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On The Convergence Rate of Non-Linear Consensus Networks with Delays

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Abstract: We consider a generic non-linear consensus model and prove convergence results to a common value together with prescribed rate of convergence. Instead of a Lyapunov approach we consider a functional metric space and make a fixed point theory argument using contraction mappings. We are restricted to the case of static networks.

1. INTRODUCTION

Collective dynamics of autonomous agents has become over the past decade one of the most important and active research areas. The analysis of networks and it's dynamics has been the center of attention for many communities such as the Control, the Applied Mathematics and Physics or the Computer Science. Network dynamics of autonomous agents who interact exchanging information in order to achieve a common value in distributed way has drawn such an interdisciplinary attention see for example Blondel et al. [2005], Moreau [2004], Cucker and Smale [2007], Matei et al. [2008], Papachristodoulou et al. [2010], Munz et al. [2008], Olfati-Saber and Murray [2004], Motsch and Tadmor [2011] and references therein. These networks are known as agreement or *consensus* networks of agents. The majority of work include linear systems with agents each of which exchanges information about their state with their "neighbours" so that they asymptotically converge to the same value (consensus value). The classic model in continuous time linear dynamics proposed in literature is

$$\dot{x}_i = \sum_{j\sim i} a_{ij}(x_j - x_i) \tag{MDL}$$

where $j \sim i$ stands for agent j adjacent to i. The communication weights $a_{ij} > 0$ model the strength of the effect of j to i. This is a very well known and easily analysable system especially when the communication weights are positive constants. One can use fundamental results from Algebraic Graph Theory to fully analyse it's stability (see for example Mesbahi and Egerstedt [2010]). The main drawback of the system above is that it is over-simplistic in two perspectives. The first is the time invariant linearity of the communication scheme and the second is the synchronous propagation of information among communicating agents.

1.1 Related literature and contribution of this work

Although there are a lot of results in discrete time delayed consensus dynamics, the respective continuous time models lack this privilege. The reason of the progress in discrete time is mainly due to the seminal work of Blondel et al. [2005] and the technique of state space augmentation. This allows for the convergence analysis of the dynamics including the rate of convergence. This technique although prominent in discrete time dynamics, it is unclear how to implement it in continuous time dynamics. To the best of our knowledge the literature in delayed continuous time consensus dynamics is relatively poor. Our interpretation for this is that the Lyapunov methods for such systems are rather difficult. The introduction of delay in the value of x_j in (MDL) (so that $x_j(t)$ is replaced by $x_j(t-\tau)$) imposes a great deal of asymmetry in the dynamics of the system, making the design of a successful Lyapunov candidate function too difficult.

In a number of papers, we acknowledge the difficulty of the Lyapunov method and propose a diiferent approach; this of Fixed Point Theory argument. In Somarakis and Baras [2013b] we discuss a simple LTI consensus scheme with multiple and distributed delays and and in Somarakis and Baras [2013c] we do so with an LTV consensus scheme. The approach is through contraction mappings. We apply the Contraction Mapping Principle in a complete metric space of solutions satisfying certain asymptotic properties (consensus point and convergence rate) and obtain both delay and symmetry-dependent results.

In this work we draw our attention to the work of Papachristodoulou et al. [2010] and their asymptotic results of a non-linear consensus scheme with multiple delays which reads

$$\dot{x}_i = \sum_{j\sim i} A_{ij} f_{ij}(x_j(t-\tau_{ji}) - x_i(t))$$

The authors there made a passivity assumption for $f_{ij}(x)$ and proved delay-independent asymptotic stability of the consensus space using the Invariance Principle with the Lyapunov function $V = \max_i x_i - \min_i x_i$. This is indeed a non-linear version of the papers presented in Somarakis and Baras [2013b,c]. However, the problem with this approach is that it tells us nothing neither about the consensus point nor about the rate of convergence to it. The contribution of this work is an attempt to make

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some progress towards this challenge, by using Fixed Point Theory methods. We carry on the assumptions made in Papachristodoulou et al. [2010], amplify them rather moderately and exploit to their convergence results to create a Fixed Point Argument based on Contraction Mappings. It is remarked that our work focuses exclusively on static topologies only. In the discussion section we will make some comments on how one would adapt the techniques of this work, to switching networks.

1.2 Organization of the paper

The paper is organized as follows: In the Section (2) we introduce the necessary notation and framework of the theories and techniques that will come in hand, in Section (3) we introduce the model as initial value problem and state the assumptions on which our analysis and results rely. In section (4) we conduct the rigorous analysis and conclude with the main result. We close with Section (5) where several important remarks for this and future work are outlined. Due to space limitations a few proofs were omitted or sketched as well as several minor (to our understanding) algebraic steps. For detailed results the reader is kindly referred to Somarakis and Baras [2013a].

2. NOTATION AND DEFINITIONS

2.1 Preliminaries

$\mathbb{R}^N$ is the N dimensional Euclidean space (and N is the number of agents) with the $p = 1$ Euclidean norm $||\cdot||$. By $\mathbf{1}$ we understand the N-dimensional vector of all ones. The subspace of $\mathbb{R}^N$ with the property

$$\Delta := \{\mathbf{z} \in \mathbb{R}^N : \mathbf{1}c, c \in \mathbb{R}\} \tag{1}$$

is called consensus subspace and will play a key role in our analysis. The orthogonal to Δ is denoted by Δ^c. By $L^1_{[a,b]}$ we denote the space of functions that are integrable in $[a, b]$ and $C^m(A)$ is the space of functions taking values in A which have a continuous derivative up to order m. $(\mathbb{B}, |\cdot|)$ is the bracket standing for the Banach space of vector valued continuous, bounded, functions and $|\cdot|$ is the appropriate supremum norm. An important class of functions in this work is defined below:

Definition 1. We call *rate function*, any function h with the following properties

(1) $h(t) : [0, \infty) \to [1, \infty)$
(2) $1/h(t) \in L^1_{[1,\infty)}$

2.2 Algebraic Graph Theory

A weighted graph $\mathcal{G} = (V, E, W)$ consists of a set of vertices $V = \{i\}|_{i=1}^N$ and a set of edges $E \subset \{(i, j) : i, j \in V, i \neq j\}$ each member of which is attributed with a positive weight $a_{ij} \in W$. If $a_{ij} = a_{ji}$ the graph is called symmetric and if for each i there is a path of positive weights that leads to j we call $\mathcal{G}$ a connected graph. The neighbourhood of a vertex i is denoted by N_i and includes the set of vertices j such that $a_{ij} > 0$, in such case we write $j \sim i$. The degree of i is denoted by $d_i = \sum_{j\in N_i} a_{ij}$. The matrix representation of $\mathcal{G}$ that comes at hand in consensus schemes is the weighted Laplacian is due to the (weighted) Laplacian $L = D - A$ where $D = Diag[\sum_{j\in N_i} a_{ij}]$ and $A = [a_{ij}]$ are the degree and incident matrix respectively (for a rigorous introduction to the subject we refer the interested reader is referred to Godsil and Royle [2001]). In case of symmetric directed graphs, L is a positive semi-definite, symmetric matrix with eigenvalue spectrum

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N = ||L||$$

and the orthonormal eigenvector of λ_1 is $\mathbf{1}\frac{1}{N}$.

2.3 Function spaces and Fixed Point Theory

Our approach to the stability problem of our non-linear consensus model is through contraction mappings and fixed point theory. We effectively make use of the most fundamental result of this theory, known as the Contraction Mapping Principle. Due to space limitations we omit the terminology of linear and metric spaces and we simply state the result:

Theorem 2. Let $(\mathbb{S}, \rho)$ be a complete metric space and let $P : \mathbb{S} \to \mathbb{S}$. If there is a constant $\alpha < 1$ such that for each pair $y_1, y_2 \in \mathbb{S}$ we have

$$\rho(Py_1, Py_2) \leq \alpha\rho(y_1, y_2) \tag{2}$$

then there exists a unique $y \in \mathcal{M}$ with $Py = y$

The proof of this theorem can be found in any advanced analysis or ODE textbook (see Royden [1989] or Markley [2004]).

3. THE MODEL

We consider a population of $N < \infty$ autonomous agents exchanging information on each others state. The state of agent $i \in V$ is denoted by x_i and it is a C^1 function of time. The model to be considered in this work is the initial value problem

$$\begin{aligned} \dot{x}_i &= \sum_{j\in N_i} f_{ij}(x_j^\tau - x_i) \quad , \quad t > 0 \\ x_i(t) &= \phi_i(t) \quad , \quad t \in [-\tau, 0] \end{aligned} \tag{IVP}$$

where $x_j^\tau = x_j(t-\tau)$ is the information of the state of j at time $t-\tau$ which agent i receives at time t, and ϕ_i are given functions which play the role of initial data. We restrict our analysis to the case of uniform delay τ for every agent. In the discussion section we will consider the extension to multiple delays.

3.1 Hypotheses

Hypothesis 3. The graph with weights f_{ij}, between the nodes $i, j \in V$ contains a spanning tree.

We will consider the following set of assumptions. For any $j \in N_i$ we assume that:

Hypothesis 4. $f \in C^1(\mathbb{R})$ with $xf_{ij}(x) > 0$, $\forall x \neq 0$, and $\lim_{x\to 0} \frac{f_{ij}(x)}{x} = a_{ij}$

Hypothesis 5. The limiting values a_{ij} constitute a set of graph weights which correspond to a simply connected topological graph.

Hypothesis 6. For any $x(t) \in C^1(\mathbb{R})$ such that: both $|x|$ and $|\dot{x}|$ are uniformly bounded and $x(t) \to 0$ there exists a rate function $h_f(t)$ with the property

$$\sup_{t\geq 0} h_f(t)\left|\frac{f_{ij}(x(t))}{x(t)} - a_{ij}\right| < \infty \tag{3}$$

The essence behind this set of assumptions is clear. Assumption (3) ensures the necessary minimum connectivity conditions for convergence. Assumption (4) takes f_{ij}'s to be passive with a linear part which dominates near the origin. Assumption (6) describes the rate at which the functions reveal their linearity, while Assumption (5) repeats the connectivity status with the simplification that both $\frac{f_{ij}(x)}{x}$ and $\frac{f_{ji}(x)}{x}$ have the same (positive by Assumption (4)) value at the origin.

In the discussion section we will see how some of the assumptions above can (or cannot) be weakened within the theoretical framework of this paper.

3.2 Preliminary Results

We end this section by drawing two very useful results from Papachristodoulou et al. [2010]. The first one has to do with the boundedness of solutions and the second with their asymptotic behaviour.

Lemma 7. For the solutions of (IVP) it holds that $\max_i |x_i| \leq c$ where $c = \max_i |\phi_i|$

That simple, yet useful, lemma states that all solutions are bounded (hence exist for all times) by the initial data.

Theorem 8. The consensus space Δ is asymptotically stable.

3.3 Strategy

Let us now briefly outline the approach to the problem. From the known results stated above we know that any appropriate solution $\mathbf{x}(t)$ of (IVP) converges to Δ. For such fixed $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$ we implement an *exact linearisation* by taking:

$$d_{ij}(t) := \frac{f_{ij}(x_j(t-\tau) - x_i(t))}{x_j(t-\tau) - x_i(t)}$$

so that (IVP) can be written as

$$\begin{aligned}\dot{x}_i(t) &= \sum_{j\in N_i} d_{ij}(t)(x_j - x_i) - \sum_{j\in N_i} d_{ij}(t)\frac{d}{dt}\int_{t-\tau}^{t} x_j(s)ds \\ &= \sum_{j\in N_i} a_{ij}(x_j - x_i) + \sum_{j\in N_i}[d_{ij}(t) - a_{ij}](x_j - x_i) \\ &\quad - \sum_{j\in N_i} d_{ij}(t)\frac{d}{dt}\int_{t-\tau}^{t} x_j(s)ds\end{aligned} \tag{4}$$

Finally we review some fundamental results of LTI consensus dynamics theory (Mesbahi and Egerstedt [2010]): The solution of $\dot{\mathbf{y}} = -L\mathbf{y}$ is $\mathbf{y} = e^{-Lt}\mathbf{y}(0)$ and it converges to $\mathbf{1}\frac{1}{N}\sum_i y_i(0)$ exponentially fast with rate λ_2. The next bound will come at hand

$$\left|\left|e^{-L(t-s)} - \mathbf{1}\mathbf{1}^T\frac{1}{N}\right|\right| \leq \sqrt{N}e^{-\lambda_2(t-s)} \tag{5}$$

4. CONVERGENCE ANALYSIS & RESULT

In this section we will build our fixed point argument in view of Theorem (2) and state our result as a theorem, at the end.

4.1 Preliminaries

Denote by L the weighted Laplacian matrix of the graph $\mathcal{G}_\infty$ with weights a_{ij}, and by $F(t)$ the matrix with elements $d_{ij}(t) - a_{ij}$. Equation (4) in vector form is written as

$$\dot{\mathbf{x}} = -L\mathbf{x} - F(t)\mathbf{x} - D(t)\frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}(s)ds \tag{6}$$

where $D(t)$ is the adjacency matrix of $\mathcal{G}_t$ Using variation of constants we write:

$$\begin{aligned}\mathbf{x}(t) =& e^{-Lt}\phi_0 - \int_0^t e^{-L(t-s)}F(s)\mathbf{x}(s)ds \\ &- \int_0^t e^{-L(t-s)}D(s)\frac{d}{ds}\int_{s-\tau}^{s}\mathbf{x}(w)dwds \\ =& \int_0^t \left(e^{-L(t-s)} - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right)\Big(\phi(s)\delta(s) - F(s)\mathbf{x}(s) - \\ &\quad - D(s)\frac{d}{ds}\int_{s-\tau}^{s}\mathbf{x}(w)dw\Big)ds \\ &+ \mathbf{1}\frac{1}{N}\int_0^t \mathbf{1}^T\Big(\phi(s)\delta(s) - F(s)\mathbf{x}(s) - \\ &- D(s)\frac{d}{ds}\int_{s-\tau}^{s}\mathbf{x}(w)dw\Big)ds \\ =&: \mathbb{F}_{\Delta^c}(\mathbf{x}(t)) + \mathbb{F}_{\Delta}(\mathbf{x}(t))\end{aligned} \tag{7}$$

where $\delta(t)$ is the the delta function and the last step decomposes the dynamics projecting the flow onto the consensus subspace and it's complement.

4.2 The space of functions

Consider the following space of vector valued functions

$$\begin{aligned}\mathbb{M}_{\phi,h} = \{&\mathbf{y} \in C^0([-\tau,\infty),\mathbb{R}^N) : \mathbf{y} = \phi|_[-\tau,0], \\ &\exists! k_{\mathbf{y}} \in \mathbb{R} : \sup_{t\geq-\tau} h(t)||\mathbf{y}(t) - \mathbf{1}k_{\mathbf{y}}|| < \infty\}\end{aligned} \tag{8}$$

In order to implement Theorem (2), we need to create a metric space which, in addition, has to be complete. For $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{M}$ we take

$$\rho_h(\mathbf{y}_1,\mathbf{y}_2) = \sup_{t\geq-\tau} h(t)\left|\left|\left[\mathbf{y}_1(t) - \mathbf{1}k_{\mathbf{y_1}}\right] - \left[\mathbf{y}_2(t) - \mathbf{1}k_{\mathbf{y_2}}\right]\right|\right|$$

this is a weighted metric which measure the difference of two elements' projection on Δ^c . It should be pointed out that for given ϕ, ρ_h is a well defined metric on $\mathbb{M}$.

The Metric Space

Proposition 9. The metric space $(\mathbb{M}, \rho_h)$ is complete.

Proof. [Sketch] The proof follows the definition of complete metric space, i.e. for each Cauchy sequence in $\mathbb{M}$ to have a limit in $\mathbb{M}$. The technique is fairly standard and will be omitted (For full proof see Somarakis and Baras [2013a]).

Let's make some comments on why $\mathbb{M}$ would be an appropriate candidate space to look for solutions of (IVP).

From the preliminary results we know that all solutions of (IVP) are bounded by the initial data. The smoothness of f_{ij} guarantees that $\dot{x}_i$ should be bounded too and thus the first two properties of $\mathbb{M}$ follow. The convergence to a point in Δ is also guaranteed by the preliminary results. There are two extra features in $\mathbb{M}$: The first is the convergence rate which is controlled by $h(t)$ and the second one is that it is asked for the consensus point to be essentially governed by the orbit $\{\mathbf{x}(t), t \geq -\tau\}$. The reason for these supplements is that on the one hand it is exactly the convergence rate the purpose of this work and on the other hand the exact linearisation technique transformed our initial non-linear autonomous system, to a linear non-autonomous one. Then the consensus point should include information from both the varying weights and the "initial states at every moment" (i.e. the whole orbit). Note however that we are free to choose (and will do so) consensus points $k_{\mathbf{x}}$ which satisfy a particular condition.

The operator Having defined our space of functions $(\mathbb{M}, \rho)$ inside which we aim to find a solution of (IVP), we consider now the operator with which we will work. This is defined for $\mathbf{x} \in \mathbb{M}$ and $t \geq -\tau$

$$(P\mathbf{x})(t) = \begin{cases} \phi(t) & -\tau < t < 0 \\ \mathbb{F}_{\Delta^c}(\mathbf{x}(t)) + \mathbb{F}_{\Delta}(\mathbf{x}(t)), & t > 0 \end{cases} \tag{9}$$

where $\mathbb{F}$ are as defined in (7).

The next result describes the sufficient conditions under which Theorem (2) can be applied, effectively proving our result (Theorem (12)).

Proposition 10. The operator P has the following properties:

(1) P is continuous in t
(2) $P : \mathbb{M} \times [\tau, \infty) \to \mathbb{M}$ if

$$\sup_t h(t) e^{-\lambda_2 t} < \infty \quad , \quad \sup_t h(t) \int_t^\infty \frac{ds}{h(s)} < \infty \tag{10}$$

(3) Denote by $\mathbf{LD} := \sup_t ||LD(t)||$, $\dot{\mathbf{D}} = \sup_t ||\frac{\mathbf{1}\mathbf{1}^T}{N} \dot{D}(t)||$ $\mathbf{F} = \sup_t h_f(t) ||F(t)||$, $\mathbf{F_1} = \sup_t h_f(t) ||\frac{\mathbf{1}\mathbf{1}^T}{N} F(t)||$. The operator P is a contraction in $\mathbb{M}$ if there exists $\alpha \in [0, 1)$ such that

$$\sup_t \sum_{i=2}^N h(t) e^{-\lambda_i t} \int_0^t \frac{e^{\lambda_i s}}{h_f(s)} \int_{s-\tau}^s \frac{dw}{h(w)} ds (\mathbf{F} + \mathbf{LD}) + \\ + \sup_t h(t) \int_t^\infty \frac{1}{h_f(s)} \int_{s-\tau}^s \frac{dw}{h(w)} ds (\mathbf{F_1} + \dot{\mathbf{D}}) \leq \alpha \tag{11}$$

Before proving Proposition (10) we need the following lemma:

Lemma 11. If $\mathbf{x} \in \mathbb{M}$ and

$$\tau \frac{\mathbf{1}^T \big(D(\infty) - D(0)\big) \mathbf{1}}{N + \tau \mathbf{1}^T D(0) \mathbf{1}} < \alpha \tag{12}$$

for some $\alpha \in [0, 1)$ then there exists a unique solution k of the equation

$$\begin{aligned} k = {} & \frac{\mathbf{1}^T}{N} \int_0^\infty \Big(\phi(s)\delta(s) - F(s)\mathbf{x}(s) - \\ & - D(s) \frac{d}{ds} \int_{s-\tau}^s (\mathbf{x}(w) - \mathbf{1}k) dw \Big) ds \\ = {} & \frac{\mathbf{1}^T}{N} D(0) \int_{-\tau}^0 (\phi(w) - \mathbf{1}k) dw + \mathbf{1}^T \frac{\sum_i \phi_i(0)}{N} \\ & + \int_0^\infty \frac{\mathbf{1}^T}{N} F(s)\mathbf{x}(s) ds \\ & + \frac{\mathbf{1}^T}{N} \int_0^\infty \dot{D}(s) \int_{s-\tau}^s (\mathbf{x}(w) - \mathbf{1}k) dw \Big) ds \end{aligned} \tag{13}$$

Proof. [sketch] Given $\mathbf{x} \in \mathbb{M}$ one can easily prove that the operator

$$\mathbb{Q}_{\mathbf{x}}(k) = \frac{\mathbf{1}^T \int_0^\infty \dot{D}(s) \int_{s-\tau}^s (\mathbf{x}(w) - \mathbf{1}k) dw ds}{N + \tau \mathbf{1}^T D(0) \mathbf{1}}$$

defined on $(\mathbb{R}, |\cdot|)$ is a contraction under condition (12) so that $\mathbb{Q}(k) = k$ for some unique k. The operator $\mathbb{Q}$ is equivalent to (13) on $\mathbb{M}$ (integration by parts).

This lemma characterizes the condition of consensus points needed to analyse (IVP) using Fixed Point Theory and we are now ready to begin with the proof of Proposition (10).

Proof. [of Prop. (10)] By definition P is both continuous in t and equal to ϕ in $[-\tau, 0]$. To show that P maps $\mathbb{M}$ onto $\mathbb{M}$ for $t > 0$ we evidently need to prove that $\sup_t h(t) |(P\mathbf{x})(t) - \mathbf{1}k_{P\mathbf{x}}| < \infty$ for some unique $k_{P\mathbf{x}} \in \mathbb{R}$.

For $\mathbf{x} \in \mathbb{M}$, as $t \to \infty$:

$$F_{\Delta^c}(\mathbf{x}(t)) \to 0$$

This is due to the fact $e^{-Lt} - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ is $L^1_{[0,\infty)}$ (by Hypothesis 4, with rate λ_2) and $\phi(t)\delta(t) - F(t)\mathbf{x}(t) - D(s)\frac{d}{ds}\int_{t-\tau}^t \mathbf{x}(w)dw$ as a function of time tends to zero in the t-limit. ($F(t) \to 0$ and $\frac{d}{dt}\int_{t-\tau}^t \mathbf{x}(w)dw \to 0$ as $\mathbf{x} \in \mathbb{M}$). Then $\mathbb{F}_{\Delta^c}(t)$ is a convolution of an L^1 function with a function that goes to zero. So we are left with the dynamics on Δ which is essentially the integral $\int_0^t \mathbf{1}^T D(s) \frac{d}{ds} \int_{s-\tau}^s \mathbf{x}(w) dw ds$ which converges to Δ since the integrand is the product of an L^1 function with a function that vanishes. So let $\mathbf{1}k_{P\mathbf{x}}$ denote the limit of $(P\mathbf{x})(t)$. Lemma (11) tells that under assumption (13) $k_{P\mathbf{x}} = k_{\mathbf{x}}$, since $k_{\mathbf{x}}$ satisfies (13). This is a crucial point to prove rate of convergence of $(P\mathbf{x})(t)$. Indeed for $t > 0$

$$\begin{aligned} ||(P\mathbf{x})(t) - \mathbf{1}k_{P\mathbf{x}}|| &= ||(P\mathbf{x})(t) - \mathbf{1}k_{\mathbf{x}}|| \\ &\leq ||\mathbb{F}_{\Delta^c}(\mathbf{x}(t))|| + \\ &+ \left|\left| \mathbf{1}\frac{1}{N} \int_t^\infty \mathbf{1}^T D(s) \frac{d}{ds} \int_{s-\tau}^s \mathbf{x}(w) dw ds \right|\right| \end{aligned}$$

the first term is bounded on condition that

$$\sup_t h(t) e^{-\lambda_2 t} < \infty$$

the rate of the second term is handled as follows: Note that for $i \sim j$

$$\int_t^\infty d_{ij}(s)(x_j(s)-x_j(s-\tau))ds =$$
$$= \int_t^\infty (d_{ij}(s)-a_{ij})(x_j(s))-x_j(s-\tau))ds+$$
$$+ \int_t^\infty a_{ij}(x_j(s))-x_j(s-\tau))ds$$

the first term is bounded if $\sup_t h(t)\int_t^\infty \frac{ds}{h_f(s)h(s)} < \infty$ and the second term if $\sup_t h(t)\int_t^\infty \frac{ds}{h(s)} < \infty$. Both conditions merge to the latter one and we all in all require that

$$\sup_t h(t)\int_t^\infty \frac{ds}{h(s)} < \infty$$

The last step is to show that P is a contraction. (This it obviously true in $[-\tau, 0]$). Take $t > 0$, and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}$. From Hypotheses (4), (6) one can easily observe that for any fixed solution, $D(t) \to D(\infty)$ and $\dot{D}(t)$ is a bounded function of time which vanishes as $t \to \infty$.

For simplicity in the notation take: $\mathbf{x}_{12}(t) := [\mathbf{x}_1(s) - \mathbf{1}k_{\mathbf{x}_1}] - [\mathbf{x}_2(s) - \mathbf{1}k_{\mathbf{x}_2}]$. Recall the first form of (7) and the second form of (13). Then

$$\left|\left|\left[(P\mathbf{x}_1)(t) - \mathbf{1}k_{P\mathbf{x}_1}\right] - \left[(P\mathbf{x}_2)(t) - \mathbf{1}k_{P\mathbf{x}_2}\right]\right|\right| \leq$$
$$\leq \int_0^t \left|\left|e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right|\right| \cdot ||F(s)|| \cdot ||\mathbf{x}_{12}(s)||ds+$$
$$+ \int_t^\infty \left|\left|\frac{\mathbf{1}\mathbf{1}^T}{N}F(s)\right|\right| \cdot ||\mathbf{x}_{12}(s)||ds$$
$$+ \int_0^t \left|\left|e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right|\right| ||LD(s)|| \int_{s-\tau}^s ||\mathbf{x}_{12}(w)||dwds$$
$$+ \int_t^\infty ||\frac{\mathbf{1}\mathbf{1}^T}{N}\dot{D}(s)|| \int_{s-\tau}^s ||\mathbf{x}_{12}(w)||dwds$$
$$= T_1 + T_2 + T_3 + T_4$$

The therms T_i are bounded as follows (recall the bold letter definitions):

$$\begin{aligned}
T_1 &\leq \mathbf{F}\sqrt{N}e^{-\lambda_2 t}\int_0^t \frac{e^{\lambda_2 s}}{h_f(s)h(s)}ds\rho_h(\mathbf{x}_1, \mathbf{x}_2)\\
T_2 &\leq \mathbf{F_1}\int_t^\infty \frac{ds}{h_f(s)h(s)}\rho_h(\mathbf{x}_1, \mathbf{x}_2)\\
T_3 &\leq \mathbf{LD}\sqrt{N}e^{-\lambda_2 t}\int_0^t \frac{e^{\lambda_2 s}}{h_f(s)}\int_{s-\tau}^s \frac{dw}{h(w)}ds\rho_h(\mathbf{x}_1, \mathbf{x}_2)\\
T_4 &\leq \dot{\mathbf{D}}\int_t^\infty \frac{1}{h_f(s)}\int_{s-\tau}^s \frac{dw}{h(w)}ds\rho_h(\mathbf{x}_1, \mathbf{x}_2)
\end{aligned} \tag{14}$$

In view of the definition of metric:

$$\rho_h(P\mathbf{x}_1, P\mathbf{x}_2) = \sup_t h(t)\left|\left|\left[(P\mathbf{x}_1)(t) - \mathbf{1}k_{P\mathbf{x}_1}\right] - \left[(P\mathbf{x}_2)(t) - \mathbf{1}k_{P\mathbf{x}_2}\right]\right|\right|$$

the operator P is a contraction under condition (11).

To sum up:

Theorem 12. Consider the initial value problem (IVP). Under Hypotheses (3) (4), (5),(6) and conditions (12),(10), (11) the solutions of the system converge to the consensus subspace with rate $1/h(t)$.

4.3 Exponential Convergence

The authors acknowledge that the conditions (12),(10), (11) are either hard to be verified or very restrictive. In case a little more is known about the rate h_f is exponential say $h_f(t) = e^{\gamma t}$ then there is no reason why one should not consider $h(t) = e^{\beta t}$ as well (for $\beta, \gamma > 0$). Then we can state without proof the corresponding condition as a corollary:

Corollary 13. The convergence conditions (10) and (11) reads

$$\lambda_2 > \gamma + \beta$$
$$\sqrt{N}\frac{e^{\beta\tau} - 1}{\beta}\frac{(\mathbf{F} + \mathbf{LD})}{\lambda_2 - \gamma - \beta} + \frac{e^{\beta\tau} - 1}{\beta(\beta + \gamma)}(\mathbf{F_1} + \dot{\mathbf{D}}) \leq \alpha \tag{15}$$

for some $\alpha \in [0, 1)$.

5. DISCUSSION

We conclude this paper with some important remarks, extensions and challenges for future work. The key role to this work is the assumptions and how restrictive they are. Fixed Point Theory does not require one to look for a global energy function that takes care of the asymmetry of the dynamical system. The price one pays is much more work in the analysis and clearly more conservative assumptions.

5.1 Assumptions and Symmetry

These assumptions however reflect exactly this lack of both symmetry and linearity to our model. More specifically, one can readily observe the constants $\mathbf{LD}, \dot{\mathbf{D}}, \mathbf{F}, \mathbf{F}_1$ and understand their role in the strengthening of the assumptions. The more assymetric is the system the harder the analysis becomes but, more importantly, the larger this constants get. It should also be noted that these results are far from global, all the constants mentioned above heavily rely or c (the magnitude of the initial conditions).

On the other hand, the advantage of this approach is that it reveals a great deal of the system's aspects and hence gives the designer the ability to implement elaborate control techniques. For example one can ask for rate bounds given the necessary delay and vice versa. Another example could be, given initial condition, how should the rate function of the non-linear part, behave so that consensus is achieved with preassigned rate and delay. Let's now talk a bit more about the sources of assymetry that were taken care by the Hypotheses:

Assymetric Limiting Weights We assumed in Hypothesis (5) that the asymptotic linear dynamics are on a symmetric graph. This can be dropped by assuming a directed graph which should contain a spanning tree. Then the L seizes to be symmetric with real spectrum. The first eigenvalue is still zero and the second has positive real part. Also the consensus matrix in the symmetric case $\frac{1}{N}\mathbf{1}\mathbf{1}^T$ is replaced by $\mathbf{1}\mathbf{c}^T$ where $\mathbf{c}$ is the left eigenvector of L. The main difficulty occurs in proving the contraction property. There the bound (5) does not hold and one needs to deal with generalized eigenspaces.

Multiple Delays It was our intention to stay within a uniform delay assumption, so as to have a rigorous and tractable analysis within the proceedings' page number limitations. One, however, may consider either agent based delays τ^i or even connection based τ^i_j (see Somarakis and Baras [2013b]). This will worsen all the bold constants as one needs to consider sums of the form $\sum_{i=1}^{N} \sum_{j \in N_i} A_i^j \int_{t-\tau_j^i}^{t} \mathbf{x}(s) ds$ where A_i^j is a sparse matrix with only one positive element(at the (i, j) position).

5.2 *Fully non-linear functions*

The fact that f_{ij} are taken to be passive does not mean, of course, that they ought to have a linear part. It must be noted that if there is no linear part, contraction mappings cannot be used. One needs to use other Fixed Point Theorems, such as the Schauder-Tichonoff Theorem, which is based on compactness, rather than completeness. The price to pay for this is that such theorem is much harder to implement, whereas it does not guarantee uniqueness.

5.3 *Switching Networks*

We did not include the case of switching network topology and we leave it for future work. Such an approach should include a family of metric spaces, each one attributed to a state of the switching signal. The sufficient assumption for convergence is the idea of recurrent connectivity [Papachristodoulou et al., 2010] and we would furthermore require certain smoothness conditions on the switching signal so that the operators are well defined and continuous in each one of the metric spaces.

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