State-Space Models for Infinite-Dimensional Systems

JOHN S. BARAS, MEMBER, IEEE, ROGER W. BROCKETT, FELLOW, IEEE, AND PAUL A. FUHRMANN

Abstract—Distributed effects are present in almost all physical systems. In some cases these can be safely ignored but there are many interesting problems where these effects must be taken into account. Most infinite dimensional systems which are important in control theory are specifiable in terms of a finite number of parameters and hence are, in principle, amenable to identification. The state-space theory of infinite dimensional systems has advanced greatly in the last few years and is now at a point where real applications can be contemplated. The realizability criteria provided by this work can be employed effectively in the first step of the identification procedure, i.e., in the selection of an appropriate infinite dimensional model. We show that there exists a natural classification of nonrational transfer functions, which is based on the character of their singularities. This classification has important implications for the problem of finite dimensional approximations of infinite dimensional systems. In addition, it reveals the class of transfer functions for which there exist models with spectral properties closely reflecting the properties of the singularities of the transfer functions. The study of models with infinitesimal generators having a connected resolvent sheds light on some open problems in classical frequency response methods. Finally, the methods used here allow one to see the finite dimensional theory itself more clearly as the result of placing it in the context of a larger theory.

I. Introduction

DESPITE a superficial similarity between the formalism for modeling finite- and infinite-dimensional systems, there are many essential differences and these differences require careful attention if one is to avoid errors and meaningless constructions. In view of the traps intrinsic in infinite-dimensional problems, there is a tendency to replace all infinite-dimensional problems which arise by finite-dimensional approximations. However there are some notable exceptions. For example the Ziegler-Nichols [19] technique for adjusting controllers in classical control theory is based on approximating whatever system one encounters by a pure delay and a second-order system—thus converting all problems into infinite-dimensional ones unless the delay happens to be zero.

The purpose of this paper is to survey the available theory for modeling linear time-invariant infinite-dimensional systems in state-space form. We consider only systems with a finite number of inputs and outputs. Regarding the practical significance of this kind of study we make the following points.

1) Systems which are infinite dimensional do not necessarily require an infinite number of experiments to identify. For example, a system whose transfer function is $e^{-\alpha s}/(s+\beta)$ certainly does not have a finite-dimensional realization but is specifiable by the two real numbers α and β.

2) There are experimental tests which will indicate that a system is not finite dimensional even though establishing that a system is finite dimensional is much harder and perhaps impossible in any empirically meaningful way. This underscores the desirability of seeing finite-dimensional problems as specializations of infinite-dimensional ones as opposed to viewing infinite-dimensional problems as extension of finite-dimensional ones.

3) With the exception of somewhat specialized techniques such as the Padé approximation methods and methods based on modal approximation, there are no methods for approximating infinite-dimensional systems by finite-dimensional ones. Moreover, it seems likely that the basis for such a theory would, by necessity, be a complete state-space theory for infinite-dimensional systems.

We restrict the discussion to systems which can be realized on state spaces which have an inner product relative to which they are Hilbert spaces. While by no means the most general setting which has been considered in the literature, this assumption leads to a theory which is compatible with the modern theory of partial differential equations, optimal control, etc.

This paper is organized as follows. In Section II we discuss the basic realizability criteria developing various analogs of the fact that a linear time-invariant system has a finite-dimensional state-space realization if and only if its transform is rational and goes to zero at infinity. There are two types of results here because one can consider realizations via

$$\dot{x}(t) = Ax(t) + bu(t): \quad y(t) = \langle c,x(t)\rangle$$

with A bounded, i.e., $\|Ax\| \leqslant k\|x\|$, or, and this is more typical, if A is not bounded but does give rise to a semigroup $e^{At}; t \geqslant 0$. In Section III we discuss the relationships between two minimal realizations of the same input-output system. Under certain assumptions we establish a state-space isomorphism theorem but also indicate how this result can fail in the infinite-dimensional case. Section IV is devoted to the important problem of finding out to what extent the input-output data determine the spectrum of the operator A. This property, which one takes for granted in the finite-dimensional case, need not hold for canonical infinite-dimensional systems. However for important classes of infinite-dimensional systems we do find that the spectrum of A is determined by the points of nonanalyticity of the transfer function.

II. Realizability Theorems

The relations between internal and external descriptions of dynamical systems constitute an essential part of

Manuscript received February 2, 1974, revised August 1, 1974. This work was supported by the U.S. Office of Naval Research under the Joint Services Electronics Program by Contract N00014-67-A-0298-0006.

J. S. Baras is with the Department of Electrical Engineering, University of Maryland, College Park, Md.

R. W. Brockett and P. A. Fuhrmann are with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass.

the analysis of modeling techniques. Here we describe these relations for several classes of distributed parameter systems. For simplicity, and for clarity of exposition, in this section and in Section IV we restrict our discussion to scalar inputs and scalar outputs. Sections III and V discuss finite-dimensional input spaces and finite-dimensional output spaces.

The input–output relations for the systems we study here are described by the standard convolution

$$y(t) = \int_0^t T(t-\sigma)u(\sigma)d\sigma \quad (1)$$

where T is a real valued function, usually called the *weighting pattern*. We always assume that T is Laplace transformable and we denote the Laplace transform by $\hat{T}$, the *transfer function*. For modeling purposes T gives a model of the system via (1). We concentrate on nonrational transfer functions.

On the other hand we consider modeling an infinite-dimensional linear system via the following dynamical equations:

$$\left.\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + bu(t) \\ y(t) &= \langle c, x(t)\rangle \end{aligned}\right\} \quad (2)$$

where $x(t)$, b, c belong to a separable Hilbert space X, with inner product $\langle\cdot,\cdot\rangle$. We assume that the operator A appearing in (2) generates a strongly continuous semigroup of bounded operators on X [12], denoted by e^{At} for $t \geqslant 0$. This means that for all $t \geqslant 0$, e^{At} is a bounded operator on X, that $e^{At_1}e^{At_2}x = e^{A(t_1+t_2)}x$ for $t_1, t_2 \geqslant 0$ and that $\lim_{t\to 0} \|e^{At}x - x\| = 0$ for all x in X. A is usually called the infinitesimal generator of the semigroup e^{At}, and its domain of definition is denoted by $\mathfrak{D}_0(A)$. The theory of strongly continuous semigroups is developed in [12]. The Hille–Yosida theorem [12] characterizes the operators A which generate strongly continuous semigroups. Standard examples are systems governed by the diffusion or the wave equation on a suitable spatial domain. We understand by (2) that $x(t)$ satisfies the integral equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma.$$

Whenever (1) and (2) represent models of the same system we must have that

$$T(t) = \langle c, e^{At}b\rangle$$

and that

$$\hat{T}(s) = \langle c, (Is - A)^{-1}b\rangle$$

for an appropriate region of the complex plane. Whenever A is a bounded operator we will say that (A,b,c) is a *bounded realization*. For A unbounded we call (A,b,c) a *regular realization*.

Remark: There is another type of realizations which are related to boundary observations and are called *balanced realizations*. For this type of realization b is restricted to belong to the domain of A, but the observations are given by $y(t) = c(x(t))$. Here c is a linear map (not bounded), defined for all x in the domain of A and such that $|c(x)| \leqslant k(\|Ax\| + \|x\|)$ for some constant k and all $x \in \mathfrak{D}_0(A)$. It turns out that the classes of weighting patterns admitting regular or balanced realizations coincide. For more details on that we refer to [3].

It is always easy to pass from the "internal" model (2) to the "external" model. The theorems that follow characterize the classes of weighting patterns (and consequently the classes of transfer functions) which admit various types of realizations, and therefore provide the answer to the converse question. To characterize the transfer functions we need to introduce the so-called Hardy spaces [13]. By $H^2(\boldsymbol{D})$ we mean the set of complex valued functions which are holomorphic in (the open unit disk) and have a Taylor series about zero with square summable coefficients. The unit circle, the boundary of $\mathbf{D}$, is denoted by $\boldsymbol{T}$. We denote by $H^2(\boldsymbol{T})$ the subspace of $L^2(\boldsymbol{T})$ consisting of functions with vanishing negative Fourier coefficients. Thus, if $f \in H^2(\boldsymbol{D})$ then $f(z) = a_0 + a_1z + a_2z^2 + \cdots$ in $\boldsymbol{D}$, and if $\phi \in H^2(\boldsymbol{T})$ then $\phi(e^{i\theta}) = b_0 + b_1e^{i\theta} + b_2e^{i2\theta} + \cdots$. The half-plane Re $s > \rho$ is denoted by Π_ρ^+. The space of functions which are analytic in Π_ρ^+ and square integrable along vertical lines in Π_ρ^+ so that

$$\sup_{\sigma>\rho} \int_{-\infty}^{+\infty} |f(\sigma + i\omega)|^2 d\omega \leqslant M < \infty$$

is usually denoted by $H^2(\Pi_\rho^+)$. When $\rho = 0$ we simply write $H^2(\Pi^+)$. Functions that are analytic and bounded in Π^+ (respectively in Π_ρ^+) form the spaces $H^\infty(\Pi^+)$ (respectively $H^\infty(\Pi_\rho^+)$). The boundary values of the elements of $H^2(\Pi^+)$ form the space $H^2(\boldsymbol{I})$ which is the image of $L_2(0,\infty)$ under the Fourier transform.

Theorem 1:

a) A weighting pattern T has a bounded realization if and only if it is an entire function of exponential order.

b) The transfer function T has a bounded realization if and only if it is analytic at infinity and vanishes there (i.e., T can be represented by its Taylor series around the point at infinity).

Proof: See Baras and Brockett [3].

Certainly rational transfer functions which vanish at infinity satisfy this criterion. Thus the results of Theorem 1 constitute an extension of the well-known realizability criteria for finite-dimensional linear systems [5]. The realization which is provided has the following extremely simple form (see, Baras and Brockett [3] and Fuhrmann [7]).

As the state space we use the sequence space l_2, of sequences $\{a_0, a_1, a_2, \cdots\}$ which are square summable, and which serves as a prototype for separable Hilbert spaces. The operator A is chosen to be a multiple of the forward shift and has the infinite matrix representation

$$A = k\begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0\cdot & \\ & & 0 & 1 & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot\ \cdot \\ & & & & \cdot\ \cdot \end{bmatrix}.$$

As vector b we choose the sequence $\{1,0,0,\cdots\}$ and as c

the sequence

$$\left\{T(0), \frac{T^{(1)}(0)}{k}, \frac{T^{(2)}(0)}{k^2}, \cdots \frac{T^{(n)}(0)}{k^n}, \cdots\right\}$$

where $T^{(l)}(0)$ is the value of the lth derivative of T at 0. The constant k can be any real number larger than the exponential order of T. We see therefore that the forward shift (modulo an unimportant scaling factor) can serve as a universal model for the dynamics of this class of systems. This result is in complete accordance with a well-known fact from operator theory; namely, a suitable number of copies of the backward shift (which has as infinite matrix representative the transpose of the matrix shown above) is known to be a universal model for bounded operators on a Hilbert space. For a precise exposition of this fact and its important consequences for the development of operator theory, the interested reader should consult Nagy–Foias (see [17, p. 277]).

Theorem 2:

a) Any weighting pattern T having a regular realization, is continuous and of exponential order. A sufficient condition for T to be realizable is that it be locally absolutely continuous (i.e., the derivative exists almost everywhere) and that its derivative be of exponential order.

b) A necessary condition for a transfer function $\hat{T}$ to have a regular realization is that it belongs to $H^2(\Pi_\rho{}^+) \cap H^\infty(\Pi_\rho{}^+)$ for some $\rho > 0$. A sufficient condition is that $\hat{T}$ and $s\hat{T} - T(0)$ belong to $H^2(\Pi_\rho{}^+)$ for some $\rho > 0$.

Proof: See Baras and Brockett [3]. A similar result appears in [1].

This theorem provides a further generalization from the previous one. The realization provided has a very simple form which is described in the sequel. If we can realize $T(t)$ we can realize $e^{-\rho t}T(t)$ for all real ρ, we can therefore assume without loss of generality that T and its derivative belong to $L_2(0, \infty)$ (after appropriate scaling by an exponential factor).

Thus we take as state space $X = L_2(0, \infty)$, and as semigroup e^{At} the *left translation semigroup* on $L_2(0, \infty)$ which acts on $x \in L_2(0, \infty)$ by

$$(e^{At}x)(\sigma) = x(\sigma + t) \qquad \sigma \geqslant 0.$$

The vector b is chosen to be T and the functional c is evaluation of a function at 0. This realization is a balanced realization and obviously

$$c[e^{At}b] = T(t + \sigma)\big|_{\sigma=0} = T(t).$$

We can then produce a regular realization, via the procedure detailed in [3].

There is a class of weighting patterns for which this regular realization takes a very simple form. Suppose that T and $\dot{T}$ belong to $L_2(0, \infty)$ and that $T(0) = 0$. Then $\hat{T}(i\omega)$, $i\omega\ \hat{T}(i\omega)$ and $(1 - i\omega)\hat{T}(i\omega)$ belong to $H^2(I)$. So

$$T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{T}(i\omega)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\frac{1}{1 + i\omega}}\, e^{i\omega t}(1 - i\omega)\hat{T}(i\omega)d\omega.$$

Therefore if we pick as state space $H^2(I)$, as A, multiplication by $i\omega$ as b the function $(1 - i\omega)T(i\omega)$ and as c the function $1/1 + i\omega$, the above formula provides a realization for T, see also Baras [4]. By applying Fourier transform we present this realization in terms of the left translation semigroup. This has as state space $L_2(0, \infty)$, as e^{At} the left translation semigroup on $L_2(0, \infty)$, as c the function e^{-t} and as b the function $T - \dot{T} = (I - d/dt)T$. Notice that the differentiation operator is the infinitesimal generator of the left translation semigroup on $L_2(0, \infty)$. We summarize this construction in the following corollary.

Corollary 2.1: Suppose T and $\dot{T}$ belong to $L_2(0, \infty)$ and $T(0) = 0$. Then T has a regular realization.

We call this realization the *translation realization* of T. We observe also the simple relation between this regular realization and the balanced realization constructed before. If (A_r, b_r, c_r) is the regular realization and (A_b, b_b, c_b) is the balanced realization

$$A_b = A_r = \frac{d}{dt}$$

$$b_r = \left(I - \frac{d}{dt}\right) T = (I - A_r)b_b$$

$$c_b(x) = x(0) = \int_0^\infty e^{-t}\left(1 - \frac{d}{dt}\right) x(t)dt = \langle c_r, (I - A_r)x\rangle.$$

The adjoint of the left translation semigroup is the right translation semigroup which acts via

$$e^{A^*t}x(\sigma) = \begin{cases} x(\sigma - t) & \sigma \geqslant t \\ 0 & \sigma < t. \end{cases}$$

Now the right translations of the function e^{-t} span $L_2(0, \infty)$. Let us denote by M_T the closed subspace of $L_2(0, \infty)$ generated by the left translations of T. Then in this case it turns out that T and $T - \dot{T}$ generate via left translations the same closed subspace of $L_2(0, \infty)$. Denoting by P_{M_T} the orthogonal projection on M_T we see that the following (henceforth called the *restricted translation realization*) also realizes T.

$$b' = P_{M_T}b_r$$
$$c' = P_{M_T}c_r$$
$$e^{A't} = P_{M_T}e^{A_rt}\big|_M.$$

Here the notation $W|M$ denotes the operator W restricted to the subspace M. Indeed

$$\langle c', e^{A't}b'\rangle = \langle P_{M_T}c_r, P_{M_T}e^{A_rt}\big|_M P_{M_T}b_r\rangle = \langle c_r, e^{A_rt}b_r\rangle = T(t).$$

The last equality implied by the definition of M_T.

The use of the left translation semigroup as a universal model for the dynamics, is in accordance with the well-known fact from the theory of semigroups of bounded operators in Hilbert spaces, which states that any asymptotically stable semigroup is modeled by a left translation semigroup in a vectorial Hilbert space (see Lax and Phillips [14]).

We finally characterize classes of transfer functions which admit realizations of a more special nature.

Definition: A scalar function ϕ is *completely monotonic* if it is infinitely differentiable in $(0, \infty)$, continuous in $[0, \infty)$ and satisfies $(-1)^n\phi^{(n)}(t) \geq 0$ for $t > 0$, [20]. A function ϕ defined on $(-\infty, \infty)$ is called *positive definite* if for all choices of real t_i, $i = 1, \cdots, n$ and all complex numbers α_i, $i = 1, \cdots, n$ we have $\sum_{i,j} \phi(t_i - t_j)\alpha_i\bar{\alpha}_j \geqslant 0$.

By Bochner's theorem [20] the positive definite functions are characterized as the Fourier transforms of finite positive Borel measures on the real line.

Definition: A realization (A,b,c) is self-adjoint if $A = A^*$ and $b = c$. It is called skew-adjoint if $A = -A^*$ and $b = c$.

Theorem 3:

a) A weighting pattern T has a self-adjoint stable realization iff T is completely monotonic.

b) A weighting pattern T has a skew-adjoint realization iff T defined on $[0,\infty)$ has an extension to $(-\infty,\infty)$ which is positive definite.

Proof:

a) T completely monotonic implies by Bernstein's theorem [20] that

$$T(t) = \int_{-\infty}^{0} e^{\lambda t} d\mu$$

for some unique finite nonnegative Borel measure μ. Consider $L_2((-\infty,0);d\mu)$ (the space of functions square integrable with respect to the measure μ) and the operator multiplication by the independent variable on $L_2((-\infty,0);d\mu)$. The projection valued measure associated with A via the spectral theorem for self-adjoint operators satisfies $(E(\sigma)f)(\lambda) = \chi_\sigma(\lambda)f(\lambda)$ where σ is a Borel set and χ is the characteristic function. Let b be the element of $L_2((-\infty,0);d\mu)$ defined by $b(\lambda) = 1$.

Then

$$\langle b,e^{At}b\rangle = \int_{-\infty}^{0} b(\lambda)e^{\lambda t}E(d\lambda)b(\lambda) = \int_{-\infty}^{0} e^{\lambda t}d\mu = T(t).$$

Conversely if

$$T(t) = \langle b,e^{At}b\rangle = \int_{-\infty}^{0} \langle b,e^{\lambda t}E(d\lambda)b\rangle = \int_{-\infty}^{0} e^{\lambda t}\langle b,E(d\lambda)b\rangle$$

$\langle b,E(d\lambda)b\rangle$ is a finite nonnegative Borel measure on the real line and the result follows by Bernstein's theorem.

b) This follows in a similar manner by Bochner's theorem [20].

Notice that if a completely monotonic function T has rational Laplace transform $\hat{T}$, then the zeros and poles of T are on the negative real axis, interlace, and the first one is a pole. Also when T is completely monotonic and T is meromorphic then the zeros and poles are on the negative real axis, interlace, and the first one is a pole. Thus we see that transfer functions like these arise from lumped or distributed RC networks.

III. The State-Space Isomorphism Theorems

The theory of finite-dimensional representations of linear systems culminates in a very elegant theorem describing the connection between any two controllable and observable realizations of the same weighting pattern. The key result being that any two such realizations differ by a choice of basis for the state space. Thus, the matrices (A,B,C) in one minimal realization of $T(\cdot)$ and the matrices (F,G,H) in a second realization are related by $A = PFP^{-1}$; $B = PG$, $C = HP^{-1}$. In the infinite-dimensional situation the question of comparing two realizations is more complicated and, except for special cases, not yet fully resolved. In this section we describe the known results and give some indications by way of examples and counterexamples, as to the limitations on the subject.

For infinite-dimensional systems the concept of controllability is less satisfactory than in the finite-dimensional case. The problem is that systems with finite-dimensional input spaces but infinite-dimensional state spaces have certain inherent limitations with respect to the reachable set, regardless of the operators A and B; these difficulties are well documented in the literature [10].

We define a realization (A,B,C) to be *controllable* if $\bigcap_{t\geq 0} \ker B^*e^{A^*t} = \{0\}$ *observable* if $\bigcap_{t\geq 0} \ker Ce^{At} = \{0\}$ and *canonical* if it is both controllable and observable. For some purposes it is convenient to use another, weaker, notion of controllability and observability. Let us assume that A is an infinitesimal generator of a group of operators. We will say that the above realization is *bilaterally controllable* (*bilaterally observable*) if $\bigcap_{t\in R} \ker B^*e^{A^*t} = \{0\}$ ($\bigcap_{t\in R} \ker Ce^{At} = \{0\}$) and *bilaterally canonical* if it is both bilaterally controllable and observable.

If there exists a bounded operator P for which the relations

$$PA = FP,\ PB = G,\ \text{and}\ C = HP$$

hold then we say that P *intertwines* the realizations (A,B,C) and (F,G,H). (It must be observed that (A,B,C) and (F,G,H) play a nonsymmetric role.) If P is one to one and has dense range then we say that the realization (F,G,H) is a *quasiaffine* transform of (A,B,C). If the intertwining operator P is boundedly invertible we say the two systems are *similar*. It is easy to check that if two systems are quasiaffine transforms of each other then they are similar.

In order to see why the state-space isomorphism question becomes more delicate in the infinite-dimensional case it suffices to look at the easiest class of examples. Consider the system

$$\dot{x}_n(t) = \lambda_n x_n(t) + b_n u(t); \qquad n = 1,2,3,\cdots$$

$$y(t) = \sum_{n=1}^{\infty} c_n x_n(t)$$

with $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ in l_2. This realizes the transfer function

$$\hat{T}(s) = \sum_{n=1}^{\infty} \frac{b_n c_n}{s+\lambda_n}.$$

If $\lambda_n \neq \lambda_m$ for $n \neq m$ and $b_n c_n \neq 0$ for all n then this system is controllable and observable. However, it can happen that $\{nc_n\}_{n=1}^{\infty}$ and $\{(1/n)b_n\}_{n=1}^{\infty}$ are both in l_2 as well, in which case

$$\dot{z}_n(t) = \lambda_n z_n(t) + \frac{1}{n} b_n u(t); \qquad n = 1,2,3,\cdots$$

$$y(t) = \sum_{n=1}^{\infty} nc_n z_n(t)$$

is also a controllable and observable realization of the given transfer function. Here x and z are related by $Px = z$ with P defined by

$$\frac{1}{n} x_n(t) = z_n(t); \qquad n = 1,2,3,\cdots.$$

This P is bounded and one to one but it does not have a bounded inverse. Thus these two realizations are not similar even though the z-system is a quasiaffine transform of the x-system.

It seems that the further conditions to add in order to make an isomorphism theorem hold can be of two types. On one hand one can ask that b and c be somehow the same size or, alternatively, that the connection between the input (output) and the state should be very tight in both the realizations. We make these remarks precise in two instances below.

Theorem 4: If (A,B,C) and (F,G,H) are two canonical realizations of the same input–output map which have the property that $A = A^*$; $B = C^*$ and $F = F^*$ and $G = H^*$ then (A,B,C) and (F,G,H) are similar and the similarity is via a unitary operator. If (A,B,C) and (F,G,H) are two bilaterally canonical realizations of the same input–output map which have the property that $A = -A^*$, $B = C^*$, and $F = -F^*$ and $G = H^*$ then (A,B,C) and (F,G,H) are similar and the similarity is via a unitary operator.

Proof: We mimic the finite-dimensional proof. Notice that since A and F are generators one has

$$\|e^{At}\| \leqslant Me^{\lambda t};\ \|e^{Ft}\| \leqslant Me^{\lambda t}$$

for some M and some $\lambda > 0$. Thus the integrals

$$W_{AB} = \int_0^\infty e^{At}BB^*e^{A^*t}e^{-4\lambda t}dt$$

and

$$M_{AC} = \int_0^\infty e^{A^*t}C^*Ce^{At}e^{-4\lambda t}dt$$

exist and define bounded self-adjoint operators. Similar remarks hold for W_{FG} and M_{FH}. Moreover it is easy to see that if the pair (A,B) is controllable then the kernel of W_{AB} is zero and if the pair (A,C) is observable then the kernel of M_{AC} is zero.

Because the input–output maps are the same for (A,B,C) and (F,G,H) we have

$$Ce^{A(t+\sigma+\rho)}B = He^{F(t+\sigma+\rho)}G.$$

Pre- and post-multiplying by $e^{-4\lambda t}e^{A^*t}C^*$ and $B^*e^{A^*\rho}e^{-4\lambda\rho}$, respectively, gives upon integration on t over $[0,\infty)$ and integration on ρ over $[0,\infty)$

$$M_{AC}e^{A\sigma}W_{AB} = \int_0^\infty e^{-4\lambda t}e^{A^*t}C^*He^{Ft}dte^{F\sigma}\int_0^\infty e^{F\rho}GB^*e^{A^*}e^{-4\lambda\rho}d\rho.$$

In view of the self-adjointness conditions in the hypothesis we can rewrite this as

$$W_{AB}e^{A\sigma}W_{AB} = N^*e^{F\sigma}N.$$

Now the polar representation (see Dunford–Schwartz's Theorem 7, [6, p. 1249]) of N is easily seen to be UW_{AB} where U is a partial isometry. The initial domain of U is the closure of the range of N^*. By controllability of (A,B) U is one to one. Now if we reverse the roles of (A,B,C) and (F,G,H) we get

$$W_{FG}e^{F\sigma}W_{FG} = N^*e^{A\sigma}N$$

using the above polar representation for N we see that controllability of (F,G) implies that U is onto. Thus U is unitary and

$$e^{A\sigma} = Ue^{F\sigma}U^*;\ U^*B = G.$$

To treat the skew-adjoint case certain modifications must be made. The fact that A and F are skew-adjoint means that they necessarily generate groups, rather than just semigroups, and that for any $\lambda > 0$ the integral

$$W_{AB} = \int_{-\infty}^\infty e^{At}BB^*e^{A^*t}e^{-\lambda|t|}dt$$

exists and defines a bounded self-adjoint operator. We then follow the above proof using the fact that

$$e^{At} = (e^{-At})^*$$

and the symmetry of the domain of integrations to get precisely the same conclusion.

Although the conclusion here is quite satisfactory the hypothesis is rather strong. An alternative hypothesis which is also restrictive but in a different way will now be described.

We observe that if a system has a finite-dimensional input space and infinite-dimensional state space then for any $t_1 < \infty$ the controllability Grammian

$$W(t_1) = \int_0^{t_1} e^{At}BB^*e^{A^*t}dt$$

is compact since the integral can be approximated in the uniform operator topology by finite rank compact operators. (Recall B has finite rank.) Hence $W(t_1)$ cannot be boundedly invertible.

This argument is invalid for $t_1 = \infty$ however, and it can happen that $W(\infty)$ exists as a bounded operator which is boundedly invertible. An example is given by the system discussed at the start of this section with $b_n = c_n = 1/n$; $\lambda_n = -1/n^2$. In this case $W(\infty)$ is an isometry

$$x'W(\infty)x = \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^2} e^{-t/n^2}x_n^2dt = \sum_{n=1}^\infty x_n^2 = \|x\|^2.$$

Following Helton [11] (see also Balakrishnan [2, p. 109]) we say that a system (A,B,C) is exactly controllable if the limit as t goes to infinity of $W(t)$ exists as a bounded operator with a bounded inverse. We call a system (A,B,C) *exactly observable* if (A^*,C^*,B^*) is exactly controllable.

Theorem 5:

a) Let (A,B,C) and (F,G,H) be two exactly controllable (observable) realizations of the same weighting pattern. Suppose in addition that both realizations are observable (controllable) then the systems are similar.

b) Let (A,B,C) and (F,G,H) be two realizations of the same weighting pattern and such that the first system is observable and exactly controllable and the second system is controllable and exactly observable, then the realization (F,G,H) is a quasiaffine transform of (A,B,C).

Part a) is due to Helton [11] whereas part b), which has a similar proof, is due to Moore.

IV. Spectral Minimality

In the absence of a general state-space isomorphism theorem, realizations which have spectral properties reflecting the singularities of the transfer function $\hat{T}$ are important. This requirement is essential from the engineering point of view and is used in several ad hoc modeling methods. Moreover, the connectedness of the resolvent set of the infinitesimal generator has important implications as far as the relationship to frequency response methods for system identification is concerned.

Let (A,B,C) be any realization of the transfer function T. We let $\rho_0(A)$ denote the principal connected component of $\rho(A)$ the resolvent set of A. Thus $\rho_0(A)$ includes some half-plane of the form $\{s|\text{Re } s > \omega\}$. Clearly there exists an analytic continuation of T to all of $\rho_0(A)$, namely, that analytic function defined by $C(sI - A)^{-1}B$ for all $s \in \rho_0(A)$. Thus for this particular analytic continuation we have the *spectral inclusion property* [3]

$$\sigma(\hat{T}) \subset \sigma_0(A)$$

where $\sigma(\hat{T})$ is the set of nonanalyticity of $\hat{T}$ and $\sigma_0(A)$ is the complement of $\rho_0(A)$.

A realization of T is called spectrally minimal if there exists an analytic continuation of $\hat{T}$ for which $\sigma(\hat{T}) = \sigma(A)$.

We proceed to analyze the question of spectral minimality in the context of restricted translation realizations and that of realization by self-adjoint systems. As might be expected, because of their extreme structural symmetry self-adjoint systems exhibit the best behavior in this respect. The situation is summed up by the following theorem.

Theorem 6:

a) Let (A,b,b) be a canonical self-adjoint realization of a weighting pattern T then the realization is spectrally minimal.

b) Let (A,b,b) be a skew-adjoint bilaterally canonical realization of a weighting pattern T for which $\rho(A)$ is connected then the realization is spectrally minimal.

Proof:

a) We indicate briefly what is involved. The spectral theorem in particular the use of projection valued measures seem essential to the solution. Let $T(s)$ be the Laplace transform of T, thus $\hat{T}(s) = \langle b,(sI - A)^{-1}b\rangle$. Let $E(\cdot)$ be the projection valued measure associated with A. Given any open interval (α,β) on the real line then $\langle E((\alpha,\beta))b,c\rangle$ can be recaptured by the following limit [6, p. 920]:

$$\langle E((\alpha,\beta))b,c\rangle = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \cdot \frac{1}{2\pi i}\int_{\alpha+\delta}^{\beta-\delta}\{\langle((\lambda - i\epsilon)I - A)^{-1}b,c\rangle - \langle((\lambda + i\epsilon)I - A)^{-1}b,c\rangle\}\,d\lambda.$$

This formula is a generalization of the Dunford–Cauchy operational calculus [6]. Now if we assume $\hat{T}$ to be analytic in (α,β) then it follows from Cauchy's theorem that $\langle E((\alpha,\beta))b,b\rangle = \|E((\alpha,\beta))b\|^2 = 0$. Since $E(\cdot)$ commutes with the semigroup e^{At} generated by A it follows that $E((\alpha,\beta))e^{At}b = 0$ for all positive t. Now the set of vectors of the form $e^{At}b$ span the state space by the assumption about controllability and hence $E((\alpha,\beta)) = 0$ or (α,β) belongs to the resolvent set of A. This together with the spectral inclusion property imply spectral minimality in this case.

b) The representation $\hat{T}(s) = \langle b, (Is - A)^{-1}b\rangle$ is valid *a priori* only for s for which $\text{Re } s > 0$. Since the infinitesimal generator is skew-adjoint its spectrum is restricted to the imaginary axis, and by assumption $\rho(A)$ is connected hence the transfer function $\hat{T}(s)$ has an analytic continuation to the left half-plane. The rest of the argument follows as before with the sole exception that the space is spanned by the set of vectors $\{e^{At}b|t \in \boldsymbol{R}\}$.

It should be noted that any completely monotonic function T defined on $[0,\infty)$ can be uniquely extended to a positive definite function T_1 on the real line by letting

$$T_1(t) = \begin{cases} T(t) & t \geqslant 0 \\ T(-t)^* & t < 0. \end{cases}$$

Thus, by Bochner's theorem, T has also a realization by a skew-adjoint system which, without loss of generality, can be taken to be bilaterally canonical. In general this realization will not be canonical and we will have no spectral minimality.

As an example, consider $T(t) = e^{-t}$ which has a one-dimensional canonical realization with -1 as the only spectral point of the generator. Now the function $T_1(t) = e^{-|t|}$ defined on $(-\infty,\infty)$ is positive definite and has the following representation

$$e^{-|t|} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega t}\frac{d\omega}{1+\omega^2}.$$

Thus the skew-adjoint realization in this case has $L^2(-\infty,\infty; d\omega/1 + \omega^2)$ as state space and multiplication by $i\omega$ as infinitesimal generator. The spectrum of the infinitesimal generator is therefore the whole imaginary axis.

Next we pass to the analysis of the case of translation realization. Generally a closed subspace of $L^2(0,\infty)$ will be called left (right) invariant if it is invariant under the left (right) translation semigroups. For a given weighting pattern $T \in L^2(0,\infty)$, M_T denotes the smallest left invariant subspace of $L^2(0,\infty)$ that includes T. We will say that T is *cyclic* or *noncyclic* according to whether M_T is equal to $L^2(0,\infty)$ or not.

The translation realization of the weighting pattern T, satisfying the assumptions of Corollary 2.1 is canonical if we take the state space to be M_T. In the cyclic case the state space becomes all of $L^2(0,\infty)$ and the infinitesimal generator of the left translation semigroup is the differentiation operator having the whole closed left half-plane as spectrum. Thus for all noncyclic functions T whose Laplace transforms $\hat{T}$ admits some analytic continuation to the left half-plane there is no hope for spectral minimality using the translation realization. So we are left with realizable functions which are noncyclic.

To facilitate analysis we apply the Fourier transform $\mathfrak{F}$ to the following direct sum decomposition $L^2(0,\infty) = M_T \oplus M_T^{\perp}$. By the Payley–Wiener theorem $L^2(0,\infty)$ is mapped onto $H^2(\Pi^+)$ and $M_T^{\perp}$ (which is right invariant) is mapped onto a subspace of $H^2(\Pi^+)$ which is invariant under multiplication by any $H^\infty(\Pi^+)$ function. The structure of these subspaces is determined by a theorem of Beurling and Lax [13],[15] and they have the form $\phi H^2(\Pi^+) = \{\phi f \mid f \in H^2(\Pi^+)\}$ where ϕ is an inner function in Π^+, i.e., a function analytic in Π^+ satisfying $|\phi(s)| < 1$ and for which almost everywhere on the boundary, the boundary values $\phi(i\omega)$ exist and satisfy $|\phi(i\omega)| = 1$. To get a feeling about inner functions we consider a left invariant subspace M spanned by a family of exponentials $\{e^{\lambda_n t} \mid \operatorname{Re}\lambda_n < 0\}$. A function $f \in L^2(0,\infty)$ is orthogonal to $e^{\lambda_n t}$ if and only if $\hat{f}$, its Laplace transform, vanishes at $-\bar{\lambda}_n$. Thus one sees that a function f is orthogonal to M if and only if its Laplace transform vanishes at all points $-\bar{\lambda}_n$ in the right half-plane. Now it turns out that if there are not too many exponentials, in the sense that $-\Sigma(\operatorname{Re}\lambda_n)/(1 + |\lambda_n|^2) < \infty$ there is a canonical inner function vanishing at these points, the function being the Blaschke product

$$B = \prod_n \frac{z + \bar{\lambda}_n}{z - \lambda_n} \frac{|1 - \bar{\lambda}_n^2|}{1 - \bar{\lambda}_n^2}$$

associated with this set of zeros. This circle of ideas is closely associated with the Müntz–Szasz theorem about density of exponentials in various spaces. The Blaschke products do not exhaust all possible inner functions and there are the singular inner functions associated with possible continuous spectrum on the imaginary axis. We omit details and refer to the excellent exposition in [13].

Now if the Fourier transform of $M_T^{\perp}$ is $\phi H^2(\Pi^+)$ it follows that $\bar{\phi}\hat{T}$ is orthogonal to $H^2(\Pi^+)$, and hence $\hat{T}$ is factorable on the imaginary axis in the form $\hat{T}(i\omega) = \phi(i\omega)\overline{h(i\omega)}$ for some $h \in H^2(\Pi^+)$. Since T generates M_T under left translations it follows that ϕ and h are relatively prime, i.e., have no common nontrivial inner factor.

We now describe how the above factorization indicates clearly the singularities of the analytic extension of $\hat{T}$ into the left half-plane. Clearly $\overline{h(i\omega)}$ has an analytic extension H into the left half-plane which actually belongs to $H^2(\Pi^-)$ and is given by $H(s) = \overline{h(-\bar{s})}$. Now $\phi(i\omega)$ has also an extension Φ which is meromorphic in Π^-, the extension being given by $\Phi(s) = \overline{\phi(-\bar{s})^{-1}}$. Thus for points in the left half-plane we have $\hat{T}(s) = \Phi(s)H(s) = [H(s)]/[\overline{\phi-(\bar{s})}]$, which clearly exhibits the meromorphic character of $\hat{T}$ in the left half-plane. It follows from the above discussion that the prime factorizations of $\hat{T}$ on the imaginary axis is but a generalization of representing a rational function as a ratio of two relatively prime polynomials.

To relate the singularities of $\hat{T}$ to the spectrum of the infinitesimal generator of the left translation semigroup when restricted to the left invariant subspace M_T we have to find the relation between the spectrum and the inner function ϕ. If M is spanned by the exponentials $e^{\lambda_n t}$ then the infinitesimal generator has point spectrum given by the set $\{\lambda_n \mid n \geqslant 0\}$ together with continuous spectrum given by the set of all finite accumulation points of the λ_n. Thus it coincides with the set of all points λ_n where $B(-\bar{\lambda}_n) = 0$ where B is the Blaschke product introduced before. In general the result is analogous and is covered by a theorem of Moeller [16]. The spectrum of the infinitesimal generator is the set of all points λ in the left half-plane for which $\phi(-\bar{\lambda}) = 0$ together with the set of all points λ on the imaginary axis where ϕ is not analytically continuable into the left half-plane. Thus in this case the spectrum of the infinitesimal generator coincides exactly with the singularities of the transfer function $\hat{T}$. To summarize we have the following (see also [4]).

Theorem 7: Let $T \in L^2(0,\infty)$ be a noncyclic weighting pattern satisfying the assumptions of Corollary 2.1. Then the restricted translation realization of T constructed in Section II is spectrally minimal.

V. Some Remarks on Finite Input/Finite Output Systems

Most of the results obtained in the previous section can be pushed further to encompass the case of matrix weighting patterns and matrix transfer functions. While some of the results generalize in a straightforward way there are natural complications arising from the noncommutativity involved.

To see in the most direct way how the theory of invariant subspaces enters naturally we consider a weighting pattern $T(t)$ that is $n \times m$ matrix valued with the matrix elements being $L^2(0,\infty)$ functions. For simplicity we will assume that $\hat{T}_{ij} \in H^2(\Pi^+) \cap H^\infty(\Pi^+)$. We define the controllability operator $\mathfrak{C}$ on the set Δ of all $\boldsymbol{C}^m$-vector valued functions whose coordinate functions have compact support and belong to $L^2(0,\infty)$. The controllability operator is defined by $(\mathfrak{C}u)(t) = \int_0^\infty T(t + \sigma)u(\sigma)u\sigma$. By our assumption $\mathfrak{C}$ is well defined and bounded. The closure of the range of $\mathfrak{C}$ which we denote by M_T is a left translation invariant subspace of $L^2(0,\infty;\mathbf{C}^n)$ the set of all $\mathbf{C}^n$ valued functions with $L^2(0,\infty)$ coordinates. Clearly the null space of $\mathfrak{C}$ is a right translation invariant subspace of $L^2(0,\infty;\mathbf{C}^m)$. By taking Fourier transforms we get in a natural way two invariant subspaces in $H^2(\Pi^+;\mathbf{C}^n)$ and $H^2(\Pi^+;\mathbf{C}^m)$, respectively, the invariance being under multiplication by all bounded matrix valued analytic function. As in the scalar case these subspaces have the form $QH^2(\Pi^+;\ \mathbf{C}^n)$ and $Q_1H^2(\Pi^+;\mathbf{C}^n)$, respectively. The functions Q and Q_1 are contractive analytic functions in Π^+ with the boundary values being almost everywhere partial isometries with a fixed initial space. Thus $Q = 0$ corresponds to the cyclic scalar case. If the initial space of the partial isometry is $\mathbf{C}^n$ then almost everywhere $Q(i\omega)$ is unitary and we call such functions inner. The functions T giving rise in this way to inner functions will be called strictly noncyclic. It should be noted that there are many functions which are neither cyclic nor strictly noncyclic. The class of strictly noncyclic functions admits a theory parallel to the one described in the previous section. Most importantly the translation realization in this case turns out to be spectrally minimal. The transfer function $\hat{T}$ admits two relatively prime factorizations on the imaginary axis of the form $\hat{T}(i\omega) = Q(i\omega)H(i\omega)^* = H_1(i\omega)^*Q_1(i\omega)$. In these factorizations Q and Q_1 are matrix inner functions of sizes $n \times n$ and $m \times m$, respectively. These inner functions are closely related. They carry the information about the singularities of $\hat{T}$ in the left half-plane. It follows from these factorizations that $\hat{T}$ admits an analytic extension to Π^+ which is meromorphic and of bounded type there. Thus spectral minimality of the translation realization is related to the meromorphic properties of $\hat{T}$. The two factorizations of $\hat{T}$ are generalizations of the type of numerator denominator type of factorization considered by Rosenbrock [18]. The advantage of our approach, besides being able to handle a large class of nonrational transfer functions, is that such factorizations immediately give rise to state-space realizations by first-order systems. All of the above can be done for the case of discrete systems and we refer the reader to Fuhrmann [8],[9].

References

[1] A. V. Balakrishnan, "System theory and stochastic optimization," NATO Adv. Study Inst. Network and Signal Theory, 1972.
[2] A. V. Balakrishnan, *Introduction to Optimization Theory in a Hilbert Space.* Berlin, Germany: Springer-Verlag, 1971.
[3] J. S. Baras and R. W. Brockett, "H^2 functions and infinite dimensional realization theory," *SIAM J. Contr.*, Feb. 1975. (An earlier version appeared in the *1972 IEEE Decision and Control Conf.*, p. 355.)
[4] J. S. Baras, "On canonical realizations with unbounded infinitesimal generators," in *Proc. 11th Allerton Conf. Circuits and Systems Theory*, p. 1.
[5] R. W. Brockett, *Finite Dimensional Linear Systems.* New York: Wiley, 1970.
[6] N. Dunford and J. Schwartz, *Linear Operators*, vol. 1 and 2. New York: Wiley, 1958.
[7] P. A. Fuhrmann, "On realization of linear systems and applications to some questions of stability," *Math. Syst. Theory*, to be published.
[8] ——, "Realization theory in Hilbert space for a class of transfer functions," to be published.
[9] ——, "On spectral minimality of the shift realization," to be published.
[10] H. O. Fattorini, "On complete controllability of linear systems," *J. Differential Equations*, vol. 3, pp. 391–402, 1967.
[11] J. W. Helton, "Discrete time systems, operator models and scattering theory," to be published.
[12] E. Hille and R. Phillips, *Functional Analysis and Semigroups*, (Colloq. Publ.) 2nd ed. New York: Amer. Math. Soc., 1957.
[13] K. Hoffman, *Banach Spaces of Analytic Functions.* Englewood Cliffs, N.J.: Prentice-Hall, 1962.
[14] P. D. Lax and R. Phillips, *Scattering Theory.* New York: Academic, 1967.
[15] P. D. Lax, "Translation invariant spaces," *Acta Math.*, vol. 101, 1959.
[16] J. W. Moeller, "On the spectra of some translation invariant spaces," *J. Math. Anal. Appl.*, vol. 4, pp. 276–296, 1962.
[17] B. Sz. Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space.* Amsterdam, The Netherlands: North Holland and American Elsevier, 1970.
[18] H. H. Rosenbrock, *State-Space and Multivariable Theory.* New York: Wiley, 1970.
[19] J. H. Ziegler and N. B. Nichols, "Optimum setting for automatic controllers," *Trans. ASME*, vol. 64, pp. 759–768, 1942.
[20] D. V. Widder, *The Laplace Transform.* Princeton, N.J.: Princeton Univ. Press, 1946.

John S. Baras (M'73) was born in Piraeus, Greece, on March 13, 1948. He received the Electrical Engineering Diploma from the National Technical University of Athens, Athens, Greece in 1970, and the M.S. and Ph.D. degrees in applied mathematics from Harvard University, Cambridge, Mass., in 1971 and 1973, respectively.

He is currently an Assistant Professor of Electrical Engineering at the University of Maryland, College Park, where he is involved in graduate teaching and research in control and system theory. His major research interests are the analysis and control of linear and nonlinear distributed parameter systems and applications.

Roger W. Brockett (S'62–M'63–SM'73–F'74) received the Ph.D. degree from Case Western Reserve University, Cleveland, Ohio.

From 1963 to 1969 he taught in the Department of Electrical Engineering at the Massachusetts Institute of Technology. He is presently Gordon McKay Professor of Applied Mathematics in the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass., where he participates in the interdisciplinary Decision and Control Program through teaching and research. He is coeditor (with H. H. Rosenbrock) of the series *Studies in Dynamical Systems* (Camden, N.J.: Nelson) and is author of the book *Finite Dimensional Linear Systems* (New York: Wiley) and coeditor (with D. O. Mayne) of *Geometric Methods in System Theory* (Dordrect, The Netherlands: Reidel), 1973. He has held post-doctoral positions at Warwick University, Imperial College, and the University of Rome, and a Guggenheim Fellowship for study in mathematical system theory.

Dr. Brockett is recipient of the American Automatic Control Council's Donald P. Eckman Award.

Paul A. Fuhrmann received the M.Sc. degree (mathematics) from the Hebrew University, Jerusalem, Israel, in 1963 and the Ph.D. degree in mathematics from Columbia University, New York, N.Y., in 1967.

From 1967 to 1972 he was with the Department of Mathematics at Tel Aviv University, Tel Aviv, Israel. He spent two years (1973 and 1974) as Research Fellow in Applied Mathematics at the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. Currently he is a Senior Lecturer at the Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel. His main interests are in the fields of operator theory and systems theory.