}$$

Moreover, if $u_o(x)$ is continuous and bounded, then

$$u(t,x)=\int_{-\infty}^{\infty}\Gamma(t,x;t_0,z)u_o(z)\,dz \tag{2.18}$$

is a bounded solution of (2.15).

In addition, we will need a technical result which identifies the dominant behavior of the potentials of (2.3) and its adjoint. Let

$$c^k_{adj}=c^k-b^k_x+a_{xx} \tag{2.19a}$$

$$c^k_{ess}=\frac{1}{2}g^2\left(\psi^k_x-h_xy+f/g^2\right)^2-\frac{1}{2}\left(h^2+f^2/g^2\right) \tag{2.19b}$$

$$c^k_\delta=c^k-c^k_{adj}=\left(a_x-f+2yah_x-2a\psi^k_x\right)_x. \tag{2.19c}$$

Note

$$c^k=c^k_{ess}+\left[a\left(\psi^k_{xx}-yh_{xx}\right)-\gamma^k+c^k_\delta\right]. \tag{2.20}$$

Lemma 2: Suppose A1)–A5), B1)–B6) hold. Let $\{t_k\}_{k=0}^\infty$, $t_0=0$ be monotone increasing and assume $\alpha>0$, $\beta_2>0$, $\beta_2>|\beta_1^k|$ for all $k\geqslant 0$. Then, for $i=0,1$

$$\left[a\left(\psi^k_{xx}-yh_{xx}\right)+ic^k_\delta(t,x)-\gamma^k\right](t,x)=o\left(c^k_{ess}\right) \tag{2.21}$$

uniformly in $t\in(t_k,t_{k+1})$ for all $k\geqslant 0$, and for each Hölder continuous path of the observation process $\{y(t), t\geqslant 0\}$.

Proof: In view of (2.2), (2.19), (2.20) it suffices to show that

$a_{xx}, f_x, (g^2h_x)_x, g^2h_{xx}, g^2(f/g^2)_x$ in case B5a)

$a_{xx}, f_x, (g^2h_x)_x, g^2h_{xx}, gh_x, g_xh$ in case B5b)

are all $o(c^k_{ess})$. This is immediate from the definition of c^k_{ess} and A4), B2)–B5). (For details, see Appendix I.) Q.E.D.

Proof of Theorem 1: From Lemma 1 it suffices to choose the parameters α, β_2, $\{\beta_1^k, t_k, \gamma^k\}_{k=0}^\infty$ with $t_0=0$, $\lim_{k\to\infty}t_k=+\infty$, so that the potentials c^k, c^k_{adj} are nonpositive on $(t_k,t_{k+1})\times R$ for all $k\geqslant 0$. If the potentials are bounded above for $\gamma^k=0$, then for $\gamma^k>0$ sufficiently large, they are nonpositive. From Lemma 2 the potentials are bounded above whenever c^k_{ess} is bounded above. From the triangle inequality and the definition of the constants η_i, c^k_{ess} is bounded above provided

$$\left.\begin{array}{l}|\alpha-1|\eta_1<\frac{1}{2}\\ |\beta_2\pm(\beta_1^k-y)|\eta_2<\frac{1}{2}\end{array}\right\}\text{ in case B5a)}$$
$$|\beta_2\pm\beta_1^k|<1 \qquad \text{in case B5b).} \tag{2.22}$$

The $\pm$ condition arises since $\phi_2(x)$ may not be sign definite.

Recall that η_1, η_2 are finite by B3), B5). Let $0<\epsilon<\frac{1}{4}\eta_2$ and define

$$t_0=0$$
$$t_{k+1}=\inf_{t>t_k}\{t: |y(t)-y(t_k)|=\epsilon\}. \tag{2.23}$$

[5] This last condition is satisfied whenever $h(x)$ is a polynomial.

Since the covariance of the observation process is nondegenerate, it follows that $\lim_{k\to\infty} t_k = +\infty$. In case B5a), take $\alpha = 1$, $\beta_1^k = y(t_k)$, and $0 < \beta_2 < \frac{1}{4}\eta_2$ and in case B5b) take $\alpha = \beta_1^k = 0$ and $0 < \beta_2 < 1/\eta_2$. Then (2.22) holds, and the parameters γ^k may be chosen large enough so that Lemma 1 applies. Q.E.D.

Remark: In some cases it is not necessary to let the parameters depend on the y path. For example, if $g(x)$ is sublinear in growth and h is of polynomial growth with $h = 0(f/g)$, then case B5a) holds, $\eta_2 = 0$, and we can take $t_1 = +\infty$.

Proof of Theorem 2: Let δ, ϵ be constants with

$$0 < \epsilon < \frac{1}{4}(\theta_2 - \tilde{\theta}_2)$$
$$0 < \delta < \frac{1}{4}\min[\theta_1 - \tilde{\theta}_1, \theta_2 - \tilde{\theta}_2]$$

and define the sequence of stopping times $\{t_k\}_{k=0}^{\infty}$ by (2.23). Also, let

$$\left.\begin{aligned} \alpha &= \frac{1}{2} + \frac{1}{2}(\theta_1 + \tilde{\theta}_1) \\ \beta_1^k &= y(t_k) \\ \beta_2 &= \frac{1}{2}(\theta_2 + \tilde{\theta}_2) \end{aligned}\right\} \text{ in case B5a)}$$

$$\left.\begin{aligned} \alpha &= \beta_1^k = 0 \\ \beta_2 &= \frac{1}{2}(\theta_2 + \tilde{\theta}_2) \end{aligned}\right\} \text{ in case B5b)} \qquad (2.25)$$

and

$$\tilde{\psi}^k(x) = \psi^k(x) + \delta\left[\phi_1(x) + [1 + \phi_2^2(x)]^{1/2}\right]$$
$$\tilde{u}^k(t,x) = u^k(t,x)\exp\left(\delta\left[\phi_1(x) + [1 + \phi_2^2(x)]^{1/2}\right]\right). \qquad (2.26)$$

Then $\tilde{u}^k$ also satisfies an equation of the form (2.3). Let $\tilde{c}^k$, $\tilde{c}_{adj}^k$ be the potentials of this equation and its adjoint, and let $\tilde{c}_{ess}^k$ denote the function in (2.19b) with ψ^k replaced by $\tilde{\psi}^k$. The assumption (2.9) guarantees that the initial data $u^k(t_k, x)$, $\tilde{u}^k(t_k, x)$ of these equations are bounded provided

$$\left.\begin{aligned} &\alpha \leq \theta_1, \quad \alpha + \delta \leq \theta_1 \\ &\beta_2 \pm (\beta_1^k - y(t_k)) \leq \theta_2, \\ &\beta_2 \pm (\beta_1^k - y(t_k)) + \delta \leq \theta_2 \end{aligned}\right\} \text{ in case B5a)}$$
$$\beta_2 \pm \beta_1^k \leq \theta_2, \quad \beta_2 \pm \beta_1^k + \delta \leq \theta_2 \qquad \text{in case B5b)} \qquad (2.27)$$

for all $k \geq 0$. As in the proof of Theorem 1, there exist parameters γ^k such that c^k, $\tilde{c}^k$, c_{adj}^k, $\tilde{c}_{adj}^k$ are nonpositive whenever c_{ess}^k, $\tilde{c}_{ess}^k$ are bounded above. A sufficient condition for the latter is that there exist constants A_i, $i = 1,2$, $0 < A_1 + A_2 < 1$, such that for all $k \geq 0$,

$$\left.\begin{aligned} &|\alpha - 1|\eta_1 < A_1, \\ &|\alpha - 1 + \delta|\eta_1 < A_1 \\ &|\beta_2 \pm (\beta_1^k - y)|\eta_2 < A_2, \\ &|\beta_2 \pm (\beta_1^k - y) + \delta|\eta_2 < A_2 \end{aligned}\right\} \text{ in case B5a)}$$
$$|\beta_2 \pm \beta_1^k| < A_2, \quad |\beta_2 \pm \beta_1^k + \delta| < A_2 \qquad \text{in case B5b).} \qquad (2.28)$$

These inequalities are satisfied by $A_i = \eta_i \theta_i$. From Lemma 1 u^k, $\tilde{u}^k$ exist and are bounded, U^k is given by (2.11). Since $\delta > 0$, $u^k(t,x) \to 0$ as $|x| \to \infty$. Applying the maximum principle as in [15], [16], u^k is unique in the class of functions which tend to zero as $|x| \to \infty$. (The method of [15], [16] only applies to this class.) In the original coordinates this unique class consists of those functions satisfying

$$U(t,x)\exp\left[\psi^k(x) - \gamma^k t - h(x)y(t)\right] \to 0 \quad \text{as } |x| \to \infty. \qquad (2.29)$$

Hence, $U(t,x)$ is unique in the class of functions satisfying (2.10) provided

$$\left.\begin{aligned} &\alpha > \tilde{\theta}_1 \\ &\beta_2 \pm (\beta_1^k - y) > \tilde{\theta}_2 \end{aligned}\right\} \text{ in case B5a)}$$
$$\beta_2 \pm \beta_1^k > \tilde{\theta}_2 \qquad \text{in case B5b).} \qquad (2.30)$$

The choice of parameters (2.24), (2.25) satisfies all the conditions (2.27), (2.28), (2.30) and the result follows. Q.E.D.

Proof of Theorem 3: The lower bound in (2.14) is a simple consequence of the comparison theorem for parabolic equations (see, e.g., [16]). Suppose $w(t,x), w_i(t,x)$, $i = 1,2$, satisfy

$$w_1(r,x) \leq w(r,x) \leq w_2(r,x)$$
$$(Tw_1)(t,x) \leq (Tw)(t,x) \leq (Tw_2)(t,x)$$
$$(t,x) \in (r,s) \times R \qquad (2.31)$$

where $T = \partial/\partial t - \mathfrak{L}$ is a parabolic operator and that $w, w_1, w_2 \to 0$ as $|x| \to \infty$ uniformly for $t \in (r,s)$. Then $w_1(t,x) \leq w(t,x) \leq w_2(t,x)$ for all $(t,x) \in (r,s) \times R$.

Now let the parameters $\alpha, \beta_2, \{\beta_1^k, t_k, \gamma^k\}_{k=0}^{\infty}$ be defined as in the proof of Theorem 2, and let (r,s) be any one of the intervals (t_k, t_{k+1}), $k \geq 0$. Let $\mathfrak{L}$ be the operator in the transformed robust equation (2.3), and define

$$w(t,x) = u^k(t,x)$$
$$w_i(t,x) = \exp\left[\lambda_i t - \mu_i \tilde{\phi}(x)\right], \qquad i = 1,2 \qquad (2.32)$$

for some constants λ_i, μ_i to be chosen and

$$\tilde{\phi}(x) = \phi_1(x) + [1 + \phi_2^2(x)]^{1/2}. \qquad (2.33)$$

Let $\hat{c}_{ess}^k(i)$ be the function in (2.19b) with ψ^k replaced by

$\mu_i \tilde{\phi}(x)+\psi^k$. Then by Lemma 1,

$$Tw_i = -\hat{c}^k_{ess}(i)+\lambda_i + o\left(\hat{c}^k_{ess}(i)\right). \tag{2.34}$$

In case B5a), $-\hat{c}^k_{ess}(1)$ is bounded above if, for all $|x|$ sufficiently large,

$$\begin{aligned}|-\operatorname{sgn}(x)(f/g^2)(\mu_1+\alpha-1)\\ +\operatorname{sgn}(x)h_x\left(\beta_1^k - y+\operatorname{sgn}(h)(\beta_2+\mu_1)\right)|\\ > \frac{1}{g}\left[h^2+(f^2/g^2)\right]^{1/2}.\end{aligned} \tag{2.35}$$

Because $\nu_1+\nu_2>0$, this is ensured by choosing μ_1 so large that

$$(\mu_1+\alpha-1)\nu_1>1 \qquad \text{if } \nu_1>0 \tag{2.36a}$$

$$\left(\mu_1+\beta_2\pm\left(\beta_1^k-y\right)\right)\nu_2>1 \qquad \text{if } \nu_2>0. \tag{2.36b}$$

Similarly, in case B5b), $\nu_1=0$, $\nu_2=1$, and $-\tilde{c}^k_{ess}$ (1) is bounded above if μ_1 is chosen so large that (2.36b) holds with $y(t)$ formally set equal to zero. Also, $\tilde{c}^k_{ess}$ (2) is bounded above if $\mu_2=\delta$ [recall (2.24)]. Hence, Tw_1, $-Tw_2$ are bounded above, and there exist constants λ_i so that (2.31) holds. This implies that (2.14) holds for $t\in(t_k,t_{k+1})$ for some constants M_i^k, K_i^k. The proof is completed by repeating this argument for each of the intervals (t_k,t_{k+1}), $t_k<T$ and defining

$$\begin{aligned} K_1 &= \max_k\left(K_1^k\right), & M_1 &= \min_k\left(M_1^k\right)\\ K_2 &= \min_k\left(K_2^k\right), & M_2 &= \max_k\left(M_2^k\right).\end{aligned} \tag{2.37}$$

Q.E.D.

III. Examples

To illustrate our results and make contact with other recent work on nonlinear filtering (e.g., [7]–[10]), we consider a class of systems with polynomial f, h. In particular, we consider the case of a Wiener process observed through a polynomial sensor. We also obtain a new uniqueness result for a generalization of the bilinear problem studied in [17].

Example 1: Polynomial coefficients.

Let f, h be polynomials with f of odd degree and stable, i.e.,

$$\begin{aligned} f(x) &= \sum_{i=0}^{2q-1} f_i x^i, & F &\triangleq -f_{2q-1}>0\\ h(x) &= \sum_{j=0}^{s} h_j x^j, & H &\triangleq h_s \neq 0\end{aligned} \tag{3.1}$$

where q, s are positive integers. Suppose

$$g(x)=G(1+x^2)^{r/2}, \qquad G>0 \tag{3.2}$$

where $r\in[0,\infty)$. Our conditions for existence and uniqueness and estimates of the asymptotic behavior of the density depend on whether or not $g(x)$ is globally Lipschitz and on the degree of $h(x)$ relative to the degree (or stability) of $f(x)$. There are two cases covered by Theorems 1–3.[6]

Case 1: $r\in[0,1]$, $q>r$, $s\geq 1$, $q\geq 1$.

The restrictions A1)–A5), B1)–B6) applied here require $g(x)$ to satisfy a linear growth constraint $r\in[0,1]$, that f be at least a cubic polynomial, $q\geq 2$, when $g(x)$ is of linear growth, $r=1$, and that $h(x)$ be nonconstant.[7] In defining the transformation (2.2) only the asymptotic behavior ($|x|\to\infty$) played a role in the proofs of Theorems 1–3. Therefore, $\psi^k(x)$ may be defined in terms of the monomials

$$\begin{aligned}\phi_1(x) &= (F/G^2)x^{2(q-r)}/2(q-r)\\ \phi_2(x) &= \begin{cases} Hx^s, & r\geq 1 \text{ or } r<1 \text{ and } r+s\leq 2q-1\\ (H/G)x^{s-r+1}/s-r+1, & \\ \quad r<1 \text{ and } r+s>2q-1. & \end{cases}\end{aligned} \tag{3.3}$$

Then the dominant behavior of the potentials of the transformed robust equation (2.3) and its adjoint are characterized by the constants

$$\eta_1=\nu_1=\begin{cases}0, & r+s>2q-1\\ 1, & r+s<2q-1\\ \dfrac{F/G}{\left(H^2+f^2/G^2\right)^{1/2}}, & r+s=2q-1\end{cases} \tag{3.4a}$$

$$\eta_2=\nu_2=\begin{cases}0, & r=1,\ r+s<2q-1\\ sG, & r=1,\ r+s>2q-1\\ \dfrac{sGH}{\left(H^2+F^2/G^2\right)^{1/2}}, & r=1,\ r+s=2q-1\\ 0, & r<1,\ r+s\leq 2q-1\\ 1, & r<1,\ r+s>2q-1.\end{cases} \tag{3.4b}$$

Notice that $\eta_1+\eta_2>0$.

Now suppose that the initial density $p_o(x)$ in (1.2) satisfies

$$p_o(x)\exp\left[\theta_1\phi_1(x)+\theta_2|\phi_2(x)|\right]=0(1) \tag{3.5}$$

for some positive constants θ_1, θ_2 that satisfy $0<\theta_1\eta_1+\theta_2\eta_2<1$. Then from Theorem 2 the DMZ equation has a unique solution in the class of functions satisfying

$$U(t,x)\exp\left[\tilde{\theta}_1\phi_1(x)+\tilde{\theta}_2|\phi_2(x)|\right]=o(1) \tag{3.6}$$

[6]In case 1, $gh_x=0(h)$, whereas in case 2, $h=o(gh_x)$. The conditions on the parameters q, r, s are obtained from assumptions A4), B2)–B5), using the fact that when $F_i(x)$, $i=1,2$, are asymptotic to monomials of degrees p_i, then $F_2=o(F_1)$ [resp. $F_2=0(F_1)$] if and only if $p_1<p_2$ [resp. $p_1\leq p_2$]. Also, the condition $s\geq 1$ satisfies B6), while the restriction that $f(x)$ is odd and stable is a consequence of A3).

[7]This case includes the linear filtering problem when $f(x)=ax+b$, $a<0$, $g(x)=1$, $h(x)=cx+d$; when $a>0$, assumption A3) is violated.

for all $t \geqslant 0$ and $\tilde{\theta}_i \in (0, \theta_i)$, $i = 1,2$. Suppose in addition, that

$$M \exp\left[-\hat{\theta}_1 \phi_1(x) - \hat{\theta}_2 |\phi_2(x)|\right] \leqslant p_o(x) \qquad (3.7)$$

for some positive constants $\hat{\theta}_i$, M. Then from Theorem 3, for any $0 \leqslant t_1 \leqslant t_2$, there exist constants M_i, K_i depending on the observation path such that

$$M_1 \exp\left[-K_1 |x|^\rho\right] \leqslant U(t,x) \leqslant M_2 \exp\left[-K_2 |x|^\rho\right] \qquad (3.8)$$

where

$$\rho = \begin{cases} s - r + 1, & r < 1 \text{ and } r + s > 2q - 1 \\ \max[s, 2(q-r)], & \text{otherwise.} \end{cases} \qquad (3.9)$$

Although it is not covered in the present case, the situation $r = 0$, f and its first two derivatives are bounded, and h is asymptotic to a nonconstant polynomial, can be easily treated by adapting the arguments in Theorems 1–3. In particular, the inequalities (3.6), (3.8) hold with $\rho = s + 1$, and this result overlaps [8], [9]. For example, if $f = 0$, $g = 1$, $h(x) = h_s x^s$, $h_s \neq 0$, then

$$\rho = s + 1$$
$$0 < K_2 < |h_s|/(s+1). \qquad (3.10)$$

This was obtained by Sussmann for $s = 3$ in [9].

Case 2: $r > 1$, $q > r + \frac{1}{2}s$, $s \geqslant 1$, $q \geqslant 2$.

Here $g(x)$ is of superlinear growth, $f(x)$ is at least a cubic polynomial, and $h(x)$ is dominated, as indicated, by the dynamics of the state process. In this case the asymptotic behavior of the conditional density is the same as that of the *a priori* density [of $x(t)$]. To see this, let $\phi_2 = o(\phi_1)$. Thus, the dominant part (as $|x| \to \infty$) of the potentials of the transformed robust equation (2.3) and its adjoint are given by

$$c_{ess}^k = \frac{1}{2} g^2(x) \psi_x^k \left[\psi_x^k - (f/g^2)\right](x) + o(f/g^2)(x). \qquad (3.11)$$

Since $\eta_1 = 1$, $\eta_2 = 0$, we can in fact take $\psi^k(x) = \alpha \phi_1(x)$, independent of k. Thus, the results (3.6), (3.8) of case 1 hold with $\theta_2 = \tilde{\theta}_2 = 0$ and $\rho = 2(q - r)$.

There are two interesting classes of problems with polynomial coefficients not covered here: 1) when g has superlinear growth ($r > 1$) and h is strongly nonlinear ($s \geqslant 2(q - r)$); and 2) when g has at most linear growth ($r \leqslant 1$) and f is linear and unstable (i.e., $f_1 > 0$). In these cases it may be possible to obtain results by selecting time-varying weight functions more complex than those used here.

Example 2: Bilinear filtering problem. Consider the system

$$dz(t) = f(z(t))\,dt + z(t)\,d\alpha(t)$$
$$dy(t) = h(z(t))\,dt + d\beta(t)$$
$$z(0) = z_o, \quad y(0) = 0, \quad 0 \leqslant t \leqslant T < \infty \qquad (3.12)$$

with z_o having density $p_o(z)$, and z_o, α, β mutually independent as before. Since $z(t)$ will eventually be trapped in either the positive or negative half space, we shall arrange that $z(t) \in [0, \infty)$ by taking $f \in C^1(0, \infty)$ satisfying

$$\text{C1)} \begin{cases} |f(z)| \leqslant K(1+z) & \text{for some } K > 0 \qquad (3.13a) \\ f(0) \geqslant 0 & \qquad (3.13b) \end{cases}$$

and by taking $p_o(z)$ defined on $(0, \infty)$ and continuous and integrable there. We also assume that $h \in C^2(0, \infty)$ with h_z, h_{zz} locally Hölder continuous (and so, bounded at zero).

We impose the following growth conditions on f, h.

C1′) $f_z(z)$ is bounded and locally Hölder continuous

C2) $\lim_{z \downarrow 0} [f(z)/z] > 0$

C3) $\lim_{z \to \infty} |h(z)|/\log z = +\infty$

C4) $\lim_{z \to \infty} [h_{zz}(z)/h_z^2(z)] = 0$

C5) for some constants K_i, M_i, $i = 1,2$

$$M_1 + K_1 |z h_z(z)| \leqslant |h(z)| \leqslant M_2 + K_2 |z h_z(z)|.$$

Note that these conditions are satisfied when $f(z)$ is affine and $h(z)$ is a nonconstant polynomial; we do not consider the case $h(z)$ constant. The assumptions that f, h and their derivatives are bounded at the origin are made for convenience only. They can be relaxed by introducing more complex growth restrictions. The other assumptions C2)–C5) are essential (to our method).

The DMZ equation associated with (3.12) is

$$dU(t,z) = \left[\frac{1}{2}(z^2 U)_{zz} - (fU)_z - \frac{1}{2} h^2 U\right] dt + hU\,dy(t)$$
$$U(0,z) = p_o(z), \qquad (t,z) \in [0,T] \times [0,\infty). \qquad (3.14)$$

Because the generator for the diffusion x in (3.12) is not uniformly elliptic, our theorems are not directly applicable. If we make the logarithmic change of coordinates $x = \log z$, then (3.8) becomes

$$dx(t) = \left(e^{-x} f(e^x) - \frac{1}{2}\right) dt + d\tilde{\alpha}(t)$$
$$dy(t) = h(e^x)\,dt + d\tilde{\beta}(t) \qquad (3.15)$$

and Theorems 1–3 can be applied to this system. Alternately, we can change coordinates directly in the DMZ equation. Let $W(t,x) = U(t, e^x)$, $x \in R$. Using (3.14), we have

$$dW(t,x) = \left\{\frac{1}{2} W_{xx} + \left[3/2 - e^{-x} f(e^x)\right] W_x + \left[1 - f_z(e^x) - \frac{1}{2} h^2(e^x)\right] W\right\} dt + h(e^x) W\,dy(t)$$
$$W(0,x) = p_o(e^x). \qquad (3.16)$$

The methods of Besala [11] as used in the proofs of Theorems 1 and 2 can be directly applied to the robust version of this equation. This is the line of attack which we shall take.

These two approaches are inherently different because the robust form of (3.16) does not "solve" the filtering problem (3.15). In fact, (3.16) does not correspond to any filtering problem of the form (1.1). Nevertheless, the proofs of Theorems 1–3 go through when the weight functions $\psi^k(x)$ are defined by (2.2) with $x = \log z$ and

$$\begin{aligned}\phi_1(z) &= -\int_1^z f(\xi)/\xi^2\, d\xi \\ \phi_2(z) &= h(z).\end{aligned} \tag{3.17}$$

The transformations

$$\begin{aligned}V(t,x) &= W(t,x)\exp[-h(e^x)y(t)] \\ u^k(t,x) &= V(t,x)\exp[\psi^k(x)-\gamma^k t]\end{aligned} \tag{3.18}$$

map (3.16) into an equation of the form (2.3) where $a(x) = \frac{1}{2}$ and

$$\begin{aligned}b^k(x) &= -\psi_x^k + \frac{3}{2} - f(z)/z + yzh_z(z)|_{z=e^x} \\ c^k(x) &= \frac{1}{2}\left[(\psi_x^k)^2 - \psi_{xx}^k\right] - \gamma^k - \psi_x^k \\ &\quad \cdot [3/2 - f(z)/z + yzh_z(z)] \\ &\quad + \frac{1}{2}z^2\left[y^2h_z^2(z) + yh_{zz}(z)\right] \\ &\quad + yh_z(z)[2z - f(z)] + 1 \\ &\quad - f_z(z) - \frac{1}{2}h^2(z)|_{z=e^x}.\end{aligned} \tag{3.18}$$

Note that $\psi^k(x)$ diverges to $+\infty$ as $|x| \to \infty$ by assumptions C2), C3). In Section II this was ensured by assumptions A3), B6) which preclude unstable, linear f. Since we are considering the general bilinear problem, we must allow ϕ_2 to diverge to $-\infty$ more slowly than ϕ_1 diverges to $+\infty$. Also because b_x, ψ_{xx}^k in (3.18) are asymptotic to linear combinations of $[e^x h_z(e^x)]_x$ and $[e^{-x}f(e^x)]_x$, assumptions C1)–C5) ensure that the potentials c^k, c_{adj}^k (associated with the adjoint of the equation for u^k) are both asymptotic to (as $|x| \to \infty$)

$$c_{ess}^k(x) = \begin{cases} \frac{1}{2}\alpha(\alpha-2)f^2(z)/z^2 + o(f^2/z^2)|_{z=e^x} \\ \quad \text{as } x \to -\infty \\ \frac{1}{2}\left\{\left[\beta_1^k h(z) + \beta_2(1+h^2(z))^{1/2}\right]_x - yzh_z(z)\right\}^2 \\ \quad - \frac{1}{2}h^2(z) + o(h^2)|_{z=e^x} \quad \text{as } x \to +\infty. \end{cases} \tag{3.19}$$

Define

$$\begin{aligned}\nu_1 = \eta_1 &= \lim_{z \downarrow 0} |f/z| / \left[h^2 + f^2/g^2\right]^{1/2} \\ &= 1 \quad \text{by C2)} \\ \nu_2 &= \lim_{z \to +\infty} \inf z|h_z(z)| / \left[h^2 + f^2/g^2\right]^{1/2} \in (0,\infty) \\ \eta_2 &= \lim_{z \to +\infty} \sup z|h_z(z)| / \left[h^2 + f^2/g^2\right]^{1/2} \in (0,\infty).\end{aligned} \tag{3.20}$$

Because $\min(\nu_1, \nu_2) > 0$ by C5), the remainder of the arguments in the proofs of Theorems 1–3 go through, with appropriate simplifications due to the dichotomy of (3.19). Details are given in Appendix II. This proves the following.

Theorem 4: Suppose C1)–C5) hold. Let $\theta_i > 0$, $0 < \theta_1 + \eta_2\theta_2 < 1$, *and suppose* $p_o(z)$ *satisfies*

$$p_o(z) \leqslant M_1 \exp\left[\theta_1 \int_1^z (f(\xi)/\xi^2)\, d\xi - \theta_2 |h(z)|\right] \tag{3.21}$$

for all $z \in (0,\infty)$ *and some* $M_1 > 0$. *Then for any* $\hat{\theta}_i < \theta_i$ *the DMZ equation (3.14) has a unique solution in the class of functions satisfying, for all* $t \geqslant 0$

$$\lim_{z \to 0,\infty} \sup U(t,z)\exp\left[-\hat{\theta}_1 \int_1^z (f(\xi)/\xi^2)\, d\xi + \hat{\theta}_2|h(z)|\right] = 0. \tag{3.22}$$

Moreover, if there exist constants M_2, $\hat{\theta}_i$, $0 < \hat{\theta}_i < \theta_i$ *such that for all* $z \in (0,\infty)$

$$M_2 \exp\left[\hat{\theta}_1 \int_1^z (f(\xi)/\xi^2)\, d\xi - \hat{\theta}_2|h(z)|\right] \leqslant p_o(z) \tag{3.23}$$

then for all $t \geqslant 0$ *the solution* $U(t,z)$ *is asymptotic to*

$$\exp\left[\int_1^z (f(\xi)/\xi^2)\, d\xi - |h(z)|\right] \tag{3.24}$$

in the sense of (2.14) in Theorem 3.

For example, when $f(z) = az + b$, assumption C2) implies either $b > 0$ or $a > 0$ and $b = 0$. Then

$$\phi_1(z) = bz^{-1} - b - a\log z \tag{3.25}$$

and, whenever (3.21), (3.22) are satisfied, $U(t,z)$ is asymptotic to

$$\begin{cases} \exp[-|h(z)| - b/z], & \text{if } b > 0 \\ z^a \exp[-|h(z)|], & \text{if } b = 0 \text{ and } a > 0. \end{cases} \tag{3.26}$$

Finally, we mention two examples involving coefficients with rapid growth (faster than any polynomial) to which our theorems apply.

Example 3: f is any stable, odd polynomial, g is a constant, and $h(x) = \exp[(1+x^2)^{1/2}]$.

Example 4: f, g satisfy Khas' minskii's test for explosions in a trivial manner [14]

$$\lim_{|x| \to \infty} -\int_0^x (f/g^2)(\xi)\, d\xi = +\infty$$

and there is a constant $r > 0$ such that $h(x) = 0$, $\forall |x| > r$. If $p_o(x)$ satisfies the conditions of Theorem 3, then (2.2), (2.14) show that for all $t \geqslant 0$ the conditional density is asymptotic to the *a priori* density as $|x| \to \infty$.

IV. Conclusions

Apart from guaranteeing that a large class of nonlinear filtering problems with unbounded coefficients are well-posed, the conditions derived here are potentially useful in numerical treatment of filtering problems. The bounds on the tail behavior in Theorems 3 and 4 show that the conditional densities in many problems are rapidly asymptotic to zero as $|x| \to \infty$. This information can be used to select and shape the finite domain ($(t, x) \in \mathfrak{D}$) over which discrete approximations to the density are fabricated; and it can serve to provide estimates of the numerical error associated with a given numerical technique. Of course, the bounds also identify the class of functions in which the conditional density may be found if the initial density is in the appropriate class.

The class of functions in which we have shown uniqueness is perhaps "smaller" than one would like. We have not, for instance, shown uniqueness in the class of nonnegative L_1 functions, the largest class of densities. This limitation is shared by the related work in [8], [9]. For further results on this issue, see [18].

As mentioned in the introduction the treatment of some multidimensional signal and observation processes [$(x, y) \in R^n \times R^m$ in (1.1)] may be accomplished with a straightforward modification of our arguments. For example, the exponential transformation to robust form (1.4) is

$$V(t,x) = \exp\left[-\langle h(x), y(t)\rangle_{R^m}\right] U(t,x) \quad (4.1)$$

and the derivation of the robust equation is standard [3]. An analysis of some multidimensional problems is reported in [18].

Finally, having identified the proper weight functions, the smoothness assumptions on f, g, h can be relaxed considerably. Thus, the necessary differentiations can be replaced by existence of weak derivatives and one can establish most of our results in weighted Sobolev spaces, using exactly the same weights presented here.

Appendix I

We give details of the proof of Lemma 2, and indicate the derivation of inequalities (2.22), (2.28), (2.35), (2.36) in the proofs of Theorems 1–3. Throughout we drop the superscript k.

Define

$$c_1(x,t) = \frac{1}{2}g^2(\psi_x)^2 - \psi_x(g^2h_xy - f) + \frac{1}{2}g^2(h_x)^2y^2 - fh_xy - \frac{1}{2}h^2 \quad (I.1.a)$$

$$c_2(x,t) = -\frac{1}{2}g^2\psi_{xx} - (g^2)_x\psi_x + \frac{1}{2}g^2h_{xx}y + (g^2)_xh_xy + \frac{1}{2}(g^2)_{xx} - f_x - \gamma \quad (I.1.b)$$

$$c_3(x,t) = \frac{1}{2}g^2(\psi_{xx} - yh_{xx}) - \gamma \quad (I.1.c)$$

and note that $c_1 = c_{ess}$, $c = c_1 + c_2$, $c_{adj} = c_1 + c_3$. [Compare (2.4), (2.19).] Also, c_2 is a linear combination of

$$(g^2)_{xx}, f_x, y(t)(g^2h_x)_x, (g^2\psi_x)_x \quad (I.2.a)$$

$$y(t)g^2h_{xx}, g^2\psi_{xx} \quad (I.2.b)$$

and c_3 is a linear combination of the terms in (I.2.b). (Here certain terms in (I.1.b) have been rewritten using the product rule for differentiation.) Letting $F(x) = \min[h^2(x), (f^2/g^2)(x)]$, it suffices for the proof of Lemma 2 to show that $c_2, c_3 = o(F)$. Those terms in (I.2) which do not involve ψ are $o(F)$ by assumptions A4), B2), B4). In case B5) a), ψ_x is a linear combination of

$$f/g^2, h_x, h_xh/[1+h^2]^{1/2}. \quad (I.3)$$

Noting that $h/[1+h^2]^{1/2} = 0(1)$, the terms in (I.2) which do involve ψ are $o(F)$ by assumptions A4), B2). In case B5b), $g^2\psi_{xx}$ and $(g^2\psi_x)_x$ are both linear combinations of gh_x, g_xh, and hence are $o(h^2)$ by the additional hypothesis in B5b).

In the proofs of Theorems 1–2, the inequalities (2.22), (2.28) imply c_{ess} is bounded above as follows. Define

$$M_i(x) = g(\phi_i)_x/[h^2 + f^2/g^2]^{1/2}, \qquad i = 1,2 \quad (I.4.a)$$

$$N(x) = g[1+\phi_2^2]_x^{1/2}/[h^2+f^2/g^2]^{1/2} \quad (I.4.b)$$

and note that $N(x) = O(M_2(x))$. Let α, β_1, β_2 correspond to the generic function $\psi(x)$ defined in (2.2). When case B5a) holds, c_{ess} is bounded above if there exists a constant $R > 0$ such that

$$|(\alpha-1)M_1 + (\beta_1 - y(t))M_2 + \beta_2 N| < 1, \qquad \text{for all } |x| > R. \quad (I.5)$$

By the triangle inequality, (I.5) holds if for some constants A_i, $0 < A_1 + A_2 < 1$,

$$|\alpha - 1||M_1| \leqslant A_1 \qquad \text{for all } |x| > R \quad (I.6.a)$$

$$|\beta_2 N + (\beta_1 - y(t))M_2| \leqslant A_2 \qquad \text{for all } |x| > R. \quad (I.6.b)$$

Inequalities (2.22), (2.28) follow by taking limits supremum in (I.6) for various values of α, β_1, β_2, and noting that B3) implies η_1, η_2 are finite. When case B5b) holds, the analysis is similar, with $M_1 = 0$ and (I.5) replaced by

$$|\beta_1 M_2 + \beta_2 N| < 1 \qquad \text{for all } |x| > R. \quad (I.7)$$

The proof of Theorem 3 requires that $\hat{c}^k_{ess}(1)$ be bounded below for μ_1 sufficiently large. [See (2.34).] When case B5a) holds, this follows from the existence of positive constants μ_1, R, such that

$$|(\mu_1 + \alpha - 1)M_1 + (\beta_1^k - y)M_2 + (\mu_1 + \beta_2)N| > 1. \quad (I.8)$$

If $M_2 = o(M_1)$ [i.e., $h = o(f/g)$], then $\nu_2 = 0$, $\nu_1 = 1$, and (I.8) is implied by (2.36a). Similarly, if $\nu_1 = 0$, then (I.8) follows from (2.36b). If both ν_1, ν_2 are positive then (I.8) follows from (2.36), provided the functions

$$(\mu_1 + \alpha - 1)M_1 \tag{I.9.a}$$

$$(\beta_1^k - y)M_2 + (\mu_1 + \beta_2)N \tag{I.9.b}$$

have the same algebraic sign for $|x| > R$. Because $\beta_2 > |\beta_1^k - y(t)|$, (I.9.b) has the same sign as $N(x)$. Also, (I.9.a) has the same sign as $M_1(x)$ provided $\mu_1 + \alpha - 1 > 0$. Since $-f(x)\mathrm{sgn}(x)$ and $h_x(x)\mathrm{sgn}(xh(x))$ are nonnegative for $|x|$ sufficiently large and

$$(M_1N)(x) = \frac{-\mathrm{sgn}(x)fg}{[h^2 + f^2/g^2]^{1/2}} \cdot \frac{\mathrm{sgn}(x)gh_xh/[1+h^2]^{1/2}}{[h^2 + f^2/g^2]^{1/2}} \tag{I.10}$$

it follows that for R sufficiently large, $(M_1N)(x) > 0$ for all $|x| > R$. Hence, (2.36) implies (I.8).

The analysis when case B5b) holds is simpler; since $M_1 = 0$, it suffices to take limits infimum in (I.8) with $y(t)$ set formally equal to zero. [Compare (I.7).]

Appendix II

The proof of Theorem 4 differs only slightly from the proof of Theorems 1–3. We shall indicate those changes needed to 1) show $\psi^k(x)$ diverges to $+\infty$ as $|x| \to \infty$ whenever $\alpha > 0$, $\beta_2 > |\beta_1^k|$, 2) prove an analog of Lemma 1 for c_{ess} as defined in (3.19), and 3) select the parameters $\alpha, \beta_2, \{t_k, \beta_1^k, \gamma^k\}_{k=0}^{\infty}$.

1) Recall the transformation $z = \exp(x)$ and the definition of ψ^k in (3.17), and note that $z \to 0$ as $x \to -\infty$. By C2) there exists a constant ϵ such that $f(z) \geq \epsilon z$ for $z \in (0, \epsilon)$. Hence,

$$\int_0^1 f(\xi)/\xi^2 \, d\xi = +\infty \tag{I.11}$$

and since h is bounded at the origin, $\psi^k(x) \to +\infty$ as $x \to -\infty$ whenever $\alpha > 0$. Also, by C1) there exists a constant K such that

$$\left| \int_1^z -f(\xi)/\xi^2 \, d\xi \right| \leq K \log(z) \qquad \text{for } z \geq 1. \tag{I.12}$$

Since $\log(z) = o(h(z))$ as $z \to +\infty$ by C3), it follows that $\psi^k \to +\infty$ as $x \to +\infty$ whenever $\beta_2 > |\beta_1^k|$.

2) Using the chain rule for differentiation, compute

$$\psi_x^k(x) = \alpha(-f(z)/z) + \beta_1^k z h_z(z) + \beta_2 z\left(h_z h/[1+h^2]^{1/2}\right)(z)\Big|_{z=e^x} \tag{I.13}$$

and let c_{ess}^k be any locally Hölder continuous function satisfying (3.19). Recall $c^k(x,t)$ in (3.18), and denote $c_{adj}^k(x,t) = (c^k - b_x^k)(x,t)$. To prove $c^k, c_{adj}^k = o(c_{ess}^k)$ it is sufficient to show C1)–C5) ensure that

$$f_z, fh_z, zh_z, z^2h_{zz}, f/z \tag{I.14}$$

are all $o((\psi_x^k)^2)$, while

$$z^2(h_z)^2, h^2 = o(f(z)/z^2) \qquad \text{as } z \downarrow o$$
$$f^2/z^2 = o(h^2) \qquad \text{as } z \to +\infty. \tag{I.15}$$

For example, fh_z is bounded at the origin, while C1), C5) imply $fh_z = 0(h)$ as $z \to +\infty$. But $h = o(h^2)$ as $z \to +\infty$ by C3), whence $fh_z = o(h^2)$ as $z \to +\infty$. The remaining verifications are similar.

3) The proof of Theorem 4 differs from the proofs of Theorems 1–3 in the form of the inequalities (2.22), (2.28). (Note that the present analysis corresponds to case B5a) in Section II.) The inequality (2.22) becomes

$$\alpha \in (0,2)$$
$$|\beta_2 \pm (\beta_1^k - y)|\eta_2 < 1 \tag{I.16}$$

while (2.28) becomes

$$\alpha, \alpha + \delta \in (0,2)$$
$$|\beta_2 \pm (\beta_1^k - y)|\eta_2 < 1,$$
$$|\beta_2 \pm (\beta_1^k - y) + \delta|\eta_2 < 1. \tag{I.17}$$

Nevertheless, since $\nu_1 = 1$, $\nu_2 > 0$, $\theta_1 \leq 1$, $\theta_2\eta_2 \leq 1$, the choice of parameters (2.24), (2.25) proves Theorem 4.

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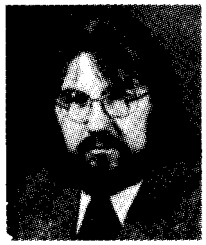

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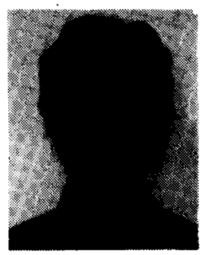

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