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Observing a Linear Process over Analog Erasure Channels using Multiple Sensors: Necessary and Sufficient Conditions for Mean-square Stability

Vijay Gupta, Nuno C. Martins and John S. Baras

***Abstract*— Consider a discrete-time linear time-invariant process being observed by two sensors, which are connected to an observer via links that can be modeled as erasure channels. If a link transmits successfully then a finite-dimensional vector of real numbers is conveyed from the sensor to the controller. If an erasure event occurs, then any information conveyed over the link is lost. This paper addresses the problem of designing the maps that specify the processing at the sensors and the observer to calculate the minimum mean square error estimate of the state of the process. We derive necessary and sufficient conditions for the existence of maps such that the estimate error is bounded in the second moment sense.**

I. Introduction

Recently a lot of attention has been directed towards processes being observed or controlled across wireless links or communication networks that may also be used for transmitting other unrelated data (see, e.g., [1], [10] and the references therein). The estimation and control performance in such systems is severely affected by the properties of the communication channels.

In this work, we are specifically interested in the problem of estimation and control across communication links that exhibit data loss. Preliminary work in this area has largely concentrated on the case when only one sensor is present. A good overview of works dealing with the effect of the erasure links on estimation and control as well as designing a compensator at the observer end to estimate the data when the link drops packets has been provided in [10], [6], [12]. This paper takes a more general view by allowing the sensors to encode or pre-process information prior to transmission and by utilizing the fact that a typical network / communication data packet allows transmission of extra data apart from that required inside a traditional control loop. As has been shown in [7], [6], this approach can yield significant improvements in terms of stability and performance. Moreover, for a given performance level, it can also lead to a reduced amount of communication.

In this work, we extend the principle to the case when multiple sensors are present. Suppose a process is observed using two sensors that transmit the data over packet-erasure links to an observer that is interested in the minimum mean square error (MMSE) estimate of the process. If the sensors can share their measurements, there is effectively only one sensor. We look at the case when cooperation between the sensors is either not permitted, or occurs over an erasure link. We solve for the conditions on the links and the dynamics of the process that allow for the estimation error covariance to be bounded. The conditions are shown to be necessary and sufficient. We then extend this result to more than two sensors being present. As shown in [8], the results also have relevance for the control of a plant using multiple sensors over packet erasure links.

The problem involving the presence of multiple sensors transmitting data in an aperiodic fashion is much more complicated than the problem involving only a single sensor. The problem of finding optimal encoding algorithms for the multi-sensor case and analyzing their performance is similar to the problems of fusion of data from multiple sensors and track-to-track fusion that have long been open. Many approaches have been suggested for this problem, some representative examples being [9], [3], [13], [11]. However, these approaches assume a fixed communication topology among the nodes with a link, if present, being perfect. In our case, information is erased randomly by the communication channels. This random loss of information reintroduces the problem of correlation between the estimation errors of various nodes [2] and renders the approaches proposed in the literature sub-optimal. In particular, it is known [4] that techniques based on combining state estimates based on each sensor's own local measurements are not optimal. There are special cases for which the solution is known, e.g., when the process noise is absent [14] or when one of the sensors transmits data over a channel that does not erase information [7].

The paper is organized as follows. We begin in the next section by describing the problem set-up and a summary of the stabilizability results for the case when two sensors transmit data over erasure channels. We then prove the necessity of these conditions in Section III-A. Then, in Section III-B, we prove that the conditions are sufficient as well, by presenting a sub-optimal algorithm that stabilizes the system. Section IV presents some generalizations.

II. Problem Formulation

Consider the set-up of Fig 1. Let the process be described by a discrete-time state-space representation of the type:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{w}(k), \ k \geq 1 \tag{1}$$

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The authors are with the Institute for Systems Research, University of Maryland College Park, College Park, MD 20742, USA. (vijay@cds.caltech.edu, nmartins@isr.umd.edu, baras@isr.umd.edu)

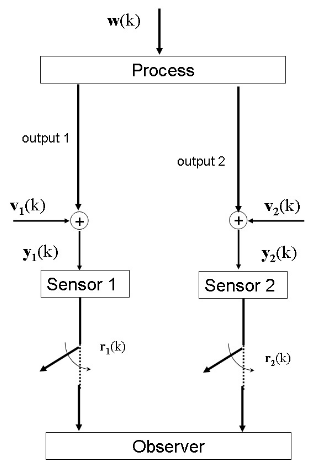

Fig. 1. Basic framework for estimation using two remote sensors, in the presence of erasure channels. The process and measurement noises are represented by $\mathbf{w}(k)$ and $\mathbf{v}_{1 \text{ or } 2}(k)$, respectively. Erasures in the links between the sensors and the observer are governed by $\mathbf{r}_{1 \text{ or } 2}(k)$.

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the process state and $\mathbf{w}(k)$ is the process noise assumed to be white, Gaussian, zero mean with covariance $\Sigma_w > 0$. The initial state $\mathbf{x}(0)$ is a zero mean Gaussian random variable with covariance matrix Σ_0. The process state is observed using two sensors that generate measurements of the form

$$\mathbf{y}_1(k) = C_1\mathbf{x}(k) + \mathbf{v}_1(k), \ k \geq 0 \quad (2)$$
$$\mathbf{y}_2(k) = C_2\mathbf{x}(k) + \mathbf{v}_2(k), \ k \geq 0 \quad (3)$$

where $\mathbf{y}_1(k) \in \mathbb{R}^{m_1}$ and $\mathbf{y}_2(k) \in \mathbb{R}^{m_2}$. The measurement noises $\mathbf{v}_1(k)$ and $\mathbf{v}_2(k)$ are also assumed to be white, Gaussian, zero mean with positive definite covariance matrices $\Sigma_{v,1}$ and $\Sigma_{v,2}$ respectively. Throughout this work, we adopt the following assumption:

Assumption 1: For simplicity, we assume that the pairs (A, C_1) and (A, C_2) are not observable. In addition, we assume that the overall system is observable, i. e., that (A, C) is observable, where $C^T = \left[\begin{array}{cc} C_1^T & C_2^T \end{array} \right]$.

Assumption 1 corresponds to the more difficult scenario where the controller might have to combine the information gathered from $\mathbf{y}_1$ and $\mathbf{y}_2$. Later we show that the stability analysis, for the case where (A, C_1) and (or) (A, C_2) are observable, constitutes a particular case of our analysis. Thus Assumption 1 comes at no loss of generality.

Definition 2.1: (**Erasure Link Model**) Consider that $\{\mathbf{r}_1(k)\}_{k=0}^{\infty}$ and $\{\mathbf{r}_2(k)\}_{k=0}^{\infty}$ represent Bernoulli stochastic processes taking values in the set $\{\mathbf{1}, \emptyset\}$ and characterized by a probability mass function of the following type:

$$p_{i,j} \stackrel{def}{=} Pr\left(\mathbf{r}(k) = (i,j)\right), \ (i,j) \in \{\mathbf{1}, \mathbf{0}\}^2$$

where $\mathbf{r}(k) \stackrel{def}{=} (\mathbf{r}_1(k), \mathbf{r}_2(k))$. The process $\mathbf{r}(k)$ governs the state of the links that connect the sensors to the observer. More specifically, the relationship between sensor i's output $\mathbf{s}_i(k)$ and the observer's input $\mathbf{z}_i(k)$ is described by:

$$\mathbf{z}_i(k) = \begin{cases} \emptyset & \text{if } \mathbf{r}_i(k) = \mathbf{0} \\ \mathbf{s}_i(k) & \text{if } \mathbf{r}_i(k) = \mathbf{1} \end{cases}, \ i \in \{1, 2\} \quad (4)$$

where we adopt the symbol $\emptyset$ to represent erasure, i.e., it indicates that the information sent from sensor i to the observer was lost.

Note that, in general, we do *not* assume that the erasure events in the channels are uncorrelated. However, we presuppose that the sources of randomness $\mathbf{x}(0)$, $\{\mathbf{r}(k)\}_{k=0}^{\infty}$, $\{\mathbf{v}(k)\}_{k=0}^{\infty}$ and $\{\mathbf{w}(k)\}_{k=0}^{\infty}$ are mutually independent.

The sensors are described by a functional structure $\mathbb{S}_q$. At every time step k, sensor i calculates and transmits a vector $\mathbf{s}_i(k)$, as

$$\mathbf{s}_i(k) = \begin{cases} \mathcal{S}\left(i, k, \mathbf{y}_i(0), \ldots, \mathbf{y}_i(k)\right) & k \geq 1 \\ \mathcal{S}\left(i, 0, \mathbf{y}_i(0)\right) & k = 0 \end{cases} \quad (5)$$

where i is in the set $\{1, 2\}$ and $\mathbf{s}_i(k)$ takes values in $\mathbb{R}^q$. In the sequel, we will also refer to the sensor maps as encoding algorithms or information processing algorithms and to the sensors as encoders.

Definition 2.2: (**Observer class**) Consider stochastic processes $\mathbf{z}_1(k)$ and $\mathbf{z}_2(k)$ taking values in $\mathbb{R}^q \bigcup \{\mathbf{1}, \emptyset\}$. We define the observer class $\mathbb{K}$ as the set of all observers with the following structure:

$$\hat{\mathbf{x}}(k) = \mathcal{K}(k, \mathbf{z}_1(0), \mathbf{z}_2(0), \ldots, \mathbf{z}_1(k-1), \mathbf{z}_2(k-1)) \quad (6)$$

where $\hat{\mathbf{x}}(k)$ denotes the estimate of the process state $x(k)$. Given the description of the plant and the erasure link statistics, specified by the probability mass function $p_{i,j}$, we want to investigate conditions for the existence of sensor maps and an observer that estimate the process state in the following sense.

Definition 2.3: (**Stability criterion**) Consider the set-up of Figure 1 and assume that the matrices A, B, C_1, C_2 and the erasure link statistics $p_{i,j}$ are given. A selection of observer $\mathcal{K}$, integer q and sensor maps $\mathcal{S}_1$ and $\mathcal{S}_2$, in the set $\mathbb{S}_q$, is stabilizing if and only if the following holds:

$$\sup_{k \geq 0} E_{\sigma(k), \mathbf{x}(0)} \left[(\hat{\mathbf{x}}(k) - \mathbf{x}(k))' \left(\hat{\mathbf{x}}(k) - \mathbf{x}(k)\right) \right] < \infty, \quad (7)$$

where $\mathbf{x}(k)$ is the state of the plant and $\sigma(k) \stackrel{def}{=} \{\mathbf{r}(i), \mathbf{v}(i), \mathbf{w}(i)\}_{i=0}^{i=k}$ is used to indicate that the expectation is taken with respect to all independent sources of randomness. Thus, the error covariance for the mmse estimate of the process state is bounded at all times.

We now summarize the stability results for the problem formulated above. The proofs will be presented later in the paper. We will rely on the following result that can be proven by applying the canonical structure theorem.

Proposition 2.1: Consider an n dimensional linear and time-invariant system satisfying Assumption 1 and let $\mathbf{y}_1(k)$ and $\mathbf{y}_2(k)$, taking values in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, constitute a bi-partition of the system's output. We can always construct a state-space representation with the structure (1)-(3), where the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ are written in one and only one of the following forms, which we refer to as **type I** and **type II**. The first possibility, denoted as **type I**, is given by:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ \mathbf{0}^{n_2\times n_1} & A_{2,2} \end{bmatrix} \quad (8)$$

$$C_1 = \begin{bmatrix} \mathbf{0}^{m_1\times n_1} & C_{1,2} \end{bmatrix} \qquad C_2 = \begin{bmatrix} C_{2,1} & \mathbf{0}^{m_2\times n_2} \end{bmatrix} \quad (9)$$

where $A_{i,i} \in \mathbb{R}^{n_i\times n_i}$, $C_{i,j} \in \mathbb{R}^{m_i\times n_i}$ and $n_1 + n_2 = n$. The following is the second possibility (**type II**) :

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ \mathbf{0}^{n_2\times n_1} & A_{2,2} & A_{2,3} \\ \mathbf{0}^{n_3\times n_1} & \mathbf{0}^{n_3\times n_2} & A_{3,3} \end{bmatrix} \quad (10)$$

$$C_1 = \begin{bmatrix} \mathbf{0}^{m_1\times n_1} & C_{1,2} & C_{1,3} \end{bmatrix} \quad (11)$$

$$C_2 = \begin{bmatrix} C_{2,1} & \mathbf{0}^{m_2\times n_2} & C_{2,3} \end{bmatrix} \quad (12)$$

where $A_{i,i} \in \mathbb{R}^{n_i\times n_i}$, $C_{i,j} \in \mathbb{R}^{m_i\times n_i}$ and $n_1+n_2+n_3 = n$.

Remark 2.1: In the above representations (of types I or II), $A_{1,1}$ describes the dynamics of the state subspace that is not observable from $\mathbf{y}_1(k)$, while the modes that are not observable by $\mathbf{y}_2(k)$ follow the dynamics of $A_{2,2}$. If the representation is of type II, then $A_{3,3}$ specifies the dynamics of the modes that are observable by both $\mathbf{y}_1(k)$ and $\mathbf{y}_2(k)$.

Using this result, we can state the necessary conditions for stabilizability of the system as follows. The proof is provided in Section III-A.

Theorem 2.2: (**Necessary Conditions for Stabilizability**) Consider the problem set-up of Fig 1. In addition, assume that the plant satisfies Assumption 1 and that the statistics of the erasure links are specified by a given probability mass function $Pr(\mathbf{r}(k) = (i, j))$, with $(i, j) \in \{\mathbf{1}, \emptyset\}^2$ that is independent of the time index k. If the state-space representation can be written as in (8)-(9) (type I) then there exists an observer in the class $\mathbb{K}$, a positive integer q and sensors in the class $\mathbb{S}_q$ such that the closed loop system is stable only if the following inequalities hold:

$$\varrho(A_{1,1})^2 Pr(\mathbf{r}_2(k) = \emptyset) \quad < \quad 1 \quad (13)$$

$$\varrho(A_{2,2})^2 Pr(\mathbf{r}_1(k) = \emptyset) \quad < \quad 1, \quad (14)$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If, instead, the state-space representation is of type II, i. e. of the form (10)-(12), then necessary conditions for stabilization also include the following inequality:

$$\varrho(A_{3,3})^2 Pr\left(\mathbf{r}(k) = (\emptyset, \emptyset)\right) < 1. \quad (15)$$

Remark 2.2: The case when Assumption 1 does not hold and the system is observable using only one sensor has already been considered in the literature [7]. Our results can be applied to this case if we adopt the convention that the spectral radius of an empty matrix is 0. Thus, e.g., if the entire state is observable from $\mathbf{y}_1(k)$, then the spectral radius of $A_{1,1}$ is assumed to be 0. A similar statement can be made about the sufficiency conditions given below as well. Thus we will assume that Assumption 1 holds in our analysis from now on.

It turns out that the above conditions are also sufficient for stabilizability for sensors in the class $\mathbb{S}_q$. We have the following result that will be proven in Section III-B.

Theorem 2.3: (**Sufficient conditions for stabilizability**) Consider the problem set-up of Figure 1. In addition, assume that the plant is observable and that it satisfies Assumption 1. In addition, let the statistics of the erasure link, given by the probability mass function $Pr(\mathbf{r}(k) = (i, j))$, $(i, j) \in \{\mathbf{1}, \emptyset\}^2$, be given. If the state space representation can be written as in (8)-(9) (type I), then there exists an observer of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q$ such that the feedback system is stable, if the following two inequalities hold:

$$\varrho(A_{1,1})^2 Pr(\mathbf{r}_2(k) = \emptyset) \quad < \quad 1 \quad (16)$$

$$\varrho(A_{2,2})^2 Pr(\mathbf{r}_1(k) = \emptyset) \quad < \quad 1 \quad (17)$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II, i.e. it is of the form (10)-(12), then stability is assured by requiring that the following additional inequality also holds:

$$\varrho(A_{3,3})^2 Pr\left(\mathbf{r}(k) = (\emptyset, \emptyset)\right) < 1. \quad (18)$$

Remark 2.3: We have not assumed that the sensors at time step k have access to any acknowledgements from the observer about the vectors $s_i(j)$ transmitted at any time step $j < k$. As proved in [8], the conditions given in Theorems 2.2 and 2.3 remain necessary and sufficient for stability even if such access were allowed. In this sense, the conditions are the least conservative.

Remark 2.4: The stabilizability conditions make intuitive sense. The quantity $\varrho(A_{1,1})^2$ measures the rate of increase of the second moment of the modes that are observable using only sensor 2. To keep the estimate error covariance of these modes bounded, we need the information from sensor 2 to arrive at a large enough rate. Equation (13) formalizes this relation. Inequalities (14) and (15) can be similarly interpreted.

III. PROOFS OF THEOREMS 2.2 AND 2.3

At any time k, define the time-stamp corresponding to sensor i as $t_i(k) = \max\{j \mid j \le k-1, \quad r_i(j) = \mathbf{1}\}$. Thus the time-stamp denotes the latest time at which transmission was possible from sensor i. Using the time-stamp, define the maximal information set $\mathcal{I}_i^{\max}(k)$ for each sensor as

$$\mathcal{I}_i^{\max}(k) = \{y_i(0), y_i(1), \cdots, y_i(t_i(k))\}.$$

The maximal information set is the largest set of measurements from sensor i that the controller can possibly have access to at time k. For any encoding algorithm $\mathcal{A}$ followed by the sensors, we will also define the information set corresponding to sensor i at time k as

$$\mathcal{I}_i^{\mathcal{A}}(k) = \{\mathbf{z}_i(0), \ldots, \mathbf{z}_i(k-1)\},$$

where $\mathbf{z}_i(m)$ is the output of the communication link corresponding to the sensor i at time m, when the algorithm

$\mathcal{A}$ is followed. For any encoding strategy,

$$\mathbb{I}_i^{\mathcal{A}}(k) \subseteq \mathbb{I}_i^{\max}(k),$$

where $\mathbb{I}_i^{\mathcal{A}}(k)$ is the smallest sigma algebra generated by $\mathcal{I}_i^{\mathcal{A}}(k)$. Consider two encoding algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$ that guarantee at every time step

$$\mathbb{I}_1^{\mathcal{A}_1}(k) \subseteq \mathbb{I}_1^{\mathcal{A}_2}(k), \qquad \mathbb{I}_2^{\mathcal{A}_1}(k) \subseteq \mathbb{I}_2^{\mathcal{A}_2}(k).$$

With the optimal mmse estimator for the two algorithms, it is obvious that a necessary condition for the algorithm $\mathcal{A}_1$ to be stabilizable is that the algorithm $\mathcal{A}_2$ is stabilizable. Now consider an algorithm $\bar{\mathcal{A}}$ under which, at every time step k the encoder for sensor i transmits the set

$$S_i(k) = \{y_i(0), y_i(1), \cdots, y_i(k)\}.$$

Note that the algorithm $\bar{\mathcal{A}}$ does not specify valid sensor maps $\mathbb{S}_q$ since the dimension of the transmitted vectors cannot be bounded by any constant q. However, if algorithm $\bar{\mathcal{A}}$ is followed, at any time step k, the decoder (and the controller) would have access to the maximal information sets $\mathcal{I}_1^{\max}(k)$ and $\mathcal{I}_2^{\max}(k)$. This implies that for any other encoding algorithm $\mathcal{A}$, a necessary condition for the system to be stabilizable is that the system be stabilizable when the information sets $\mathcal{I}_i^{\max}(k)$'s are available to the observer, that calculates the mmse optimal estimate.

A. Necessary Conditions for Stabilizability

We shall need the following result that can be proved along the lines of Theorem 4 in [5].

Proposition 3.1: Consider the system in equation (1) being observed by a sensor of the form

$$\bar{y}(k) = \bar{C}\mathbf{x}(k) + \bar{v}(k),$$

where $\bar{v}(k)$ is white Gaussian noise with zero mean and covariance R. Let $f(X)$ denote the Ricatti recursion corresponding to this sensor as applied on the matrix X, thus,

$$f(X) = AXA^T + \Sigma_w - AX\bar{C}^T\left(\bar{C}X\bar{C}^T + R\right)^{-1}\bar{C}XA^T. \tag{19}$$

Further, let $f^m(X)$ denote the above Ricatti recursion applied m times on the matrix X. Finally, let p be a scalar. Then, the sum

$$S = X + pf(X) + p^2f^2(X) + p^3f^3(X) + \cdots + p^mf^m(X), \tag{20}$$

is bounded as $m \to \infty$ if and only if $p \mid \varrho(\bar{A}) \mid^2 < 1$, where $\varrho(\bar{A})$ is the spectral radius of the state subspace that is unobservable from $\bar{y}(k)$. In particular, if the matrix $\bar{C} = 0$, so that the Ricatti recursion (19) corresponds to the Lyapunov recursion $f(X) = AXA^T + \Sigma_w$, then the sum (20) converges if and only if $p \mid \varrho(A) \mid^2 < 1$, where $\varrho(A)$ is the spectral radius of matrix A.

Proof of Theorem 2.2: Necessary conditions for stabilizability using algorithm $\bar{\mathcal{A}}$ will yield necessary conditions for stabilizability using any other algorithm in the class $\mathbb{S}_q$. For ease of notation, we define the Ricatti operators $f_1(.)$, $f_2(.)$ and $f_\emptyset(.)$ in a fashion similar to equation (19) when sensor 1, sensor 2 and no sensor is used, respectively. We also define $f_1^m(.)$, $f_2^m(.)$ and $f_\emptyset^m(.)$ analogously. Finally we define $M(k)$ to be the error covariance of the mmse estimate of $x(k+1)$ when all the measurements from sensors 1 and 2 till time step k are available. Because of the assumption on observability of (A, C), $M(k)$ converges exponentially to a steady-state value denoted by $M^\star$.

We will condition the expected error covariance $E[P(k)]$ of the estimate of the state $x(k)$ on events $\mathcal{E}_{mn}$ where the subscript m denotes the time at which the last transmission was successfully received from sensor 1 and n denotes the time at which the last transmission was successfully received from sensor 2. Obviously $0 \le m, n \le k$. We also allow the indices to attain the value -1 to denote the event when transmission from the corresponding sensor was never possible till time k. Denote the error covariance conditioned on the event $\mathcal{E}_{mn}$ happening by P_{mn}. P_{mn} is the error covariance in estimating $x(k+1)$ based on measurements $y_1(0)$, $y_1(1)$, $\cdots$, $y_1(m)$ from sensor 1 and $y_2(0)$, $y_2(1)$, $\cdots$, $y_2(n)$ from sensor 2. Let p_{mn} be the probability of the event $\mathcal{E}_{mn}$ occurring. We can thus write

$$E[P(k)] = \sum_{m=-1}^{k}\sum_{n=-1}^{k} p_{mn}P_{mn}.$$

Since each term in the above summation is positive semi-definite, a necessary condition for the sum to be bounded is that any subsequence in the sum is bounded. We will consider three particular subsequences and show that the conditions in (13-15) are necessary for stabilizability. First consider the sequence

$$\begin{aligned} S_1(k) &= \sum_{m=0}^{k} p_{mk}P_{mk} \\ &= Pr(\mathbf{r}_1(k) = \mathbf{1})Pr(\mathbf{r}_2(k) = \mathbf{1})\Big(M(k) \\ &\quad + Pr(\mathbf{r}_1(k) = \emptyset)f_2\left(M(k-1)\right) + \cdots \\ &\quad + (Pr(\mathbf{r}_1(k) = \emptyset))^k f_2^k\left(M(0)\right)\Big). \end{aligned}$$

Since $M(k)$ converges exponentially to $M^\star$ as $k \to \infty$, we can substitute $M^\star$ for the conditional error covariances to study the convergence. Thus, we obtain

$$\lim_{k\to\infty} S_1(k) = Pr(\mathbf{r}_1(k) = \mathbf{1})Pr(\mathbf{r}_2(k) = \mathbf{1}) \sum_{m=0}^{\infty}\left(Pr(\mathbf{r}_1(k) = \emptyset)\right)^m f_2^m\left(M^\star\right).$$

Thus, using Proposition 3.1, we can prove that this sum converges only if (13) holds. In a similar fashion, we can prove that the condition in (14) is necessary by considering the sub-sequence $S_2(k) = \sum_{n=0}^{k} p_{kn}P_{kn}$. Finally, the sub-sequence $S_2(k) = \sum_{n=0}^{k} p_{nn}P_{nn}$ yields the necessary condition

$$\varrho(A)^2 Pr\left(\mathbf{r}(k) = (\emptyset, \emptyset)\right) < 1. \tag{21}$$

Note that $\varrho(A) = \max\{\varrho(A_{i,i})\}$. The proof is now complete since

1) if $\varrho(A) = \varrho(A_{3,3})$, equation (21) reduces to (15);
2) if either $\varrho(A) = \varrho(A_{1,1})$ or $\varrho(A) = \varrho(A_{2,2})$, equation (21) is subsumed by either equation (14) or (13). Moreover, equation (21) implies (15).

B. Sufficient Conditions for Stabilizability

We now present the proof of Theorem 2.3 by considering a particular algorithm in the class $\mathbb{S}_q$. Due to Proposition 2.1, we can consider the system to be either of **type I** or of **type II**. We can also partition the state space $x(k)$ of the process in one of two ways.

1) If the system is of **type I**, denote
$$x(k) = \begin{bmatrix} x_1(k)^{n_1\times 1} \\ x_2(k)^{n_2\times 1} \end{bmatrix}. \qquad (22)$$
2) If the system is of **type II**, denote
$$x(k) = \begin{bmatrix} x_1(k)^{n_1\times 1} \\ x_2(k)^{n_2\times 1} \\ x_3(k)^{n_3\times 1} \end{bmatrix}. \qquad (23)$$

Now consider the following algorithm. At each time step k

- *Encoder for Sensor 1:*
  - If the system is of **type I**, sensor 1 calculates and transmits the estimate $\hat{x}_2^{loc,1}(k)$ of the modes $x_2(k)$ of the process using its local measurements $y_1(0)$, $y_1(1)$, $\cdots$, $y_1(k)$.
  - If the system is of **type II**, sensor 1 calculates and transmits the estimate $\hat{x}_2^{loc,1}(k)$ and $\hat{x}_3^{loc,1}(k)$ of the modes $x_2(k)$ and $x_3(k)$ of the process using its local measurements $y_1(0)$, $y_1(1)$, $\cdots$, $y_1(k)$.
- *Encoder for Sensor 2:*
  - If the system is of **type I**, sensor 2 calculates and transmits the estimate $\hat{x}_1^{loc,2}(k)$ of the modes $x_1(k)$ of the process using its local measurements $y_2(0)$, $y_2(1)$, $\cdots$, $y_2(k)$.
  - If the system is of **type II**, sensor 2 calculates and transmits the estimate $\hat{x}_1^{loc,2}(k)$ and $\hat{x}_3^{loc,2}(k)$ of the modes $x_1(k)$ and $x_3(k)$ of the process using its local measurements $y_2(0)$, $y_2(1)$, $\cdots$, $y_2(k)$.
- *Decoder:*
  - If the system is of **type I**, the decoder maintains an estimate $\hat{x}_1(k)$ of the modes $x_1(k)$ and $\hat{x}_2(k)$ of the modes $x_2(k)$. At every time step k,
    1) if $r_1(k-1) = \emptyset$, $\hat{x}_1(k) = A\hat{x}_1(k-1)$, else $\hat{x}_1(k) = \hat{x}_1^{loc,2}(k)$;
    2) if $r_2(k-1) = \emptyset$, $\hat{x}_2(k) = A\hat{x}_2(k-1)$, else $\hat{x}_2(k) = \hat{x}_2^{loc,1}(k)$.

    The estimate $\hat{x}(k)$ is constructed by stacking the estimates $\hat{x}_1(k)$ and $\hat{x}_2(k)$.
  - If the system is of **type II**, the decoder maintains estimates $\hat{x}_1(k)$, $\hat{x}_2(k)$ and $\hat{x}_3(k)$ of the modes $x_1(k)$, $x_2(k)$ and $x_3(k)$ respectively. At every time step k,
    1) if $(r_1(k-1), r_2(k-1)) = (\mathbf{1},\mathbf{1})$,
    $$\begin{aligned}\hat{x}_1(k) &= \hat{x}_1^{loc,2}(k)\\ \hat{x}_2(k) &= \hat{x}_2^{loc,1}(k)\\ \hat{x}_3(k) &= \hat{x}_3^{loc,1}(k);\end{aligned}$$
    2) if $(r_1(k-1), r_2(k-1)) = (\mathbf{1},\emptyset)$,
    $$\begin{aligned}\hat{x}_1(k) &= \hat{x}_1^{loc,2}(k)\\ \hat{x}_2(k) &= A\hat{x}_2(k-1)\\ \hat{x}_3(k) &= \hat{x}_3^{loc,1}(k);\end{aligned}$$
    3) if $(r_1(k-1), r_2(k-1)) = (\emptyset,\mathbf{1})$,
    $$\begin{aligned}\hat{x}_1(k) &= A\hat{x}_1(k-1)\\ \hat{x}_2(k) &= \hat{x}_2^{loc,1}(k)\\ \hat{x}_3(k) &= \hat{x}_3^{loc,2}(k);\end{aligned}$$
    4) if $(r_1(k-1), r_2(k-1)) = (\emptyset,\emptyset)$,
    $$\begin{aligned}\hat{x}_1(k) &= A\hat{x}_1(k-1)\\ \hat{x}_2(k) &= A\hat{x}_2(k-1)\\ \hat{x}_3(k) &= A\hat{x}_3(k-1).\end{aligned}$$

    The estimate $\hat{x}(k)$ is constructed by stacking the estimates $\hat{x}_1(k)$, $\hat{x}_2(k)$ and $\hat{x}_3(k)$.

We shall now prove that under the conditions (16-18), the estimate $\hat{x}(k)$ of the state $x(k)$ is stable.

Proof of Theorem 2.3 We give the proof if the system is of **type II**. The proof for **type I** is similar. By construction, the estimates $\hat{x}_2^{loc,1}(k)$, $\hat{x}_1^{loc,2}(k)$, $\hat{x}_3^{loc,1}(k)$ and $\hat{x}_3^{loc,2}(k)$ are stable. Denote the corresponding error covariance matrices by $K_1(k)$, $K_2(k)$, $K_3(k)$ and $K_4(k)$ respectively.

1) For the modes $x_3(k)$, the error covariance evolves as follows:
$$P_3(k) = \begin{cases} K_3(k) & \text{w. pr. } Pr(\mathbf{r}_1(k) = \mathbf{1}) \\ K_4(k) & \\ \quad \text{w. pr. } Pr(\mathbf{r}_1(k) = \emptyset)Pr(\mathbf{r}_2(k) = \mathbf{1}) & \\ A_{3,3}P_3(k-1)A_{3,3}^T + Q_3 & \\ \qquad \text{w. pr. } Pr(\mathbf{r}(k) = (\emptyset,\emptyset)), & \end{cases}$$
where Q_3 is covariance matrix of the process noise entering the evolution of the modes $x_3(k)$. If (18) is satisfied, the error for the modes $x_3(k)$ will be stable.
2) For the modes $x_2(k)$, the error covariance in estimating the modes $x_3(k)$ can thus be considered to be additional noise with bounded covariance. The error covariance for these modes evolves as
$$P_2(k) = \begin{cases} K_2(k) \text{ w. pr. } Pr(\mathbf{r}_1(k) = \mathbf{1}) \\ A_{2,2}P_2(k)A_{2,2}^T + Q_2 \\ \qquad \text{w. pr. } Pr(\mathbf{r}_1(k) = \emptyset), \end{cases}$$
where Q_2 denotes the covariance of noise and error through the estimation of modes $x_3(k)$. If (17) is satisfied, the error for the modes $x_2(k)$ is stable.
3) A similar argument shows that if (16) is satisfied, the error for the modes $x_1(k)$ will be stable.

IV. Extensions and Generalizations

It is fairly obvious that the proof techniques of Theorems 2.2 and 2.3 can be generalized to the case when N sensors are present. We have the following stability result.

Proposition 4.1: Consider the process in (1) being observed by N sensors, such that the i-th sensor generates measurements according to the model

$$y_i(k) = C_i x(k) + v_i(k), \qquad 1 \leq i \leq N.$$

The sensors transmit data over erasure channels, with the packet erasure in the i-th channel being denoted by $\mathbf{r}_i = \emptyset$. Consider the 2^N possible ways of choosing m out of the N sensors, for all values of m between 0 and N. For the j-th such way, let the sensors chosen be denoted by $n_1, n_2, \cdots, n_j$ and sensors not chosen by $m_1, m_2, \cdots, m_{N-j}$. Denote by $\mathcal{C}^j$ the matrix formed by stacking the matrices $C_{m_1}, C_{m_2}, \cdots, C_{m_{N-j}}$. Finally, denote by ϱ^j the spectral radius of the unobservable part of matrix A when the pair $(A, \mathcal{C}^j)$ is put in the observer canonical form. A necessary and sufficient condition for the existence of a positive integer q, an encoding algorithm of the type $\mathbb{S}_q$ and an observer that stabilize the process is that the following 2^N inequalities be satisfied:

$$Pr\left(\mathbf{r}_{n_1} = \emptyset, \mathbf{r}_{n_2} = \emptyset, \cdots, \mathbf{r}_{n_j} = \emptyset\right) \mid \varrho^j \mid^2 < 1,$$

for all $1 \leq j \leq 2^N$.

We can also consider the case when sensors transmit information not over erasure channels, but over networks of erasure links, provided there is a provision for time-stamping the packet. A special case of the network arises when each sensor transmits data over a single link to the controllers. However, in addition, the sensors can cooperate by communicating with each other over an erasure link.

Proposition 4.2 (Cooperation over an erasure link): Consider the set-up of Figure 1 with an additional bidirectional erasure link connecting the two sensors. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. Let the plant be observable and that its state-space representation be of type I or type II. Also, let the erasures over the link connecting the two sensors be according to a Bernoulli process with the erasure event at time k denoted by $\mathbf{r}_3(k) = \emptyset$. If the state space representation is of type I, then there exists an observer of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q$ such that the feedback system is stable if and only if the following inequalities hold

$$\varrho(A_{2,2})^2 \max\Big(Pr(\mathbf{r}_1(k) = \emptyset), \; Pr(\mathbf{r}_2(k) = \emptyset, \mathbf{r}_3(k) = \emptyset)\Big) < 1$$

$$\varrho(A_{1,1})^2 \max\Big(Pr(\mathbf{r}_2(k) = \emptyset), \; Pr(\mathbf{r}_1(k) = \emptyset, \mathbf{r}_3(k) = \emptyset)\Big) < 1,$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II then the necessary and sufficient conditions for stabilizability include the following additional inequality:

$$\varrho(A_{3,3})^2 Pr(\mathbf{r}_1(k) = \emptyset, \mathbf{r}_2(k) = \emptyset) < 1. \tag{24}$$

V. Conclusions

In this paper, we considered the problem of observing a process using measurements from multiple sensors that transmit information over erasure links. We identified necessary and sufficient conditions that allow for the plant to be stabilized in a mean square sense using transmission of a vector of constant dimension from the sensors to the controller. We also considered various extensions such as sensors being able to co-operate over an erasure channel.

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