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Christoforos Somarakis, John S. Baras, and Evripides Paraskevas

ON THE DYNAMICS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

Christoforos Somarakis[1], John S. Baras[2] and Evripides Paraskevas[3]

The Institute For Systems Research
Univerisity Of Maryland
College Park, MD 20740, USA

Abstract. We discuss the dynamics of general linear functional differential equations with solutions that exhibit asymptotic constancy. We apply fixed point theory methods to study the stability of these solutions and we provide sufficient conditions of asymptotic stability with emphasis on the rate of convergence. Several examples are provided to illustrate the claim that the derived results generalize, unify and in some cases improve the existing ones.

1. Introduction

The long term behavior of dynamical processes lies in the core of every sector of the applied sciences and engineering. In the theory of delayed differential equations, probably the simplest example one can discuss is the evolution of

$$\dot{x}(t) = -ax(t) + bx(t-\tau). \tag{1.1}$$

There is hardly a textbook in the field not mentioning this equation at the chapter of stability of solutions [21, 19, 22]. It is indeed, in that particular chapter where the text focuses on equations of the type (1.1) with the parameters a, b satisfying $a > 0$ and $|b| \leq a$. In such case, the zero solution is shown to be stable for any $\tau > 0$. In the vast majority of these texts, the next sentence goes pretty much as follows: *"Moreover, if $|b| < a$, then the zero solution is asymptotically stable.[...]"*. In any advanced textbook or technical paper one may find the proof that the asymptotic stability of such systems is exponential. This can be shown using any of the known stability analysis methods. It is the purpose of the next subsection, to briefly review these tools in the stability analysis of Eq. (1.1) for $|b| < a$.

1.1. **The case $|b| < a$.** Eq. (1.1) is states as the following initial value problem

$$\begin{cases} \dot{x}(t) &= -ax(t) + bx(t-\tau), & t \geq 0 \\ x(t) &= \phi(t), & t \in [-\tau, 0] \end{cases} \tag{1.2}$$

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[1]csomarak@umd.edu
[2]baras@umd.edu
[3]evripar@umd.edu

where $a > 0$ and $b \in \mathbb{R}$ such that $|b| < a$. It is the latter condition which guarantees that both $x(t)$ and $e^{\gamma t}x(t)$ are qualitatively equal for $\gamma > 0$ small enough. If $x(t)$ is a solution of (1.2) then $y(t) = e^{\gamma t}x(t)$ satisfies:

$$\begin{cases} \dot{y}(t) & = -(a-\gamma)y(t) + be^{\gamma\tau}y(t-\tau), \qquad t \geq 0 \\ y(t) & = e^{\gamma t}\phi(t), \qquad t \in [-\tau, 0] \end{cases} \tag{1.3}$$

We choose $\gamma > 0$ small enough so that

$$\frac{|b|e^{\gamma\tau}}{a-\gamma} < 1 \tag{1.4}$$

and such γ always exist only because $|b| < a$. We will review the main stability analysis methods for the asymptotic behavior of y with respect to the zero solution as asymptotic stability of y implies exponential stability of x with rate γ.

1.1.1. *Frequency Methods.* We choose the direct method (§2.2.3 of [22]). The quasi-polynomial of (1.3) is

$$a(s, e^{-\tau s}) = s + (a-\gamma) - be^{\gamma\tau}e^{-s\tau} = a_0(s) + a_1(s)e^{-s\tau}$$

with $a_0(s) = s + (a-\gamma)$ and $a_1(s) = -be^{\gamma\tau}$. At $\tau = 0$, $a(s,1) = s + (a-\gamma) - b$ and it is stable $a(s,1) = 0$ if and only if $\Re(s) < 0$. Then

$$\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| = \frac{|b|e^{\gamma\tau}}{\sqrt{\omega^2 + (a-\gamma)^2}} < 1, \qquad \omega \in \mathbb{R}.$$

in view of (1.4). Then there is no solution of $a(s, e^{s\tau}) = 0$ with $\Re(s) > 0$ and the stability result follows.

1.1.2. *Liapunov-Krasovskii*[19]. For the problem (1.3), the appropriate functional to be selected is $V(\phi) = \frac{1}{2}\phi^2(0) + \mu\int_{-\tau}^{0}\phi^2(\theta)d\theta$ so that

$$\dot{V}(\phi) = -(a-\gamma-\mu)\phi^2(0) + |b|e^{\gamma\tau}\phi(0)\phi(-\tau) - \mu\phi^2(-\tau) \leq 0$$

if $\mu \in \left(\frac{|b|e^{\gamma\tau}}{2}, a - \gamma - \frac{|b|e^{\gamma\tau}}{2}\right)$ which exists in view of (1.4). Then the result follows in view of (1.4) and Theorem 2.1, in [19].

1.1.3. *Liapunov-Razumkhin*[19]. The Lyapunov function in this case is $V(x) = \frac{x^2}{2}$. Then

$$\dot{V}(x(t)) \leq -(a-\gamma)x^2(t) + |b|e^{\gamma\tau}|x(t)| \cdot |x(t-\tau)| \leq -(a-\gamma-|b|e^{\gamma\tau})x^2(t) \leq 0$$

whenever $|x(t)| \geq |x(t-\tau)|$. Then the result follows in view of (1.4) and Theorem 4.1, in [19].

1.1.4. *Fixed Point Theory*[20]. This technique does not directly rely on the transformation $y(t) = e^{\gamma t}x(t)$. The condition $|b| < a$ suffices to prove that the solution operator defined by inverting (1.2) as

$$(\mathcal{Q}x)(t) = \begin{cases} e^{-at}\phi(0) + b\int_0^t e^{-a(t-s)}x(s-\tau)ds, & t \geq 0 \\ \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.5}$$

is a contraction in the complete metric space $(\mathbb{M}, \rho)$ where

$$\mathbb{M} = \{x \in C^0[-\tau, \infty) : x = \phi|_{[-\tau,0]}, \sup_{t \geq -\tau} e^{\gamma t}|x(t)| < \infty\}$$

and $\rho(x_1, x_2) = \sup_{t\geq 0} e^{\gamma t}|x_1(t) - x_2(t)|$, whenever condition (1.4) holds. Then by the Contraction Mapping Principle, $\mathcal{Q}$ attains a unique fixed point in $\mathbb{M}$. This is a de facto proof of the asymptotic convergence of the solution to zero (see p. 41 of [20]).[4]

There is no doubt that the critical condition which characterizes (1.1) as "simple" is $|b| < a$ and consequently condition (1.4). Neither the constancy of the weights a and b nor the nature of delay (large, time/state dependent) play any substantial role in the asymptotic behavior of the solution x. So long as the magnitude of the undelayed term dominates the magnitude of the retarded term the effect of delay acts as a harmless perturbation of the the solutions in the qualitative sense. This property can be readily generalized to more complex systems and it is therefore generally exploited in different contexts for the establishment of delay-independent stability results [22]. But the crucial condition (1.4) seizes to hold when $a = b$.

1.2. **The case $b = a$.** At this "bifurcation" value, a number of new phenomena occur. Since Eq.(1.4) does not hold, one cannot use the transformation $y(t) = e^{\gamma t}x(t)$. Also Eq. (1.1) reads

$$\dot{x}(t) = -ax(t) + ax(t-\tau) \tag{1.6}$$

and every real constant is a solution. In particular if from (1.2), $\phi \in \Delta$ where $\Delta = \{y \in C^0[-\tau, 0] : y(t) \equiv \text{const.}, t \in [-r, 0]\}$ then the solutions stay in Δ for all times. However, no solution in Δ is asymptotically stable in the classical sense: if $y_1, y_2 \in \Delta$ with $|y_1 - y_2| < \epsilon$ for any $\epsilon > 0$, with $\phi = y_1$ then $y(t) \equiv y_1$ and it will never converge to y_2. As a result none of the above methods is applicable any more. For instance, the direct method of Section 1.1.1 gives for Eq. (1.2)

$$\frac{a}{\sqrt{\omega^2 + a^2}} \leq 1, \qquad \omega \in \mathbb{R}$$

which by no means imply asymptotic stability of the solutions, let alone the convergence rate. However, we may still conclude stability. The same result occurs for the Liapunov-Krasovski and Liapunov-Rhazumikhin methods, whereas the mapping $\mathcal{Q}$ in the Fixed Point Theory approach seizes to be a contraction for any value of τ.

It is this class of delayed differential equations for which Invariance Principles may take over in the Liapunov-based approaches and prove asymptotic stability of the solutions with respect to the invariant set Δ [3]. These techniques however suffer from the standard drawback that they provide no information on the rate at which the solutions convergence to such an invariant subset. A feature that is of utmost importance for real-world problems.

1.2.1. *A brief history of* $\dot{x} = -ax + ax_t$. To the best of our knowledge, equations of such type have so far appeared in the literature from two different fields of applied science.

The Cooke-Yorke model. The first occurrence of this equation dates back in 1973 with the seminal work of Cooke and Yorke [10]. The authors developed a theory of biological growth and epidemics by introducing and analyzing the system

$$\dot{x} = g(x(t)) - g(x(t-\tau)) \tag{1.7}$$

where g is an arbitrary Lipschitzian function. The authors proved that whenever the solution of Eq.(1.7) exists in the large, it approaches asymptotically a constant value. Their work has ever since attracted enormous attention from the mathematical community and caused an abundance of convergence results of these types

[4]This approach does not include however the step of the stability of solutions with respect to the classical definition. This part must be handled separately (usually with an $\epsilon - \delta$ argument).

of functional differential equations, known in the literature as *equations with asymptotic constancy of solutions* [11, 12, 16, 15, 14, 13, 17, 18, 20]. A similar field of study where such equations appear is this of the motion of a classical radiating electron [5]. In the following we will review a number of past works that emphasize both on the asymptotic stability and on the rate of convergence.

In [12, 11] the authors develop conditions which ensure that all solutions of certain functional differential equations are asymptotically constant as $t \to \infty$. Equation (1.6) is a special case of their work for which it must hold that

$$|a\tau| < 1 \tag{1.8}$$

so that solutions x tend to a constant so that $x \in L^1$. In [11], it is proved that the rate is exponential with exponent $\gamma \in (0, -\frac{\ln(a\tau)}{\tau})$.

In his monograph [20], T.A. Burton explains that, since the solution operator of (1.6) can be expressed as

$$x(t) = -a\int_{t-\tau}^{t} x(s)ds + x(0) + a\int_{-\tau}^{0}\phi(s)ds,$$

condition (1.8) ensures that this solution form can be a contraction in the complete metric space $(\mathbb{M}, \rho)$ defined in Sect. 1.1.4 with the origin translated to the fixed constant

$$k = \frac{\phi(0) + a\int_{-\tau}^{0}\phi(s)ds}{1 + a\tau} \tag{1.9}$$

and γ small enough to satisfy $|a|\frac{e^{\gamma\tau}-1}{\gamma} < 1$. The approach of Burton is more general in the sense that it sheds light upon the asymptotic value k as well as it can be readily applied to non-linear versions of (1.6) such as Eq. (1.7).

Finally, Krizstin [16], developed a Liapunov-Rhazumikhin argument, based on the monotonicity of (1.6) (i.e. the fact that we can take $a > 0$). He estimated the rate of convergence without the condition (1.8). In particular, assuming

$$a\tau < \infty$$

he proved that the rate of convergence of solutions to a constant is exponential with rate proportional to $\frac{1-e^{-a\tau}}{2\tau}$ which is a delay-independent result.

The main conclusion of the discussion should be that as far as (1.6) is concerned, the convergence to a constant value is independent of the magnitude of the delay τ only when $a > 0$ but is restricted to conditions like (1.8) whenever $a < 0$.

Monotone Dynamical Systems. In a rather different vein, multi-dimensional systems with delays appear in the study of Monotone Dynamical Systems. In his monograph [7], Smith discusses systems of the type

$$\begin{aligned}\dot{x}_1(t) &= -ax_1(t) + ax_2(t-\tau)\\ \dot{x}_2(t) &= -bx_2(t) + bx_1(t-\tau)\end{aligned} \tag{1.10}$$

The dynamical systems involved, are categorized either as competitive or as cooperative, depending on their monotonicity (here again the sign of the parameters, a and b in the example above). Whenever a, b are positive the system is cooperative and the asymptotic behavior is a constant value for any bounded τ. Systems of the type of (1.10) are known from the control community as linear distributed agreement (consensus) dynamics and in the un-delayed case ($\tau = 0$) are treated with Algebraic Graph Theory methods [23] and form the core of Networked Control Theory [24]. Despite the abundance of results in the control community, the case of distributed delayed dynamics is treated only on the part of simple convergence results. It is still an open problem to estimate the rate of convergence to a constant value for the general case as a function of the delay.

1.3. **This work: Motivation and Contribution.** In a series of papers,[28, 26, 27], the authors discussed this problem in a wide variety of systems of the type (1.10) with the use of fixed point theory methods. Despite the fact that the connectivity weights a, b in Eq. (1.10) were taken positive, the delay conditions were of the type (1.8) and therefore the resulting delay bounds (as they were depicted in the contraction condition) were unnecessary restrictive. Within the context of fixed point theory this peculiarity occurs as a result of the way the solutions are expressed and the corresponding solutions operators are defined.

Motivated by this shortcoming, in this work we discuss a one dimensional variant of (1.10) which sustains similar restrictive phenomena. In fact, we revisit the topic of stability of scalar linear functional equations with asymptotically constant solutions, within the framework of fixed point theory on the base of the following observation: A more careful inspection of Eq. (1.8) reveals that this condition simply neglects the sign of a. Indeed, the solution x for (1.6) also converges to a constant exponentially fast if (1.8) holds. Moreover one may suspect that instability occurs at $|a\tau| = 1$ exactly because, it may mean $a\tau = -1$ for which value k as defined in (1.9), is not finite. From now on, we will focus on fixed point theory methods. The first remark is that for the solution of (1.6) can be expressed either as

$$x(t) = -a\int_{t-\tau}^{t} x(s)ds + \left(\phi(0) + a\int_{-\tau}^{0}\phi(s)ds\right) \tag{1.11}$$

or as

$$x(t) = e^{-a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-a(t-s)}ax(s-\tau)ds \tag{1.12}$$

for any t, t_0 with $t \geq t_0$. These are two forms with different information on the dynamics of x. Eq. (1.11) shows that the value of $x(t)$ exclusively depends on the information of x in $[t-\tau, t]$ and Eq. (1.12) shows that $x(t)$ is based on the information of x in $[t-\tau, t_0]$ whereas this form also exploits the dissipative nature of the dynamics, due to $a > 0$. The problem with Eq. (1.12) is that, unlike Eq. (1.5), it is not a contraction in any useful metric space with regards to asymptotic stability. This is exactly because $a = b$ from (1.1). Since in the Fixed Point Theory framework it is the representation of the solution as of much importance as the form of the Lyapunov function in the Lyapunov theory, we will combine (1.11) and (1.12), to obtain a new representation of the solution that in many cases will yield asymptotic stability results independent of the magnitude of the delays. Motivated by recent results in the study of stability of scalar functional differential equations with the use of fixed points [2, 8], we revisit the problem of type (1.6) and in particular we study the general equation

$$\dot{x}(t) = -\sum_i a_i(t)x(t) + a_i(t)x(t-\tau_i(t)), \qquad a_i(t), x \in \mathbb{R}$$

The novelty of our approach lies on a combination of two different expressions of the solution x so as to obtain new forms of the solution which will be used as self-mapping contraction operators in suitably designed complete metric spaces.

1.4. **Organization.** The paper is organized as follows. §2 discusses the notation that will be used throughout this work, introduces fundamental notions and definitions the Theory of functional differential equations.

In §3 we begin our study for both a_i's and τ_i's constant. In §4 we generalize to time varying a_i's and τ_i's and in §5 we revisit the same problem from a different perspective where old rate estimate are re-established [16]. In §6 we work a number of illustrative examples as applications to the derived theoretical results.

2. Notation and Definition

Throughout this work, $m \in \mathbb{Z}_+$ with i to denote a number in $\{1,\dots,m\}$, $t_0 \in \mathbb{R}$ and $\tau_i \in [t_0,\infty) \to \mathbb{R}_+$ is a continuous function such that $\lambda_i(t) := t - \tau_i(t)$ is non-decreasing with $\lim_{t\to\infty} \lambda_i(t) = \infty$, for any $i = 1,\dots,m$. Whenever the subscript i is omitted we understand the maximum over i, i.e. $\tau(t) := \max_i \tau_i(t)$ whereas $\lambda(t) = t - \tau(t) = \min_i \lambda_i(t)$. Also $\lambda^{(j)}(t)$ will denote the j^{th} composition of $\lambda(t)$ and for $t \geq t_0$ $I_{\lambda^{(j-1)}t} := [\lambda^{(j)}(t), \lambda^{(j-1)}(t)]$ with the convention $I_{\lambda^{(0)}(t)} = I_t$. For any real a, $a^+ := \max\{0,a\}$ and $a^- := \min\{0,a\}$. The space of continuous functions defined in I and taking values in S is denoted by $C^0(I,S)$. A subspace of interest is this of functions that are constant in a subset of I in particular we define $\Delta = \{x \in C^0([-\tau,0],\mathbb{R}) : x \equiv k, k \in \mathbb{R}\}$. Let $a_i|_1^m$ be a family of members in $C^0([t_0,\infty),\mathbb{R})$. Then $a_i^+(t) = \max\{0,a_i(t)\}$ and $a_i^-(t) = \min\{0,a_i(t)\}$ and $a(t) := \sum_i a_i^+(t)$. Next, for any $t_1,t_2 \geq t_0$ we denote $\phi(t_1,t_2) = e^{-\int_{t_2}^{t_1} a(s)ds}$. By *rate function* we understand any arbitrary function $h : C^0([t_0,\infty),\mathbb{R}_+)$ with the property that h is increasing and $h(t) \to \infty$. In this work, we shall restrict to either exponential or polynomial rate functions.

For $s \geq t_0$ consider the linear functional differential equation

$$\dot{y}(t) = A(t)y_t, t \geq s$$

and it's unique solution $y(t,s,\phi)$ where $\phi \in C^0(I_s,\mathbb{R})$ is the initial condition. Since for any fixed $s \geq t_0$ and $t \geq s$ $y(s,\cdot)(t) = y(t,s,\phi)$ is a continuous linear operator in $C^0(I_{t_0},\mathbb{R})$ we can associate a family of continuous linear operators $T(t,s) : C^0(I_s,\mathbb{R}) \to C^0(I_s,\mathbb{R})$, $t \geq s$ by defining for any $\phi \in C^0(I_s,\mathbb{R})$, $T(t,s)\phi = y(t,s,\phi)$. Since solutions are continuous in t,s for $t_0 < s \leq t < \infty$, $T(t,s)$ is strongly continuous for $t_0 < s \leq t < \infty$ with $T(t,t) = Id$, $T(t,s)T(s,a) = T(t,a)$ for $t_0 < a \leq s \leq t < \infty$. Additionally one can define a solution of y with ϕ being piecewise continuous function so that for any constant c, $\phi(s) = \delta(t_0 - s)c$ whenever $s \in I_{t_0}$ can be an appropriate initial condition. This extension of the solutions will come in hand in §5 where a variation of constants formula for functional equations will be used. Also $M_{t,\phi} = \left[\min_{s\in I_t}\phi(s), \max_{s\in I_t}\phi(s)\right]$ and $|M_{t,\phi}| = \max_{s\in I_t}\phi(s) - \min_{s\in I_t}\phi(s)$ stands for it's length.

2.1. **Elements of Fixed Point Theory.** The two main fixed point theorems to be applied in this work are the Banach's Contraction Mapping Principle and Shauder's first fixed point theorem that we state here for reference.

Theorem 2.1. *Let $(\mathbb{M},\rho)$ be a complete metric space and let $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. If there is a constant $\alpha < 1$ such that $\rho(\mathcal{P}\phi_1,\mathcal{P}\phi_2) \leq \rho(\phi_1,\phi_2)$, $\forall \phi_1,\phi_2 \in \mathbb{M}$, we have*

$$\rho(\mathcal{P}\phi_1,\mathcal{P}\phi_2) \leq \alpha\rho(\phi_1,\phi_2)$$

then there is a unique $\phi^* \in \mathbb{M}$ with $\mathcal{P}\phi^* = \phi^*$.

Theorem 2.2. *Let $\mathbb{M}$ be a non-empty compact convex subset of a Banach space and let $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ be continuous. Then $\mathcal{P}$ is a fixed point in $\mathbb{M}$.*

The proofs of these theorems can be found, for example, in [9].

3. Time Invariant Dynamics

The first set of results concerns the following initial value problem

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{m} -a_i x(t) + a_i x(\lambda_i(t)) \\ x(t) &= \phi(t), t \in I_0, \end{cases} \tag{3.1}$$

with $a_i \in \mathbb{R}$ constant $\lambda_i(t) = t - \tau_i$ for τ_i constant and ϕ given initial datum. We re-write (3.1) as

$$\dot{x} = -ax(t) + \sum_i a_i^+ x(\lambda_i(t)) + \sum_i |a_i^-| \frac{d}{dt} \int_{\lambda_i(t)}^t x(s)ds$$

so as to separate the dissipation dynamics from the non-dissipation ones. Then the solution x satisfies both

$$\begin{aligned} x(t) = e^{-a\tau} x(\lambda(t)) + \int_{\lambda(t)}^t e^{-a(t-s)} \sum_i a_i^+ x(\lambda_i(s))ds + \\ + \int_{\lambda(t)}^t e^{-a(t-s)} \sum_i |a_i^-| \frac{d}{ds} \int_{\lambda_i(s)}^s x(w)dwds, \qquad t \geq \tau \end{aligned} \tag{3.2}$$

and

$$x(t) = c_0 - \sum_i a_i \int_{\lambda_i(t)}^t x(s)ds \tag{3.3}$$

with $c_0 := \phi(0) + \sum_i a_i \int_{-\tau_i}^0 \phi(s)ds$. For $t \geq \tau$ we substitute $x(\lambda(t))$ of the first form with the right hand-side equivalent of the second form and we derive the following form of the solution of (3.1):

(3.4)

$$\begin{aligned} x(t) = e^{-a\tau} c_0 &+ \sum_i \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \left(e^{-a(t-w-\tau_i)} - e^{-a\tau}\right) a_i^+ x(w)dw \\ &+ \sum_i \int_{\lambda(t)}^{\lambda_i(t)} e^{-a(t-w-\tau_i)} a_i^+ x(w)dw + \sum_i |a_i^-| \int_{\lambda_i(t)}^t x(w)dw \\ &- \sum_i \int_{\lambda(t)}^t e^{-a(t-s)} a|a_i^-| \int_{\lambda_i(s)}^s x(w)dwds \\ = e^{-a\tau} c_0 &+ \sum_i \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \left[\left(e^{-a(t-w-\tau_i)} - e^{-a\tau}\right) a_i^+ - \int_{\lambda(t)}^{g_i(w)} e^{-a(t-s)} a|a_i^-|ds \right] x(w)dw \\ &+ \sum_i \int_{\lambda(t)}^{\lambda_i(t)} \left[e^{-a(t-w-\tau_i)} a_i^+ - \int_w^{g_i(w)} e^{-a(t-s)} a|a_i^-|ds \right] x(w)dw \\ &+ \sum_i \int_{\lambda_i(t)}^t |a_i^-| \left[1 - \int_w^t ae^{-a(t-s)}ds \right] x(w)dw \end{aligned}$$

where the last step is due to the change of the order of integration on the term $\sum_i \int_{t-\tau}^t e^{-a(t-s)} a|a_i^-| \int_{s-\tau_i}^s x(w)dwds$. This will be our solution operator, for $t \geq \tau$. We exploit the monotonicity of the integrand functions to arrive in the following condition

Assumption 1. *There exists $\alpha \in [0, 1)$ such that*

$$1 - e^{-a\tau} - \sum_i \left[\tau_i |a_i| e^{-a\tau} - \frac{|a_i^-|}{a} \left(1 - e^{-a\tau_i} - e^{-a\tau} + e^{-a(\tau-\tau_i)}\right) \right] \leq \alpha$$

Remark 1. We outline the following two special cases:

(1) $a_i^- \equiv 0$: The condition reduces to $1 - e^{-a\tau} - \sum_i a_i \tau_i e^{-a\tau} =: \alpha < 1$ and it is satisfied for any magnitude of $\tau_i, a_i < \infty$.
(2) $a_i^+ \equiv 0$: Then $a = 0$ and the condition reduces to $\sum_i |a_i \tau_i| \leq \alpha < 1$.

For $\gamma \in (0, a)$ we define the quantities

$$\Gamma_1(i,\gamma) = |a_i^-| e^{a\tau_i} e^{-(a-\gamma)\tau} \frac{1 - e^{-(a-\gamma)\tau_i}}{a-\gamma} - |a_i^-| e^{\gamma\tau} \frac{e^{\gamma\tau_i} - 1}{\gamma}$$

$$\Gamma_2(i,\gamma) = |a_i^-|(e^{a\tau_i} - 1) \frac{e^{-(a-\gamma)\tau_i} - e^{-(a-\gamma)\tau}}{a-\gamma}$$

$$\Gamma_3(i,\gamma) = |a_i^-| \frac{e^{(a-\gamma)\tau_i} - 1}{a-\gamma}$$

and we pick γ small enough so that

$$\sum_i a_i^+ \left(e^{\gamma\tau_i} \frac{1 - e^{-a\tau}}{a-\gamma} - e^{-a\tau} \frac{e^{\gamma\tau_i} - 1}{\gamma} \right) + \sum_{j=1}^{3} \Gamma_j(i,\gamma) \leq 1. \tag{3.5}$$

One can always find such γ to satisfy this condition since in the limit $\gamma \downarrow 0$ (3.5) coincides with Assumption 1. We state and prove now the first result of this work.

Theorem 3.1. *Under Assumption 1, the solution of* (3.1) *is exponentially asymptotically stable with respect to* Δ. *More specifically, the solution converges to*

$$k = \frac{\phi(0) + \sum_{i=1}^m a_i \int_{-\tau_i}^0 \phi(s) ds}{1 + \sum_{i=1}^m a_i \tau_i} \tag{3.6}$$

exponentially fast with exponent γ *that satisfies* (3.5).

Proof. At first, we see that k is well-defined as if, $1 + \sum_{i=1}^m a_i \tau_i \downarrow 0$ then the left hand side of the expression in the Assumption 2 gets greater than 1.

Next, we prove stability of the solution with respect to Δ. Fix $\epsilon > 0$ and k. We pick ϕ and $\delta_\tau = \delta_\tau(\epsilon, k) < \epsilon$ so that $|x(t, \phi) - k| < \delta_\tau$ for $t \in [-\tau, \tau]$. Such a δ_τ can always be found by the continuous dependence on initial conditions. Next we pick $\delta \leq \delta_\tau$ satisfying $\delta(1 + \sum_i a_i \tau_i) e^{-a\tau} + \alpha\epsilon < \epsilon$, consider the first time $t^* \geq \tau$ such that $|x(t^*, \phi) - k| = \epsilon$ and express x as in (3.4) to arrive in a contradiction. Finally, we prove exponential convergence to k by a fixed point argument. Consider the metric space $(\mathbb{M}, \rho)$ with

$$\mathbb{M} = \left\{ y \in C^0([-\tau, \infty), \mathbb{R}) : y = \tilde{x}|_{[-\tau,\tau]}, \sup_{t \geq \tau} e^{\gamma t} |y(t) - k| < \infty \right\}$$

and $\rho(x_1, x_2) = \sup_{t \geq \tau} e^{\gamma t} |x_1(t) - x_2(t)|$. It is a standard exercise to show that $(\mathbb{M}, \rho)$ is a complete metric space [20]. Next, we define the operator

$$(\mathcal{P}x)(t) = \begin{cases} \tilde{x}(t), t \in [-\tau, \tau] \\ x_{(3.4)}(t), t \geq \tau \end{cases}$$

To show that $\mathcal{P}$ is a member of $\mathbb{M}$ we observe that it is continuous and it agrees in $[-\max_i \tau_i, \tau]$ with any member of $\mathbb{M}$, by definition. Next $\sup_{t \geq \tau} e^{\gamma t} |(\mathcal{P}x)(t) - k|$ is finite, for $\gamma < a$. Finally, we show that $\mathcal{P}$ is a contraction in $(\mathbb{M}, \rho)$: For $x_1, x_2 \in \mathbb{M}$ we calculate an upper bound of $e^{\gamma t} |\mathcal{P}x_1 - \mathcal{P}x_2|$ and we arrive at

$$e^{\gamma t} |\mathcal{P}x_1(t) - \mathcal{P}x_2(t)| \leq \Gamma(t, \gamma) \rho(x_1, x_2)$$

where $\sup_{t \geq \tau} \Gamma(t, \gamma)$ is equal to (3.5). Then Theorem 2.1 can be applied concluding the proof. ∎

4. The General Model

The purpose of this section is to generalize the equation (3.1) in the case of time varying weights and delays and study the stability of solutions in a similar way.

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{m} -a_i(t)x(t) + a_i(t)x\big(\lambda_i(t)\big), t \geq t_0 \\ x(t) &= \phi(t), t \in I_{t_0}. \end{cases} \tag{4.1}$$

where $a_i(t)$ are functions to be determined and $\lambda_i(t) = t - \tau_i(t)$ as defined in §2. As opposed to §3 there is a number of reasons to focus on the dynamics of the first derivative of the state $w := \dot{x}$ for Eq. (4.1), rather than the study of x itself. It should be intuitively clear that there is no hope to try to express the asymptotic value k in a closed form due to the time-varying nature of the system. Then since x always converges to $\mathbb{R}$ whenever $\int^{\infty} w(s)ds$ exists, it is desirable to seek the solution w in (4.1) in L^1. To outline this method, let the simplified system

$$\dot{x}(t) = -a(t)x(t) + a(t)x(\lambda(t)).$$

We note that w satisfies both

$$w(t) = -a(t)\int_{\lambda(t)}^{t} w(s)ds$$

and

$$\dot{w}(t) = -a(t)w(t) + a(t)w(\lambda(t))\dot{\lambda}(t) - \dot{a}(t)\int_{\lambda(t)}^{t} w(s)ds$$

The latter is an integrodifferential equation which will be handled with the method of resolvents [25] in order to express the solution w in a suitable form. Prior to the analysis, we state the following set of assumptions:

Assumption 2. *$\forall i = 1, \ldots, m$, $a_i(\cdot) \in C^0([t_0, \infty), [-M, M])$ for some $M > 0$ and uniformly bounded integrable first derivative.*

Assumption 3. *$\forall i = 1, \ldots, m$, $\tau_i(t) \in C^1([t_0, \infty), \mathbb{R}_+)$ with $t - \tau_i(t) \to +\infty$ as $t \to +\infty$ and $\dot{\tau}_i(t) < 1$.*

This condition is a very mild one and it is involved with deterministic issues. It also implies that $t - \tau_i(t)$ are invertible functions with $g_i(\cdot)$ to denote their inverses.

Assumption 4. *$\forall i = 1, \ldots, m$ it holds that:*

$$\sup_t \int_{\lambda(t)}^{t} a_i(s)ds < \infty.$$

4.1. The dynamics of w and the solution operator.

Consequently $w = \dot{x}$ satisfies the following initial value problem

$$\begin{cases} \dot{w}(t) = \sum_{i=1}^{m} -a_i(t)w(t) + a_i(t)w(\lambda_i(t))\dot{\lambda}_i(t) - \dot{a}_i(t)\int_{\lambda_i(t)}^{t} w(s)ds, t \geq g^{(2)}(t_0) \\ w(t) = \phi_w(t), t \in [t_0, g^{(2)}(t_0)] \end{cases} \tag{4.2}$$

We separate the positive $a_i(t)$'s from the negative ones and we observe that w satisfies both

(4.3)

$$\dot{w} = \sum_{i=1}^{m} -a_i^{+}(t)w(t) + a_i^{+}(t)w(\lambda_i(t))\dot{\lambda}_i(t) - \dot{a}_i^{+}(t)\int_{\lambda_i(t)}^{t} w(s)ds - \frac{d}{dt}\left(a_i^{-}(t)\int_{\lambda_i(t)}^{t} w(s)ds\right)$$

and

$$w(t) = -\sum_{i=1}^{m} a_i(t)\int_{\lambda_i(t)}^{t} w(s)ds \tag{4.4}$$

The notation $\dot{a}_i^{+/-}$ stands for $\frac{d}{dt}(a_i^{+/-}(t))$. We take $P(s) = R(t,s)w(s)$ for a function $R(t,s)$ so that $R(t,t) \equiv 1$ to be determined. Then

$$\begin{aligned}\dot{P} &= R_s(t,s)w(s) + R(t,s)\dot{w}(s) = R_s(t,s)w(s) + R(t,s)\dot{w}_{(4.3)}(s)\\ &= R_s(t,s)w(s) + R(t,s)\Big(\sum_{i=1}^{m} -a_i^+(s)w(s) + a_i^+(s)w\big(\lambda_i(s)\big)\dot{\lambda}_i(s) - \dot{a}_i^+(s)\int_{\lambda_i(s)}^{s} w(u)du\Big)\\ &- R(t,s)\sum_{i=1}^{m}\frac{d}{dt}\Big(a_i^-(t)\int_{\lambda_i(t)}^{t} w(s)ds\Big).\end{aligned}$$

For any $t \geq g(g(t_0))$ we invert the latter equation from $\lambda(t)$ to t and we apply Fubini's theorem, integration by parts and substitute Eq. (4.4):

$$\begin{aligned}&w(t) - R(t,\lambda(t))w(\lambda(t)) = w(t) - R(t,\lambda(t))w_{(4.4)}(\lambda(t)) = \int_{\lambda(t)}^{t} R_s(t,s)w(s)ds+\\ &+\sum_{i=1}^{m}\Big[\int_{\lambda(t)}^{t} R(t,s)\Big(-a_i^+(s)w(s) + a_i^+(s)w\big(\lambda_i(s)\big)\dot{\lambda}_i(s) - \dot{a}_i^+(s)\int_{\lambda_i(s)}^{s} w(u)du\Big)ds-\\ &-\int_{\lambda(t)}^{t} R(t,s)\frac{d}{ds}\Big(a_i^-(s)\int_{\lambda_i(s)}^{s} w(u)du\Big)\Big]ds\\ &\Rightarrow\end{aligned}$$

$$\begin{aligned}&w(t) =\\ &=\sum_{i=1}^{m}\Big[\int_{\lambda_i(\lambda(t))}^{\lambda(t)}\Big(R(t,g_i(s))a_i^+(g_i(s)) - R(t,\lambda(t))a_i^+(\lambda(t)) - \int_{\lambda(t)}^{g_i(s)} R(t,u)\dot{a}_i^+(u)du\Big)w(s)ds+\\ &+\int_{\lambda(t)}^{\lambda_i(t)}\Big(\frac{1}{m}R_s(t,s) - R(t,s)a_i^+(s) + R(t,g_i(s))a_i^+(g_i(s)) - \int_{s}^{g_i(s)} R(t,u)\dot{a}_i^+(u)du\Big)w(s)ds+\\ &\int_{\lambda_i(t)}^{t}\Big(\frac{1}{m}R_s(t,s) - R(t,s)a_i^+(s) - \int_{s}^{t} R(t,u)\dot{a}_i^+(u)du\Big)w(s)ds+\\ &-\int_{\lambda(t)}^{t} R(t,s)\frac{d}{ds}\Big(a_i^-(s)\int_{\lambda_i(s)}^{s} w(u)du\Big)ds - R(t,\lambda(t))a_i^-(\lambda(t))\int_{\lambda_i(\lambda(t))}^{\lambda(t)} w(s)ds\Big]\end{aligned}$$

$$\begin{aligned}w(t) = \sum_{i=1}^{m}\Big[&\int_{\lambda_i(\lambda(t))}^{\lambda(t)}\Big(\int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i^+(u)du\Big)w(s)ds+\\ &+\int_{\lambda(t)}^{\lambda_i(t)}\Big(\frac{1}{m}R_s(t,s) + \int_{s}^{g_i(s)} R_u(t,u)a_i^+(u)du\Big)w(s)ds+\\ &\int_{\lambda_i(t)}^{t}\Big(\frac{1}{m}R_s(t,s) - a_i^+(t) + \int_{s}^{t} R_u(t,u)a_i^+(u)du\Big)w(s)ds-\\ &-a_i^-(t)\int_{\lambda_i(t)}^{t} w(s)ds + \int_{\lambda_i(\lambda(t))}^{\lambda(t)}\int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i^-(u)du w(s)ds+\\ &+\int_{\lambda(t)}^{\lambda_i(t)}\int_{s}^{g_i(s)} R_u(t,u)a_i^-(u)du w(s)ds + \int_{\lambda_i(t)}^{t}\int_{s}^{t} R_u(t,u)a_i^-(u)du w(s)ds\Big]\end{aligned}$$

From the expression of $w(t)$ above we draw our attention to the term

$$\begin{aligned}S_i = &\int_{\lambda(t)}^{\lambda_i(t)}\Big(\frac{1}{m}R_s(t,s) + \int_{s}^{g_i(s)} R_u(t,u)a_i^+(u)du\Big)w(s)ds+\\ &\int_{\lambda_i(t)}^{t}\Big(\frac{1}{m}R_s(t,s) - a_i^+(t) + \int_{s}^{t} R_u(t,u)a_i^+(u)du\Big)w(s)ds\end{aligned}$$

The first part of S_i is equal to

$$\int_{\lambda(t)}^{\lambda_i(t)} \left(\frac{1}{m}R_s(t,s) + \int_s^t R_u(t,u)a_i^+(u)du\right)w(s)ds - \int_{\lambda(t)}^{\lambda_i(t)} \left(\int_{g_i(s)}^t R_u(t,u)a_i^+(u)du\right)w(s)ds$$

and the second term of those above equals

$$-\int_{\lambda(t)}^{\lambda_i(t)} \left(\int_{g_i(s)}^t R_u(t,u)a_i^+(u)du\right)w(s)ds = -\int_{\lambda(t)}^{\lambda_i(t)} a_i^+(t)w(s)ds +$$
$$+ \int_{\lambda(t)}^{\lambda_i(t)} R(t,g_i(s))a_i^+(g_i(s))w(s)ds + \int_{\lambda(t)}^{\lambda_i(t)} \left(\int_{g_i(s)}^t R(t,u)\dot{a}_i^+(u)du\right)w(s)ds$$

All in all,

$$S_i = \int_{\lambda(t)}^{\lambda_i(t)} \left(R(t,g_i(s))a_i^+(g_i(s)) + \int_{g_i(s)}^t R(t,u)\dot{a}_i^+(u)du\right)w(s)ds +$$
$$+ \int_{\lambda(t)}^{t} \left(\frac{1}{m}R_s(t,s) - a_i^+(t) + \int_s^t R_u(t,u)a_i^+(u)du\right)w(s)ds$$

We group the components of w in $[\lambda_i(\lambda(t)), \lambda(t)]$, $[\lambda(t), \lambda_i(t)]$ and $[\lambda_i(t), t]$ and sum over i to obtain the following form of w:

$$w(t) = \sum_{i=1}^m \Bigg[\int_{\lambda_i(\lambda(t))}^{\lambda(t)} \left(\int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i^+(u)du\right)w(s)ds$$
$$+ \int_{\lambda(t)}^{\lambda_i(t)} \left(R(t,g_i(s))a_i^+(g_i(s)) + \int_{g_i(s)}^t R(t,u)\dot{a}_i^+(u)du\right)w(s)ds +$$
$$\int_{\lambda(t)}^{t} \left(\frac{1}{m}R_s(t,s) - a_i^+(t) + \int_s^t R_u(t,u)a_i^+(u)du\right)w(s)ds +$$
$$- a_i^-(t)\int_{\lambda_i(t)}^t w(s)ds + \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i^-(u)du w(s)ds +$$
$$+ \int_{\lambda(t)}^{\lambda_i(t)} \int_s^{g_i(s)} R_u(t,u)a_i^-(u)du w(s)ds + \int_{\lambda_i(t)}^t \int_s^t R_u(t,u)a_i^-(u)du w(s)ds \Bigg]$$

Consequently,

$$w(t) = \sum_{i=1}^m \Bigg[\int_{\lambda_i(\lambda(t))}^{\lambda(t)} I_1(i,t,s)w(s)ds + \int_{\lambda(t)}^{\lambda_i(t)} I_2(i,t,s)w(s)ds +$$
$$\int_{\lambda(t)}^{t} I_3(i,t,s)w(s)ds + \int_{\lambda_i(t)}^{t} I_4(i,t,s)w(s)ds \Bigg]$$

where

$I_1(i,t,s) := \int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i(u)du$

$I_2(i,t,s) := a_i^+(t) - \int_{g_i(s)}^t R_u(t,u)a_i^+(u)du + \int_s^{g_i(s)} R_u(t,u)a_i^-(u)du$

$I_3(i,t,s) := \frac{1}{m}R_s(t,s) - a_i^+(t) + \int_s^t R_u(t,u)a_i^+(u)du$

$I_4(i,t,s) := -a_i^-(t) + \int_s^t R_u(t,u)a_i^-(u)du$

If $R(t,s)$ is defined by:

$$R(t,s) = 1 - a(t)\int_s^t e^{-\int_u^t a(y)dy}du$$

where $a(t) = \sum_{i=1}^{m} a_i^+(t)$ and $R(t,t) \equiv 1$. Then we see that

$$\sum_{i=1}^{m} \int_{\lambda(t)}^{t} I_3(i,t,s)w(s)ds =$$

$$= \int_{\lambda(t)}^{t} \Big(a(t)e^{-\int_s^t a(y)dy} - \sum_{i=1}^{m} a_i^+(t) + a(t)(1 - e^{-\int_s^t a(y)dy}) \Big) w(s)ds = 0$$

so the I_3 term is canceled and finally the solution w satisfies

(4.5)

$$w(t) = \sum_{i=1}^{m} \Big[\int_{\lambda_i(\lambda(t))}^{\lambda(t)} I_1(i,t,s)w(s)ds + \int_{\lambda(t)}^{\lambda_i(t)} I_2(i,t,s)w(s)ds + \int_{\lambda_i(t)}^{t} I_4(i,t,s)w(s)ds \Big]$$

Assumption 5. *There exists* $\alpha \in [0,1]$

$$\sup_{t \geq g^{(2)}(t_0)} \sum_{i=1}^{m} \int_{\lambda_i(\lambda(t))}^{\lambda(t)} |I_1(i,t,s)|ds + \int_{\lambda(t)}^{\lambda_i(t)} |I_2(i,t,s)|ds + \int_{\lambda_i(t)}^{t} |I_4(i,t,s)|ds < \alpha \tag{4.6}$$

Theorem 4.1. *Under Assumptions 2,3,4,5 the solution of (4.1) is asymptotically stable with respect to* Δ *and the rate of convergence is governed by a rate function* h *satisfying*

$$\sup_{t \geq t_0} h(t) \sum_{i=1}^{m} \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \frac{|I_1(i,t,s)|}{h(s)} ds + \int_{\lambda(t)}^{\lambda_i(t)} \frac{|I_2(i,t,s)|}{h(s)} ds + \int_{\lambda_i(t)}^{t} \frac{|I_4(i,t,s)|}{h(s)} ds < 1$$

Proof. Since $x(t) = x(t_0) + \int_{t_0}^{t} w(s)ds$ where w is the solution of (4.2) for $\phi_w = \dot{x}$ defined on $[t_0, g(g(t_0))]$. If $w(t) \in L^1$ then obviously $k = x(t_0) + \int_{t_0}^{\infty} w(s)ds$. It suffices then to prove that w being the solution of (4.2) is integrable. Consequently it is bounded by a rate function h that is integrable. The part of stability of x with respect to Δ is identical to the proof of Theorem 3.1 and it will be omitted. Then we only need to prove the part of asymptotic convergence. Define

$$\mathbb{M}_h = \Big\{ y \in C^0([t_0,\infty),\mathbb{R}) : y = \phi_w|_{[t_0,g^{(2)}(t_0)]}, y \in L^1, \sup_{t \geq t_0} h(t)|w(t)| < \infty \Big\}$$

i.e. the space of absolutely integrable functions that vanish at rate $1/h$, together with the metric $\rho_h(y_1,y_2) = \sup_{t \geq t_0} h(t)|y_1(t) - y_2(t)|$. The couple $(\mathbb{M}_h, \rho_h)$ constitutes a complete metric space. For $w \in \mathbb{M}_h$we define the operator

$$(\mathcal{E}w)(t) = \begin{cases} \phi_w(t), t \in [t_0, g^{(2)}(t_0)] \\ w_{(4.5)}(t), t \geq g^{(2)}(t_0) \end{cases}$$

where $w_{(4.5)}$ is the right hand-side of (4.5). To show that $\mathcal{E}y$ is in $\mathbb{M}_h$ we use Assumptions 2, 3 and 4 to show that I_i are bounded. It is a tedious but straightforward exercise. Therefore $h(t)(\mathcal{E}y)(t) < \infty$ for h that for the scopes of this work can be either exponential $h(t) = e^{\gamma t}$ for some $\gamma > 0$ or sub-exponential $h(t) = \big(t + t_0\big)^{\delta}$ for some $\delta > 1$ depending on whether $\tau(t)$ is bounded or unbounded. Again it is a straightforward exercise to show that $\sup_{t \geq t_0} h(t) \int_{\lambda(t)}^{t} \frac{ds}{h(s)} < \infty$ and similarly for the rest of the two integrals of $w_{(4.5)}$.

Next, we show that $\int^{\infty} (\mathcal{E}w)(s)ds$ is finite. To see this take

$$\Big| \int_{g(g(t_0))}^{\infty} \int_{\lambda_i(\lambda(t))}^{\lambda(t)} I_1(i,t,s)w(s)dsdt \Big| \leq \int_{t_0}^{\infty} \Big(\int_{g(s)}^{g(g_i(s))} |I(i,t,s)|dt \Big) w(s)ds.$$

The inner integral is bounded in view of Assumption 4 and for $w \in L^1$, $\mathcal{E}w$ is in L^1, as well. For the rest of the terms we argue in the same way. Finally we can show that $\mathcal{E}$ is a contraction in $\mathbb{M}_h$ using Assumption 5 and Theorem 2.1 can be applied concluding the proof. □

Remark 2. Let $a_i(t) = a(t) \geq 0$ and $\dot{g}$ smooth enough. From Eq. (4.5) w satisfies

$$w(t) = \sum_{i=1}^{m} \left[\int_{\lambda_i(\lambda(t))}^{\lambda(t)} I_1(i,t,s)w(s)ds + \int_{\lambda(t)}^{\lambda_i(t)} I_2(i,t,s)w(s)ds \right]$$

where

$I_1(i,t,s) := \int_{\lambda(t)}^{g_i(s)} R_u(t,u)a_i(u)du$ and $I_2(i,t,s) := a_i(t) - \int_{g_i(s)}^{t} R_u(t,u)a_i(u)du$. Then direct calculation yields

$$\begin{aligned}
&\int_{\lambda_i(\lambda(t))}^{\lambda(t)} |I_1(i,t,s)|ds = \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \frac{a(t)a_i(g_i(s))}{a(g_i(s))} e^{-\int_{g_i(s)}^{t} a(q)dq} ds \\
&- \frac{a(t)a_i(\lambda(t))}{a(\lambda(t))} \big(\lambda(t) - \lambda_i(\lambda(t))\big) e^{-\int_{\lambda(t)}^{t} a(q)dq} - a(t) \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \int_{\lambda(t)}^{g_i(s)} e^{-\int_u^t a(q)dq} \left(\frac{a_i(u)}{a(u)}\right)' duds \\
&= \frac{a(t)a_i(g_i(\lambda(t)))}{a^2(g_i(\lambda(t)))\dot{g}_i(\lambda(t))} e^{-\int_{g_i(\lambda(t))}^{t} a(q)dq} - \\
&- \frac{(1-\dot{\tau}_i(\lambda(t)))a(t)a_i(\lambda(t))}{a^2(\lambda(t))} e^{-\int_{\lambda(t)}^{t} a(q)dq} - a(t) \int_{\lambda_i(\lambda(t))}^{\lambda(t)} e^{-\int_{g_i(s)}^{t} a(q)dq} \left(\frac{a_i(g_i(s))}{a^2(g_i(s))\dot{g}_i(s)}\right)' ds \\
&- \frac{a(t)a_i(\lambda(t))}{a(\lambda(t))} \big(\lambda(t) - \lambda_i(\lambda(t))\big) e^{-\int_{\lambda(t)}^{t} a(q)dq} - a(t) \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \int_{\lambda(t)}^{g_i(s)} e^{-\int_u^t a(q)dq} \left(\frac{a_i(u)}{a(u)}\right)' duds
\end{aligned}$$

and

$$\begin{aligned}
\int_{\lambda(t)}^{\lambda_i(t)} |I_2(i,t,s)|ds &\leq \int_{\lambda(t)}^{\lambda_i(t)} \frac{a(t)a_i(g_i(s))}{a^2(g_i(s))\dot{g}_i(s)} \frac{d}{ds} \left(e^{-\int_{g_i(s)}^{t} a(q)dq}\right) + \\
&+ \int_{\lambda(t)}^{\lambda_i(t)} a(t) \int_{g_i(s)}^{t} e^{-\int_u^t a(q)dq} \left| \left(\frac{a_i(u)}{a(u)}\right)' \right| duds \\
&\leq \frac{a_i(t)(1-\dot{\tau}_i(t))}{a(t)} - \frac{a(t)a_i(g_i(\lambda(t)))}{a^2(g_i(\lambda(t)))\dot{g}_i(\lambda(t))} e^{-\int_{g_i(\lambda(t))}^{t} a(q)dq} - \\
&- \int_{\lambda(t)}^{\lambda_i(t)} e^{-\int_{g_i(s)}^{t} a(q)dq} \left(\frac{a(t)a_i(g_i(s))}{a^2(g_i(s))\dot{g}_i(s)}\right)' ds + \\
&+ \int_{\lambda(t)}^{\lambda_i(t)} a(t) \int_{g_i(s)}^{t} e^{-\int_u^t a(q)dq} \left| \left(\frac{a_i(u)}{a(u)}\right)' \right| duds
\end{aligned}$$

Where we used the fact that $\dot{g}_i(\lambda_i(t)) = \frac{1}{1-\dot{\tau}_i(t)}$. So, summing over i and canceling the common terms, the condition of Assumption 5 reduces to

$$1 - \frac{a(t)}{a(\lambda(t))} e^{-\int_{\lambda(t)}^{t} a(s)ds} - \sum_{i=1}^{m} \frac{a(t)a_i(\lambda(t))}{a(\lambda(t))} (\lambda(t) - \lambda_i(\lambda(t))) e^{-\int_{\lambda(t)}^{t} a(s)ds} + F(\dot{a}_i, \dot{\tau}_i) < 1$$

with F

$$\begin{aligned}
F := \sum_{i=1}^{m} &\left[\frac{a_i(t)\dot{\tau}_i(t)}{a(t)} - a(t) \int_{\lambda_i(\lambda(t))}^{\lambda_i(t)} e^{-\int_{g_i(s)}^{t} a(q)dq} \left(\frac{a(t)a_i(g_i(s))}{a^2(g_i(s))\dot{g}_i(s)}\right)' ds + \right. \\
&+ \frac{\dot{\tau}_i(\lambda(t))a(t)a_i(\lambda(t))}{a^2(\lambda(t))} e^{-\int_{\lambda(t)}^{t} a(q)dq} - a(t) \int_{\lambda_i(\lambda(t))}^{\lambda(t)} \int_{\lambda(t)}^{g_i(s)} e^{-\int_u^t a(q)dq} \left(\frac{a_i(u)}{a(u)}\right)' duds + \\
&\left. + \int_{\lambda(t)}^{\lambda_i(t)} a(t) \int_{g_i(s)}^{t} e^{-\int_u^t a(q)dq} \left| \left(\frac{a_i(u)}{a(u)}\right)' \right| duds \right]
\end{aligned}$$

to be a function with the property that $F(0,0) = 0$.

Remark 3. Let simplify further to $m = 1$, $a(t) \geq 0$ and $\tau(t) \equiv \tau$. Then for $a_t = \min_{s \in I_t} a(t)$

$$a(t) \int_{t-2\tau}^{t-\tau} e^{-\int_{s+\tau}^{t} a(y)dy} ds - a(t)\tau e^{-\int_{t-\tau}^{t} a(s)ds} = \frac{a(t)}{a_t}\left(1 - e^{-a_t \tau}\right) - a(t)\tau e^{-\int_{t-\tau}^{t} a(s)ds}$$

and consequently delay independent results occur for $\dot{a} \leq 0$. We speculate that such a result holds for $m > 1$ as well.

Remark 4. If $a_i \leq 0$ then $R(t,s) \equiv 1$, $I_1 = I_2 = I_3 = 0$ while $I_4 = -a_i(t)$. Consequently, the condition of Assumption 5 reduces to $\sum_{i=1}^{m} |a_i(t)\tau_i(t)| < 1$.

5. The General Model: Another Approach

In this section we generalize the results of Sect. 3 by considering the problem

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{m} -a_i(t)x(t) + a_i(t)x\big(\lambda_i(t)\big), t \geq t_0 \\ x(t) &= \phi(t), t \in I_{t_0}. \end{cases} \tag{5.1}$$

The method of the preceding Sections relies on the observation that either x or $\dot{x}$ have an integral of motion (Eqs. (3.3) and (4.4) respectively). Here we study the stability of the solutions of Eq. (5.1), by means of fixed point theory, but this time we follow another approach. We will separate the systems parameters $a_i(t)$'s as before and we will use only one form of the solutions in segments intervals of time in which Schauder's first fixed point theorem will be applied [9]. This way we will recover the estimates of [16]. Next we will extend to the overall dynamics using a variation of parameters formula. At first we re-write (5.1) as

$$\dot{x}(t) = f_1(t, x_t) + f_2(t, x_t) \tag{5.2}$$

where

$$f_1(t, x_t) = -\sum_{i=1}^{m} a_i^+(t)x(t) + \sum_{i=1}^{m} a_i^+(t)x(t - \tau_i(t))$$
$$f_2(t, x_t) = -\sum_{i=1}^{m} a_i^-(t)x(t) + \sum_{i=1}^{m} a_i^-(t)x(t - \tau_i(t))$$

and we will study the system in two steps. At the first step only the dynamics of

$$\dot{y}(t) = f_1(t, y_t) \tag{5.3}$$

are involved. At the second step, we will obtain convergence result for (4.1) as a whole by considering the stability of (5.2) where f_2 is a perturbation whenever $f_1 \not\equiv 0$. Otherwise the results of Theorem 4.1 are directly applied.

Inverting (5.3) from $\lambda(t)$ to t we see that y satisfies

$$y(t) = \phi(t, \lambda(t))y(\lambda(t)) + \int_{\lambda(t)}^{t} \phi(t,s) \sum_i a_i^+(s)y(\lambda_i(s))ds$$

Proposition 1. *Under Assumptions 2, 3 and 4, the solution $y(t, t_0, \phi)$ of (5.3) satisfies*

1. $y(s) \in M_{\lambda(t),y}$ *for* $s \in I_t$ *and*
2. $\max_{s \in I_t} |y(s) - y(\lambda(t))| \leq (1 - e^{-\Theta})^{l_t} |M_{t_0,\phi}|$

where $\Theta = \Theta_{t_0} := \sup_{t \geq t_0} \int_{t-\tau(t)}^{t} a(s)ds$, $l_t = l \in \mathbb{Z}_+$ *such that* $\lambda^{(2l)}(t) \geq t_0$.

Proof. We begin with the proof of the first part. Fix a time $t \geq t_0$ and observe that if $\tau(t) = 0$ then the result trivially follows. So for $\tau(t) \neq 0$ we consider the solution segment $y(s, t_0, \phi)$, $s \in I_{\lambda(t)}$. Consider the space

$$\mathbb{L}_1 = \{z \in C^0(I_{\lambda(t)} \cup I_t, \mathbb{R}) : z(s) = y(s),\ s \in I_{\lambda(t)} \text{ and } z(s) \in [\min_{q\in I_{\lambda(t)}} y(q), \max_{q\in I_{\lambda(t)}} y(q)], s \in I_t\}$$

and the operator

$$(\mathcal{Q}z)(s) = \begin{cases} y(s), s \in I_{\lambda(t)} \\ \phi(s,\lambda(t))y(\lambda(t)) + \sum_{i=1}^m \int_{\lambda(t)}^s \phi(s,q)a_i^+(q)z(\lambda_i(q))dq, s \in I_t. \end{cases}$$

Now, $\mathbb{L}_1$ is a set of continuous and bounded functions defined in $I_{\lambda(t)} \cup I_t$ equipped with the supremum norm. Since

$$w(s) = \begin{cases} y(s), s \in I_{\lambda(t)}, \\ \frac{\max_{q\in I_{\lambda(t)}} y(q) - y(\lambda(t))}{\tau(t)} s + \frac{y(\lambda(t))\cdot t - \max_{q\in I_{\lambda(t)}} y(q)\lambda(t)}{\tau(t)}, s \in I_t \end{cases}$$

is a member of $\mathbb{L}_1$ the set is non-empty and we can easily show that it is convex. Obviously $|\dot{w}(s)| \leq \frac{|M_{\lambda(t),y}|}{\tau(t)}$. Observe that $\mathcal{Q} : \mathbb{L}_1 \to \mathbb{L}_1$ because

$$(\mathcal{Q}z)(s) \leq \phi(s,\lambda(t)) \max_{q\in I_{\lambda(t)}} y(q) + (1 - \phi(s,\lambda(t))) \max_{q\in I_{\lambda(t)}} y(q) = \max_{q\in I_{\lambda(t)}} y(q)$$

and similarly for the lower bound.

Clearly, $\mathcal{Q}$ is continuous in $\mathbb{L}_1$ and $|\frac{d}{ds}(\mathcal{Q}z)(s)| \leq 2Am \max_{q\in I_{\lambda(t)}} y(q)$. The maximum of the two upper bounds of $|\dot{w}|$ and $|\frac{d}{ds}(\mathcal{Q}z)(s)|$ is taken to form for an equi-continuous subset of $\mathbb{L}_1$, denoted by $\mathbb{L}_1^*$, into which both w and $(\mathcal{Q}z)$ belong. This is then compact by the Arzela-Ascoli Lemma and still non-empty and convex so that finally Theorem 2.2 can be applied and $\mathcal{Q}$ admits a fixed point in $\mathbb{L}_1^*$.

We proceed now with the second part. For $\epsilon \in [0,1)$ consider the space

$$\begin{aligned} \mathbb{L}_2^\epsilon = \{z \in \mathbb{L}_1^* : &\max_{q\in I_{\lambda(t)}\cup I_t} |z(q) - z(\lambda(t))| = \max_{q\in I_{\lambda(t)}} |z(q) - z(\lambda(t))|, \\ &|z(s) - z(\lambda(t))| \leq \epsilon \max_{q\in I_{\lambda(t)}} |z(q) - z(\lambda(t))|,\ s \in I_t\}. \end{aligned}$$

and this is a convex subset of $\mathbb{L}_1^*$. We observe that

$$\begin{aligned} |(\mathcal{Q}z)(s) - z(\lambda(t))| &\leq \int_{\lambda(t)}^t \left|\phi(s,q)\sum_{i=1}^m a_i^+(q)\big(z(\lambda_i(q)) - z(\lambda(t))\big)\right| dq \\ &\leq (1 - e^{-\Theta}) \max_{q\in I_{\lambda(t)}\cup I_t} |z(q) - z(\lambda(t))| \end{aligned}$$

we set $\epsilon := (1 - e^{-\Theta})$. Clearly, $z(s) = y(s)|_{s\in I_{\lambda(t)}}$ identical for all $z \in \mathbb{L}_2^\epsilon$. Now, if the maximizer of $|\mathcal{Q}z(q) - y(\lambda(t))|$ is in $I_\lambda(t)$ it follows that $\mathcal{Q} : \mathbb{L}_2^\epsilon \to \mathbb{L}_2^\epsilon$. If not then the maximizer lies in I_t and then

$$\begin{aligned} \max_{q\in I_{\lambda(t)}\cup I_t} |(\mathcal{Q}z)(q) - z(\lambda(t))| &= \max_{q\in I_t} |(\mathcal{Q}z)(q) - z(\lambda(t))| \\ &\leq \epsilon \max_{q\in I_{\lambda(t)}\cup I_t} |z(q) - z(\lambda(t))| \\ &= \epsilon \max_{q\in I_{\lambda(t)}} |z(q) - z(\lambda(t))| \\ &= \epsilon \max_{q\in I_{\lambda(t)}} |(\mathcal{Q}z)(q) - z(\lambda(t))| \end{aligned}$$

but this is a contradiction. Using the same argumentation as above, $\mathcal{Q}$'s fixed point lies in $\mathbb{L}_2^\epsilon$. Furthermore, as $t \geq t_0$ is arbitrary we can repeat the same argument for $t = s$ where s is a maximizer of $z(s)$ in $I_{\lambda(t)}$ and so and so forth.

This recursive argumentation can be used at most $l = l_t$ times so that $\lambda^{(2l)}(t) \geq t_0$. By the first part of the proposition,

$$\max_{q \in I_{\lambda^{(2l)}(t)}} \left| z(q) - z(\lambda^{(2l-1)}(t)) \right| \leq |M_{t_0,\phi}|$$

concluding the proof of the second part. □

Now we are ready to state and prove the first main result of this section

Theorem 5.1. *Under Assumptions 2, 3 and 4, the solution $y(t, t_0, \phi)$ of 5.3 satisfies*

$$|y(t) - k| \leq (1 - e^{-\Theta})^{l_t} |M_{t_0,\phi}|$$

for some $k \in M_{t_0,\phi}$, $\Theta = \sup_t \int_{t-\tau(t)}^{t} a(s)ds$ and $l_t = l \in \mathbb{Z}_+ : \lambda^{(2l)}(t) \geq t_0$.

Proof. From the second part of Proposition 1, we can show that for an arbitrary sequence $\{t_n\}_{n\geq 1}$ with $n \to \infty, t_n \to \infty$, $y_n = y(t_n)$ is Cauchy. Indeed fix $\varepsilon > 0$, $N > 0$, $m, n \geq N$ such that $t_m > t_n > N$: From Proposition 1 we take $\lambda(t) = t_n$ so that for $s \in [t_n, \lambda^{-1}(t_n)]$ we have $|y(s) - y(t_n)| < |M_{t_0,\phi}|\gamma^{l_N}$. This procedure will be repeated l times until t_m is reached so that the admitted estimate is calculated

$$|y(t_n) - y(t_m)| \leq |M_{t_0,\phi}|\gamma^{l_N}(1 + \gamma + \gamma^2 + \cdots + \gamma^l) \leq |M_{t_0,\phi}| \frac{\gamma^{l_N}}{1-\gamma}$$

Then for N so large that $l_N > \frac{\log(\frac{\varepsilon(1-\gamma)}{M})}{\log(\gamma)}$ the result follows. Next, since t_n is arbitrary any two such sequences can form a new one which will be Cauchy and from this we conclude that $k := \lim_t y(t) \in \mathbb{R}$

From the first part of Proposition 1 we see that k must lie in $[\mu_1(t), \mu_2(t)]$ for μ_1, μ_2 as defined in it's proof. So for any t there exists $t^* \in I_{\lambda(t)}$ such that $y(t^*) = k$ and then from the second part of Proposition 1,

$$|y(t) - y(t^*)| = |y(t) - k| \leq (1 - e^{-\Theta}) \left| \max_{s \in I_t^{(2)}} y(s) - k \right|$$

Define the sequence $t_n = t, t_{n-1} = \text{argmax}_{s \in I_t^{(2)}} y(s)$ to obtain

$$|y(t) - k| = |y(t_n) - k| \leq (1 - e^{-\Theta}) \left| y(t_{n-1}) - k \right| \leq (1 - e^{-\Theta})^{l_t} |M_{t_0,\phi}|.$$

□

Remark 5. If $\tau(t) \leq \tau < \infty$ then $\lambda^{2l}(t) \geq t - 2l\tau$ and it suffices to take $l_t = \frac{t-t_0}{2\tau} - 1$ so that

$$|y(t) - k| \leq M_{t_0,\phi}(1 - e^{-\Theta})^{-1} e^{\frac{1}{2\tau} \ln(1-e^{-\Theta}) \cdot (t-t_0)}$$

Remark 6. The obtained rate estimates are the same with the ones obtained in [16] where a Lyapunov-Rhazumikhin type of argument was used.

Based on Theorem 5.1 we proceed now to consider the overall (5.1) and we establish sufficient conditions for the asymptotic stability of solutions with respect to Δ. For this we will need the following auxiliary result:

Lemma 5.2. *Let $\xi(t, u) : [t_0, \infty) \times [t_0, \infty) \to \mathbb{R}$, $f, g : [t_0, \infty) \to \mathbb{R}$ be integrable bounded functions with the properties that*

(1) $\lim_{t\to\infty} f(t) = 0$
(2) $\lim_{t\to\infty} \xi(t, u) = 0$ *for any fixed* $u \geq t_0$.
(3) $\sup_{t\geq t_0} \int_{t_0}^{t} |\xi(t, u)g(u)|du < \infty$.

Then

$$\int_{t_0}^{t} \xi(t,u)g(u)f(u)du \to 0 \text{ as } t \to \infty$$

Proof. Fix $\varepsilon > 0$ and set $B_1 = \sup_{t \geq t_0} \int_{t_0}^{t} |\xi(t,u)g(u)|du$ and $B_2 = \int_{t_0}^{t_1} |g(u)f(u)|du$. From Property 1, there exist $t_1 > t_0$ such that $|f(t)| < \frac{\varepsilon}{2B_1}$ for $t > t_1$ and $t_2 > t_1$ such that $q(t,s) < \frac{\varepsilon}{2B_2}$ for $s \leq t_1$ and $t \geq t_2$. Then for such t

$$\begin{aligned}\int_{t_0}^{t} \xi(t,u)g(u)f(u)du &< \int_{t_0}^{t_1} |\xi(t,u)g(u)f(u)|du + \int_{t_1}^{t} |\xi(t,u)g(u)f(u)|du \\ &< \frac{\varepsilon}{2B_2} \int_{t_0}^{t_1} |g(u)f(u)|du + \frac{\varepsilon}{2B_1} \int_{t_1}^{t} |\xi(t,u)g(u)|du < \varepsilon.\end{aligned}$$

For arbitrary $\varepsilon > 0$ the result follows. □

Remark 7. Notice that if $\xi(t,s) = \xi(t-s)$ for $\xi \in L^1$ then Lemma 5.2 is the classical result of Real Analysis that the convolution of an L^1 function with a function that goes to zero, vanishes as well.

The outcome of Theorem 5.1 implies the existence of a rate function $1/q(t,t_0)$ with the property that $q(t,t_0) \to 0$ for $t \to \infty$ for any fixed t_0 such that

$$|y(t) - k_0| \leq q(t,t_0)|M_{t_0,\phi}|$$

for some limit point $k_0 \in M_{t_0,\phi}$.

Theorem 5.3. *Consider the initial value problems (5.1) and (5.3) with solutions $x(t,t_0,\phi)$ and $y(t,t_0,\phi)$ respectively and the rate function $q(t,t_0)$ associated with $y(t,t_0,\phi)$. The following cases are considered:*

Case I: $a(t) \not\equiv 0$ and there exists a rate function h with the properties:

1. $\int_{t_0}^{\infty} \frac{|a_i^-(u)|}{h(u)} du < \infty$, $i = 1, \ldots, m$
2. $\sup_u \frac{h(\lambda_i(u))}{h(u)} < \infty$, $i = 1, \ldots, m$
3. $\sup_{t \geq t_0} h(t)q(t,t_0) < \infty$.
4. *There exists t^* such that*

$$\begin{aligned}\sup_{t \geq t^*} \Bigg\{ h(t) \sum_{i,j=1}^{m} \int_{t_0+\tau(t_0)}^{\lambda_i(t)} \int_{u}^{g_i(u)} \big(q(t,s) + q(t,g_i(u))\big)|a_i^-(s)| \left(\frac{|a_j(u)|}{h(u)} + \frac{|a_j(u)|}{h(\lambda_j(u))} \right) ds du \\ + h(t) \sum_{i,j=1}^{m} \int_{\lambda_i(t)}^{\infty} \int_{u}^{g_i(u)} (q(g_i(u),s) + 1)|a_i^-(s)| \left(\frac{|a_j(u)|}{h(u)} + \frac{|a_j(u)|}{h(\lambda_j(u))} \right) ds du \Bigg\} < 1\end{aligned}$$

Case II: $a(t) \equiv 0$ and there exists a rate function h with the properties

1. $\frac{1}{h} \in L^1$
2. $\sup_{t \geq t_0 + \tau(t_0)} h(t) \sum_{i=1}^{m} |a_i^-(t)| \int_{\lambda_i(t)}^{t} \frac{ds}{h(s)} \leq 1$.

Then the solution of (5.1) converges to a constant with rate $\frac{1}{h}$.

Proof. We begin with Case I. Denote by $y(t,t_0,\phi)$ be the unique solution of (5.3). Next, consider the fixed solution of (5.1) in $\tilde{I}_{t_0} := I_{t_0} \cup [t_0, \tilde{t}_0]$, $\tilde{t}_0 = t_0 + \tau(t_0)$, $\tilde{\phi}$ as initial condition. Given ϕ, growth estimates of $x \in [t_0, \tilde{t}_0]$ can be established with the use of Gronwal's inequality or with contraction mappings. This step will be omitted. Using the variations of constants formula for functional differential equations [4], the solution of (5.1) satisfies for $t \geq \tilde{t}_0$

$$
\begin{aligned}
x(t,t_0,\phi) &= T(t,\tilde{t}_0)\tilde{\phi} - \int_{\tilde{t}_0}^{t} T(t,s)\sum_{i=1}^{m} a_i^-(s)\int_{s-\tau_i(s)}^{s}\dot{x}(u)duds \\
&= T(t,\tilde{t}_0)\tilde{\phi} + \sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{t}\int_{s-\tau_i(s)}^{s} T(t,s)a_i^-(s)a_j(u)\big(x(u)-x(\lambda_j(u))\big)duds \\
&= T(t,\tilde{t}_0)r_0 + \sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{\lambda_i(t)}\int_{u}^{g_i(u)} T(t,s)a_i^-(s)a_j(u)\big(x(u)-x(\lambda_j(u))\big)dsdu \\
&\quad + \sum_{i,j=1}^{m}\int_{\lambda_i(t)}^{t}\int_{u}^{t} T(t,s)a_i^-(s)a_j(u)\big(x(u)-x(\lambda_j(u))\big)dsdu \\
&= R_1 + R_2 + R_3
\end{aligned}
\tag{5.4}
$$

where $T(t,s)$ is the linear operator defined in §2 and

$$
r_0 := \tilde{\phi} + \sum_{i,j=1}^{m}\int_{\lambda_i(\tilde{t}_0)}^{\tilde{t}_0}\int_{\tilde{t}_0}^{g_i(u)} T(t_0,s)a_i^-(s)a_j(u)\big(\tilde{\phi}(u)-\tilde{\phi}(\lambda_j(u))\big)dsdu
$$

For any $e(s)\in\mathbb{R}$, $T(t,s)e(s)$ represents the solution of (5.3) $y(t,s,\phi^*)$ with $\phi^*(s)=\delta(s-u)e(u)$, $u\in I_s$. Consequently $|k_s|\in[0,|e(s)|]$. Consider the space

$$
\mathbb{L}_3 = \big\{z\in C^0([\lambda(t_0),\infty),\mathbb{R}) : z(s)=\tilde{\phi}(s) \text{ if } s\in\tilde{I}_{t_0}, \sup_{t\geq\tilde{t}_0} h(t)|z(t)-k_z|\leq C, |k_z|\leq W\big\}
$$

Where h is the rate function with the properties in the statement of the Theorem and C,W are positive constants to be determined. Following the same steps as in the proof of Theorem 5.1 we can show that $\mathbb{L}_3$ is a non-empty, compact, convex subset of the space of continuous bounded functions in $[\lambda(t_0),\infty)$ with the supremum norm [20]. The only significant difference we need to take into consideration is that $\mathbb{L}_3$ is a family of functions defined on the infinite interval $[\lambda(t_0),\infty)$ so that the conventional Arzela-Ascoli lemma does not apply but a certain modification of it applies in view of the rate function h (see Theorem 1.2.2 of [20]). Define the operator

$$
(\mathcal{L}x)(t) = \begin{cases} \tilde{\phi}, \tilde{I}_{t_0} \\ x_{(5.4)}(t,t_0,\phi), t\geq\tilde{t}_0 \end{cases}
$$

where $x_{(5.4)}$ is the right hand side of (5.4). $\mathcal{L}$ is continuous in t and in x. Next we need to determine $k_{(\mathcal{L}z)}$. Given $z\in\mathbb{L}_3$, we have

$$
\begin{aligned}
&R_1 \to k_0 \\
&R_2 \to \sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{\infty} e(g_i(u))\int_{u}^{g_i(u)} T(g_i(u),s)a_i^-(s)a_j(u)\big(z(u)-z(\lambda_j(u))\big)dsdu \\
&R_3 \to 0
\end{aligned}
$$

where in R_2 we used the property $T(t,s)=T(t,g_i(u))T(g_i(u),s)$. Then,

$$
k_{(\mathcal{L}z)} = k_0 + \sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{\infty} e(g_i(u))\int_{u}^{g_i(u)} T(g_i(u),s)a_i^-(s)a_j(u)\big(z(u)-z(\lambda_j(u))\big)dsdu
$$

where the last integral is well defined in view of the properties of h and Assumption 4. Next it is desirable to establish a suitable relationship between C and W. If we set

$$W := |k_0| + C\sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{\infty}\int_{u}^{u+\tau_i(u)}|a_i^-(s)|\Big(\frac{|a_j(u)|}{h(u)} + \frac{|a_j(u)|}{h(\lambda_j(u))}\Big)dsdu$$

it is easy to see that $|k_{(\mathcal{L}z)}|$ is bounded by W. The final step for proving that $\mathcal{L} : \mathbb{L}_3 \to \mathbb{L}_3$, is to show that $\sup_{t\geq\tilde{t}_0} h(t)|z(t) - k_z| \leq C$, if C is appropriately chosen. Take any $t^* \geq t_0 + \tau(t_0 + \tau(t_0))$ and for $t \geq t^*$

$$\sup_{t\geq t^*} h(t)|(\mathcal{L}z)(t) - k_{(\mathcal{L}z)}| \leq I_1 + I_2C + I_3C$$

where

$$I_1 := h(t)q(t,\tilde{t}_0)JM_{\tilde{t}_0,\tilde{\phi}}$$

$$I_2 := h(t)\sum_{i,j=1}^{m}\int_{\tilde{t}_0}^{\lambda_i(t)}\int_{u}^{g_i(u)}\big(q(t,s) + q(t,g_i(u))\big)|a_i^-(s)|\Big(\frac{|a_j(u)|}{h(u)} + \frac{|a_j(u)|}{h(\lambda_i(u))}\Big)dsdu$$

$$I_3 := h(t)\sum_{i,j=1}^{m}\int_{\lambda_i(t)}^{\infty}\int_{u}^{g_i(u)}(q(g_i(u),s) + 1)|a_i^-(s)|\Big(\frac{|a_j(u)|}{h(u)} + \frac{|a_j(u)|}{h(\lambda_j(u))}\Big)dsdu$$

From the first and second conditions of the theorem we set

$$C := \frac{I_1}{1 - I_2 - I_3} < \infty$$

so that $\mathcal{L} : \mathbb{L}_3 \to \mathbb{L}_3$. The case $a \equiv 0$ is identical to the corresponding one of Theorem 4.1.

□

6. Examples

In this section, we will discuss the results of the previous sections with several illustrative examples. It is our goal to consistently compare the applicability and strength of the results especially those between §4 and §5. In the first part of this we will focus on one dimensional examples and in the second part we will discuss a two dimensional example that nearly lies within the scopes of this work, yet it merges with the discussion in the introduction.

6.1. One dimensional examples.

As a numerical application of Theorem 3.1 consider the following delayed equation

Example 1 (Stability bounds in LTI systems)**.** *Consider the initial value problem*

$$\begin{cases} \dot{x} = -1.2x(t) + 1.2x(t-\tau) + 2.3x(t) - 2.3x(t-\sigma), t \geq 0 \\ x(t) = \phi(t), t \in [-\max\{\tau,\sigma\}, 0] \end{cases}$$

We will discuss the asymptotic behavior of solutions with respect to τ and σ and we will compare the results of §3 and §5.

$\sigma = 0$: *In this case both methods yield delay independent results, so we will compare the estimates that Theorems 3.1 and 5.1 provide for $\tau = \{1, .., 10\}$.*

(1) Theorem 3.1 asks for

$$F(\tau, 0) = 1 - e^{-1.2\tau} - 1.2\tau e^{-1.2\tau} < 1$$

which is always satisfies and thus we have delay-independent exponential stability with respect to Δ with rate $\gamma = \gamma_{3.1}$ which satisfies

$$G(\tau, \gamma_{3.1}) = 1.2e^{\gamma_{3.1}\tau}\frac{1 - e^{-1.2\tau}}{1.2 - \gamma_{3.1}} - 1.2e^{-1.2\tau}\frac{e^{\gamma_{3.1}\tau} - 1}{\gamma_{3.1}} \leq 1.$$

(2) Theorem 5.1 calculates $\Theta = 1.2\tau$ *and from Remark 5*

$$|x(t) - k| \leq |M_{0,\phi}|(1 - e^{-1.2\tau})^{-1} e^{\frac{1}{2\tau}\ln(1-e^{-1.2\tau})t}$$

so that the estimated rate is $\gamma_{5.1} = \frac{1}{2\tau}\ln(1 - e^{-1.2\tau})$.

In Fig. 1b we compare the two curves $\gamma_{3.1} : G(\tau, \gamma_{3.1}) = 1$ *and* $\gamma_{5.1}$ *for* $\tau = 1, .., 10$. *We conclude that the estimates of Theorem 3.1 clearly outperform these of Theorem 5.1. Simulations for* $\tau = 5$ *indicate that, in general both estimates are still away from the numerically verified ones by an order of 10 (see Fig. (1c)).*

$\sigma > 0$: *In this case the convergence is guaranteed for small values of* σ. *Again we will compare the particular estimates that the Theorems provide.*

(1) Assumption 1 asks for $F(\tau, \sigma) < 1$ *where*

$$F(\tau, \sigma) := 1 - e^{-1.2\tau} - \Big[1.2\tau e^{-1.2\tau} + 2.3\sigma e^{-1.2\tau} - \\ - 1.9\big(1 - e^{-1.2\sigma} - e^{-1.2\tau} + e^{-1.2(\tau-\sigma)}\big)\Big]$$

In Fig. (1a) the function $\sigma = \sigma(\tau)$ *is plotted for the values such that* $F(\tau, \sigma(\tau)) = 1$ *which is the stability bounds of the system.*

(2) From Theorem 5.3 we identify $q(t, s) = e^{-\gamma_{5.1}(t-s)}$ *with* $\gamma_{5.1} = \gamma_{5.1}(\tau) = -\frac{1}{2\tau}\ln(1 - e^{-1.2\tau})$ *and rate that is estimated by Theorem 5.1, so that condition 4 of Theorem 5.3 requires*

$$\frac{2.76(1 + e^{\gamma_{5.3}\tau}) + 5.29(1 + e^{\gamma_{5.3}\sigma})}{\gamma_{5.1}(\gamma_{5.1} - \gamma_{5.3})}\big(e^{\gamma_{5.1}\sigma} - 1 + \sigma e^{\gamma_{5.1}\sigma}\big) < 1 \tag{6.1}$$

which is always satisfied for any τ *and for* σ *small enough.*

As a numerical application take $\tau = 1$. *Fig. (1a) yields stability for* $\sigma \leq 0.325$ *so we fix* $\sigma = 0.32$. *Then condition (3.5) of Theorem 3.1 yields a rate equal to* $\gamma_{3.1} = 0.175$ *whereas Eq. (6.1) from Theorem 5.3 cannot reach this estimate. Numerical inspection of this condition gives for* $\sigma \approx 0.001$ *an estimate* $\gamma_{(5.3)} \approx 0.071$. *Fig. (1d) has the solution for* $\tau = 1, \sigma = 0.32$ *and the estimated rate is* 0.4.

Example 2. *Consider the time-varying system*

$$\begin{cases} \dot{x} = -a(t)x(t) + a(t)x(t - \tau(t)) + b(t)x(t) - b(t)x(t - \sigma(t)), t \geq t_0 \\ x(t) = \phi(t), t \in [t_0 - \tau(t_0), t_0] \end{cases}$$

under the following cases:

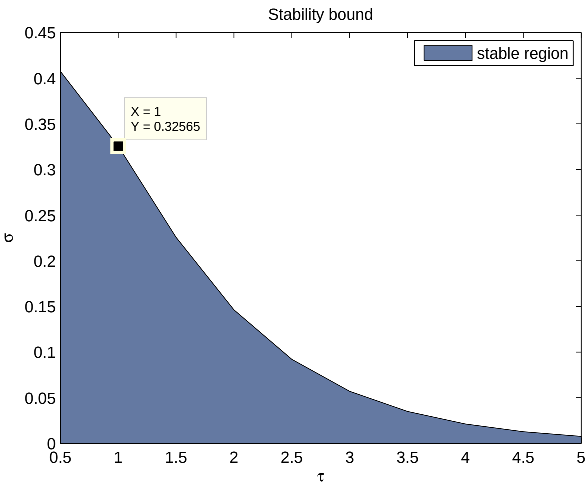
(i) $t_0 = 0$, $a(t) = \max\{0, \sin t\}$, $b \equiv 0$, $\tau(t) = \tau + 0.9\cos t$ *for* $\tau \geq 1$, *then* $\lambda(t) = t - \tau(t)$.

We begin with the application of Theorem (5.1). For this we numerically calculate $\Theta(\tau) = \sup_{t\geq 0}\int_{\lambda(t)}^{t} a(s)ds$ *and consequently the rate* $\bar{q}(\tau)$ *as it is estimated from Remark 5 (Figs. 2a and 2b respectively).*

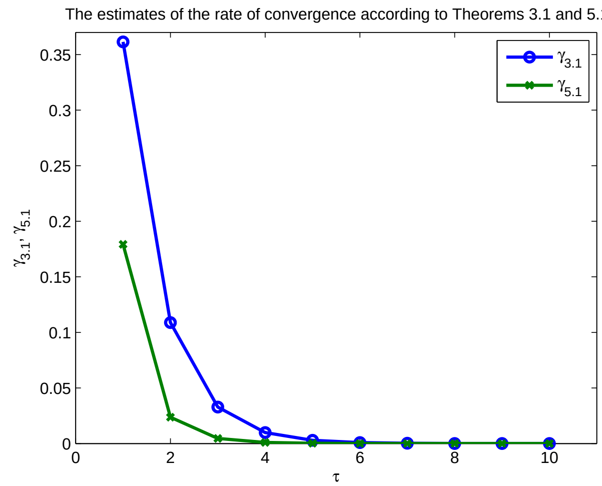
Next we will compare the result with this Theorem 4.1. Assumption 5 reads

$$F(\tau) = \sup_t F(t, \tau) < 1$$

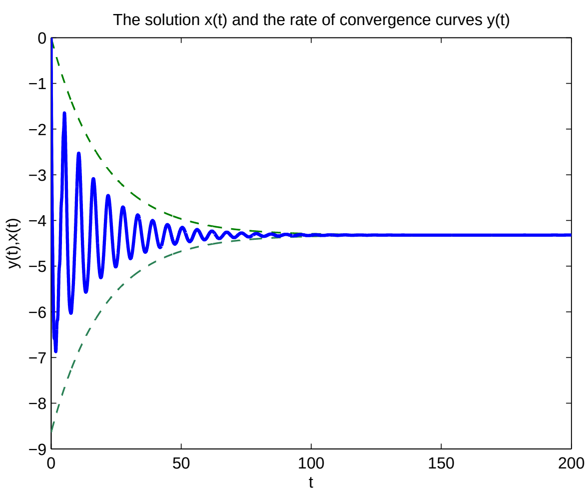
with $F(t, \tau) = a(t)\int_{\lambda^{(2)}(t)}^{\lambda(t)} e^{-\int_{g(s)}^{t} a(w)dw} ds - a(t)\big(\lambda(t) - \lambda^{(2)}(t)\big)e^{-\int_{\lambda(t)}^{t} a(w)dw}$ *and* $\lambda(t) = t - \tau - 0.9\cos(t)$. *Numerical calculation of* $F(\tau)$ *yields the values in Fig. 3 where we see that the condition is violated for* $\tau \geq 4$ *and we conclude that the results of this method are not delay-independent. We*



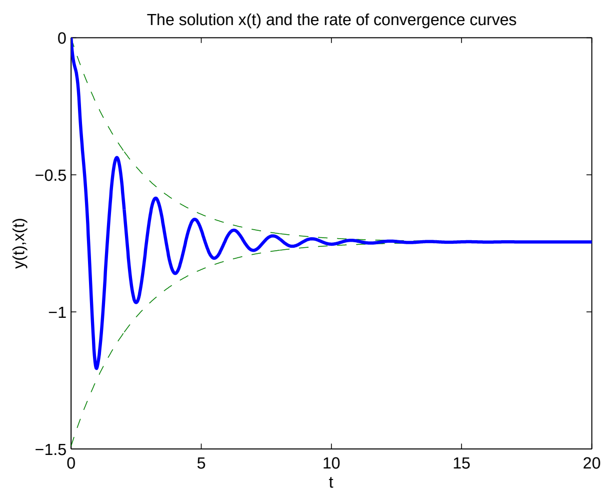
(A)



(B)

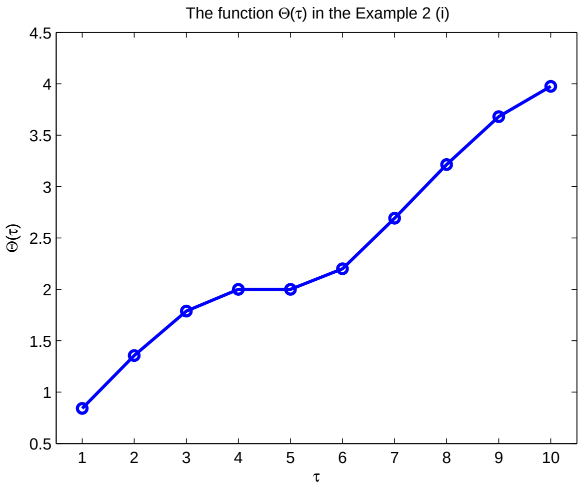


(C)

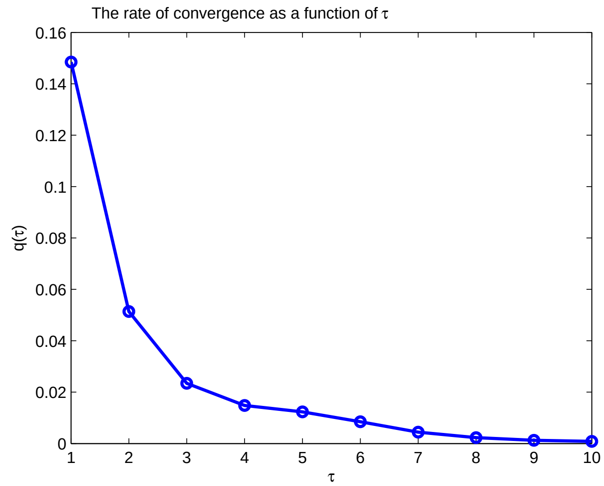


(D)

FIGURE 1. Numerical investigations for Example 1. (A) $\sigma \neq 0$. and the stability curve $\sigma = \sigma(\tau)$. The shaded region is the region of stability i.e. $(\sigma < \sigma(\tau))$. (B) The rate estimates between Theorem 3.1 (γ_1) and Theorem 5.1 (γ_2). (C) The solution of Example 1 with $\sigma = 0, \tau = 5$ and $\phi(s) = \sin(8s) + 2s$, $s \in [-5, 0]$ and the $(t, y(t))$ curve defined by $y(t) = \pm(x(0) - k)e^{-0.05t} + k$. (D) The solution of Example 1 with $\tau = 1, \sigma 0.32$ and identical initial datum as in (C). The rate curves $(t, y(t))$ are defined by $y(t) = \pm(x(0) - k)e^{-0.4t} + k$. Both simulations were carried through with MATLAB and the `ddesd` function.



(A)



(B)

FIGURE 2. Numerical investigations for Example 2(i) based on Theorem 5.1. The results are delay-independent but the estimated rates are very conservative (see also Table 1).

report, however, that further calculations for the estimation of the rate of convergence for $\tau = 1, \dots, 4$, based on Theorem 4.1, reveals improved in connection with the ones obtained with Theorem 5.1 (see Table 1). Simulation of the solution $x(t)$ is depicted in Figs. 4a with $\tau = 1$ and 4b with $\tau = 4$. We see that the numerically obtained rate estimates are close to the theoretically calculated ones from Theorem 4.1 and when $\tau = 1$.

TABLE 1. Rate Estimates for Example 2(i). $\gamma_{refthm2}$ stands for the estimates obtained via Theorem 4.1 and $\gamma_{5.1}$ stands for the estimates obtained via Theorem 5.1

Delay	*Rate* $\gamma_{4.1}$	*Rate* $\gamma_{5.1}$
1	*0.540*	*0.150*
2	*0.144*	*0.056*
3	*0.079*	*0.022*
4	*0.062*	*0.016*

(ii) $t_0 = 0$, $a(t) = \sin t$, $b(t) = b(1 + 0.3\cos t)$, *and* $\sigma(t), \tau(t)$ *are positive, bounded, invertible functions of time and classify* $\tau(t) = \tau_1$*, whenever* $a(t) > 0$ *and* $\tau(t) = \tau_2$ *otherwise. In this case we will exclusively apply the results of §5. Given* $\gamma > 0$ *the problem is to derive a sufficient condition on* $\tau_1(t), b(t), \tau_2(t), \sigma(t)$ *so as to guarantee convergence to a constant value exponentially fast with rate* γ*. Recall the notation* $a^+(t)$, $a^-(t)$*. At first, we calculate the kernel* $q(t,s)$ *that occurs from the study of the system*

$$\dot{x} = -a(t)x(t) + a(t)x(t - \tau(t))$$

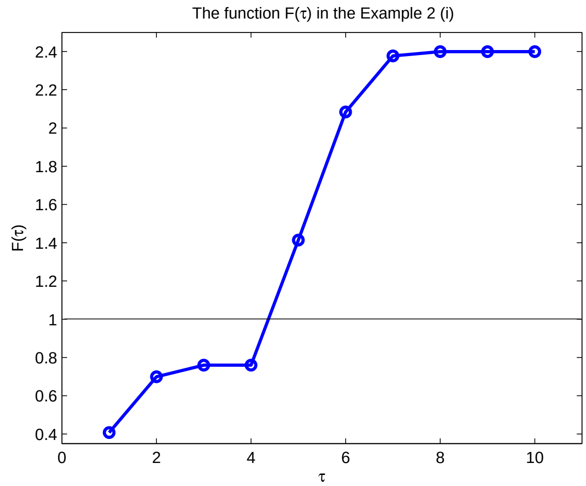
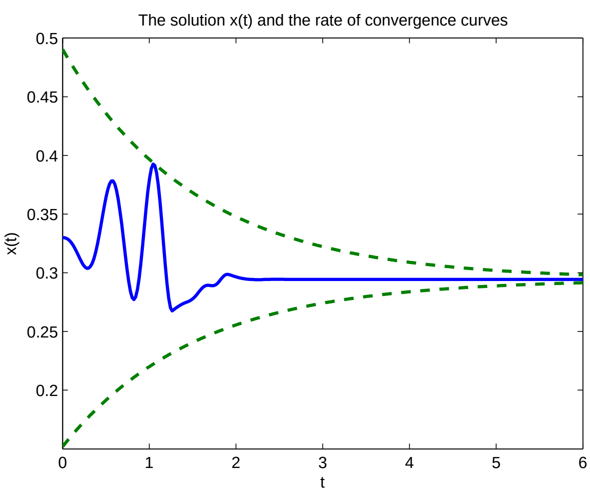
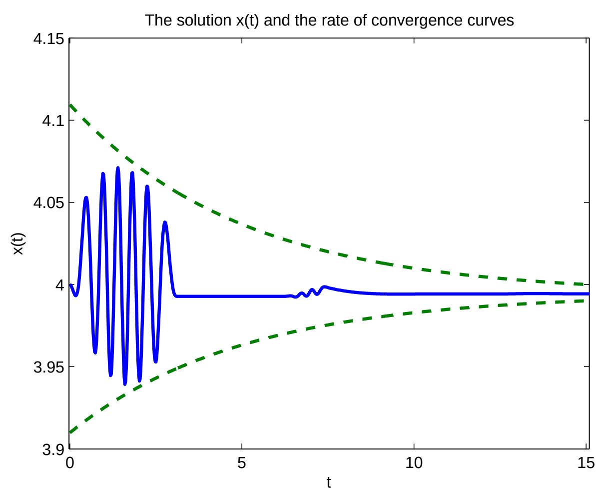


FIGURE 3. Numerical investigations for Example 2. The application of Theorem 4.1 does not yield delay independent results contrary to Theorem 5.1.



(A)



(B)

FIGURE 4. Numerical investigations for Example 2. (A) $\tau = 1$: The rate curves $(t, y(t))$ are defined by $y(t) = \pm(x(0)-k)e^{-0.65t}+k$. (B) $\tau = 4$: The rate curves $(t, y(t))$ are defined by $y(t) = \pm(x(0)-k)e^{-0.2t} + k$.

Note that

$$\sup_{t\geq 0} \int_{t-\tau(t)}^{t} a^{+}(s)ds \leq \bar{\tau}_1$$

Then

$$|x(t) - k_1| \leq |M_{0,\phi}|(1 - e^{-\bar{\tau}_1})^{-1} \cdot e^{-qt}$$

with

$$q := -\frac{\ln(1-e^{-\bar{\tau}_1})}{2\bar{\tau}_1}$$

so that $q(t,s) = (1-e^{-\bar{\tau}_1})^{-1}e^{-q(t-s)}$. *Next, for* $h(t) = e^{\gamma t}$, *we see that Properties (1)-(3) from Theorem 5.3 are satisfied and Property (4) yields the condition*

$$\Big[(e^{\gamma\bar{\tau}}+1)+b(e^{\gamma\bar{\sigma}}+1)\Big]\Big[(1-e^{-\bar{\tau}_1})^{-1}\frac{b(e^{q\bar{\sigma}}-1)+(e^{q\bar{\tau}_2}-1)}{q(q-\gamma)}+$$
$$+(1-e^{-\bar{\tau}_1})^{-1}\frac{b\bar{\sigma}+\bar{\tau}_2}{q-\gamma}+(1-e^{-\bar{\tau}_1})^{-1}\frac{b(1-e^{-q\bar{\sigma}})+(1-e^{-q\bar{\tau}_2})}{q\gamma}+\frac{b\bar{\sigma}+\bar{\tau}_2}{\gamma}\Big]<1$$

This condition is fulfilled if $\gamma < q$ *and* $b, \bar{\sigma}, \bar{\tau}_2$ *are sufficiently small.*

As a numerical application, take $a(t) = \sin t$, $\bar{\tau}_1 = 1$ *so that* $q = 0.23$, $(1-e^{-1})^{-1} = 1.59$ *and convergence is achieved with rate* $\gamma = \frac{q}{2}$ *if*

$$b \le 0.05,\ \bar{\sigma} \le 0.02,\ \bar{\tau}_2 \le 0.01.$$

(iii) $a(t) \equiv 0$. *In this case Theorems 4.1 and 5.3 coincide. We have no dissipation and the sufficient condition from the second part of Theorem 5.3 is*

$$\sup_t \frac{b(t)(e^{\gamma\sigma(t)}-1)}{\gamma} \le 1$$

(iv) $t_0 = 1$, $a(t) = \frac{1}{t+4}$, $\tau(t) = \frac{t}{4}+1$, $b(t) \equiv \frac{b}{t+5}$, $\sigma(t) = \sigma$. *In this example we have unbounded delays so the rate function we will try is going to be sub-exponential. We also ask* $\sigma < \frac{t_0}{4}+1$.

(a) For the results of §4 we choose $h(t) = (t+h_0)^\gamma$ *and we see that Theorem 4.1 imposes the condition:*

$$A(\gamma,b,\sigma) := \sup_{t\ge t_0}\big(A_1+A_2+A_3+A_4\big)(t) \le 1$$

where

$$A_1 = \frac{(t+h_0)^\gamma}{(t+4)^2}\int_{\frac{9}{16}t-\frac{7}{4}}^{\frac{3}{4}t-1}\left(\frac{4}{3}s+\frac{7}{3}-\frac{3}{4}t\right)(s+h_0)^{-\gamma}ds$$

$$A_2 = b\frac{(t+h_0)^\gamma}{(t+4)^2}\int_{\frac{3}{4}t-1-\sigma}^{\frac{3}{4}t-1}\left|\frac{3}{4}t-s-\sigma-1+\log\left(\frac{s+\sigma+5}{\frac{3}{4}t+4}\right)\right|(s+h_0)^{-\gamma}ds$$

$$A_3 = b\frac{(t+h_0)^\gamma}{(t+4)^2}\int_{\frac{3}{4}t-1}^{t-\sigma}\left|\sigma-\log\left(\frac{s+\sigma+5}{s+5}\right)\right|(s+h_0)^{-\gamma}ds$$

$$A_4 = b(t+h_0)^\gamma\int_{t-\sigma}^{t}\left|\frac{1}{t+5}-\left(t-s+\log\left(\frac{s+5}{t+5}\right)\right)\right|(s+h_0)^{-\gamma}ds$$

Let, initially, $b = 0$. *Then for* $t_0 = 1$ *we must choose* $h_0 > 19/16$, *say* $h_0 = 5/4$ *and* $A(\gamma,0,0) \le$ *for* $\gamma \le 3.21$. *Consequently the rate of convergence to a constant is of order* $t^{-2.21}$. *For* $b = 0.67, \sigma = 1$ *we calculate* $A(\gamma, 0.67, 1) = 2.4$ *so that the rate is of order* $t^{-1.4}$.

(b) Assumption 4 is satisfied since

$$\sup_{t\ge t_0}\int_{\frac{t}{4}-1}^{t} a(s)ds = \ln 4$$

Application of Theorem 5.1 requires to determine $l : \lambda^{2l}(t) \ge t_0$. *For* $t \ge t_0$ $\lambda^{2l}(t) = \left(\frac{3}{4}\right)^{2l}t - 4 + 3\left(\frac{3}{4}\right)^{2l-1}$ *so that*

$$l \le \frac{1}{2}\log_{3/4}\frac{t_0+4}{t+4}$$

so that the integer part is lower bounded by $\frac{1}{2}\log_{3/4}\frac{t_0+4}{t+4} - 1$*. Finally Theorem 5.1 gives the estimate*

$$|x(t) - k_0| \leq 1.34|M_{t_0,\phi}|\sqrt{\frac{t_0+4}{t+4}}.$$

We apply Theorem 5.3 with a candidate rate function $h(t) = (t+1)^\gamma$ *with* $\gamma < 0.5$*. We take* $a^-(t) = \frac{b}{t+5}$ *and verify that the Properties* $(1)-(3)$ *of the Theorem for such* h *are satisfied. Then for Property* (4) *we have for* $\sigma < \frac{t_0}{4}+1$

$$1.34(t+1)^\gamma \int_{\frac{5t_0}{4}+1}^{t-\sigma}\int_u^{u+\sigma}\left(\sqrt{\frac{s+4}{t+4}} + \sqrt{\frac{u+\sigma+4}{t+4}}\right)\frac{b}{s+5}ds\times$$
$$\left[\frac{1}{u+4}\left((u+1)^{-\gamma} + (u/4)^{-\gamma}\right) + \frac{b}{u+5}\left((u+1)^{-\gamma} + (u-\sigma+1)^{-\gamma}\right)\right]du+$$
$$(t+1)^\gamma\int_{t-\sigma}^{\infty}\int_u^{u+\sigma}\left(1.34\sqrt{\frac{s+4}{u+\sigma+4}}+1\right)\frac{b}{\sigma+5}ds\times$$
$$\left[\frac{1}{u+4}\left((u+1)^{-\gamma} + (u/4)^{-\gamma}\right) + \frac{b}{u+5}\left((u+1)^{-\gamma} + (u-\sigma+1)^{-\gamma}\right)\right]du$$

which is bounded by

$$J(t,\gamma,b,\sigma) := 1.34b(t+1)^\gamma\left[\int_{\frac{5t_0}{4}+1}^{t-\sigma} F_1(u,\sigma)\Gamma(u,\gamma,\sigma,b)du + \int_{t-\sigma}^{\infty} F_2(u,\sigma)\Gamma(u,\gamma,\sigma,b)du\right]$$

where

$$F1(u,\sigma) = \left(\frac{2(\sqrt{u+4+\sigma} - \sqrt{u+4})}{\sqrt{t+4}} + \sqrt{\frac{u+\sigma+4}{t+4}}\log\frac{u+\sigma+5}{u+5}\right)$$
$$F2(u,\sigma) = \left(\frac{2.78(\sqrt{u+4+\sigma} - \sqrt{u+4})}{\sqrt{u+\sigma+4}} + \log\frac{u+\sigma+5}{u+5}\right)$$
$$\Gamma(u,\gamma,\sigma,b) = \frac{(u+1)^{-\gamma} + (u/4)^{-\gamma}}{u+4} + \frac{b\big((u+1)^{-\gamma} + (u-\sigma+1)^{-\gamma}\big)}{u+5}$$

As a numerical application we take $t_0 = 1$ $b = 0.67$*,* $\sigma = 1$ *so that numerical integration with the trapezoidal rule of* $J(t, 0.2, 0.67, 1)$*, yields a supremum over* $t \geq 1$ *at* 0.81*.*

Example 3 (Direct Linearization of a non-linear system)**.** *In this example we will show how our results can be applied in the study of nonlinear systems*

$$\begin{cases}\dot{x} = a(t)f\big(x(t-\tau(t)) - x(t)\big), t \geq t_0\\ x(s) = \phi(s), s \in I_{t_0}\end{cases}$$

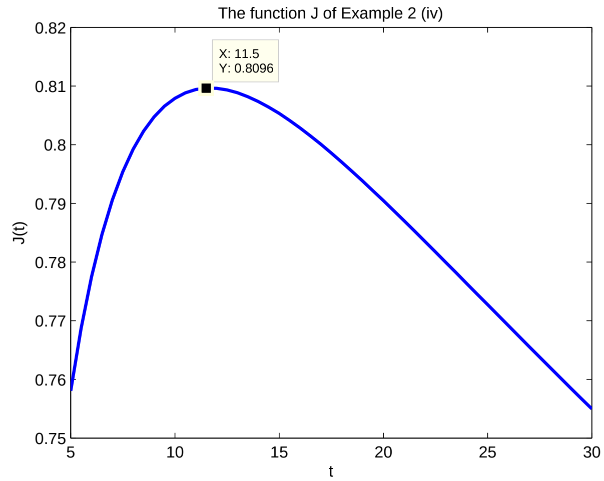
where $0 \leq a(t) \leq a_+$ *,* $\tau(t) \leq \tau$ *and* f *satisfies* $\frac{f(q)}{q} > 0$ *for* $q \neq 0$ *and* $|f(q)| \leq |q|$*.*

Claim 1. *The solutions of the system satisfy* $x(t) \in M_{t_0,\phi}$*.*

Proof. Let the first time $t^* \geq t_0$ such that $x(t^*) = \max_{s\in I_{t_0}}\phi(s)$ with $\dot{x}(t^*) > 0$. But $\dot{x}(t^*) = a(t^*)f\big(x(t^* - \tau(t^*)) - x(t^*)\big) < 0$ by virtue of the property of f, a contradiction. Similarly for the lower bound. □

Given the unique solution x *of the system* $\dot{x} = f(t, x_t)$ *we define the linear system*

$$\dot{z} = a(t)c(t)\big(z(t-\tau(t)) - z(t)\big)$$



(A)

FIGURE 5. Example 2

with $c(t) = c(x(t)) := \frac{f(x(t-\tau(t))-x(t))}{x(t-\tau(t))-x(t)} > 0$*. We observe that the solution* z *is indistinguishable from* x*. From Claim 1 we conclude that* $c_M := \sup_{x\in M_{t_0,\phi}} \frac{f(x)}{x} > 0$ *is well defined and thus* $c(t) \geq c_M$*. Then Theorem 5.1 can be applied with remark 5 and we get the estimate:*

$$|z(t) - k| \leq M_{t_0,\phi}(1 - e^{-c_M a\tau})^{-1} e^{\frac{1}{2\tau}\ln(1-e^{-c_M a\tau})(t-t_0)}.$$

We can extend this example to include negative $a(t)$ *and work similarly to the case (1) if we ask* f *to grow sub-linearly, e.g.* $f(x) = \sin(x)$*.*

6.2. **Two-dimensional delayed consensus.** This last example belongs to the family of problems which motivated this work. This is a cooperative system of two autonomous agents which exchange information with a constant uniform delay

$$\begin{cases} \dot{x}_1 &= -ax_1(t) + ax_2(t-\tau) \\ \dot{x}_2 &= -bx_2(t) + bx_1(t-\tau) \end{cases}$$

for $t \geq 0$ and $\mathbf{x}(s) = \boldsymbol{\phi}(s), s \in [-\tau, 0]$, with $a + b \neq 0$. This is the system introduced in Eq. (1.10). We will apply the results of Theorem 3.1 appropriately modified for the two-dimensional case. In vector form, the system reads

$$\dot{\mathbf{x}}(t) = -D\mathbf{x}(t) + A\mathbf{x}(t-\tau)$$

where

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

The solution $\mathbf{x}$ satisfies

$$\mathbf{x}(t) = e^{-D\tau}\mathbf{x}(t-\tau) + \int_{t-\tau}^{t} e^{-D(t-s)} A\mathbf{x}(s-\tau) ds \tag{6.2}$$

which is equivalent to Eq. (3.2). Also, if we add and subtract the terms $ay(t)$ and $bx(t)$ in each subsystem, we obtain

$$\dot{x}_1 = -ax_1(t) + ax_2(t) - a\frac{d}{dt}\int_{t-\tau}^{t} x_2(s)ds$$
$$\dot{x}_2 = -bx_2(t) + bx_1(t) - b\frac{d}{dt}\int_{t-\tau}^{t} x_1(s)ds$$

which in vector form reads

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t) - A\frac{d}{dt}\int_{t-\tau}^{t} \mathbf{x}(s)ds$$

where

$$L := \begin{bmatrix} a & -a \\ -b & b \end{bmatrix} = D - A$$

Remark 8. The matrices D, A, L are the Valency, the Adjacency and the Laplacian matrices respectively. These matrices are the main tools for matrix representation of graphs. Their use is fundamental in Algebraic Graph Theory [23]. The eigenvalues of L are $\lambda_1 = 0$ and $\lambda_2 = a + b$ and the left eigenvector of L with respect to λ_1 is $\mathbf{q}^T = (\frac{b}{a+b}, \frac{a}{a+b})$. Direct calculation of e^{-Lt} yields gives

$$e^{-Lt} = I - L\frac{1}{a+b}(1 - e^{-(a+b)t}) \Rightarrow e^{-Lt}\mathbf{z} \to \mathbf{q}^T\mathbf{z}$$

and

$$e^{-Lt}L = Le^{-(a+b)t}$$

Now, using the variation of constants and integration by parts formulae we see that $\mathbf{x}(t)$ must satisfy

$$\begin{aligned}\mathbf{x}(t) &= e^{-Lt}\boldsymbol{\phi}(0) - \int_0^t e^{-L(t-s)}A\frac{d}{ds}\int_{s-\tau}^{s}\mathbf{x}(w)dwds \\ &= e^{-Lt}\mathbf{r}_0 - A\int_{t-\tau}^{t}\mathbf{x}(s)ds + \int_0^t e^{-L(t-s)}LA\int_{s-\tau}^{s}\mathbf{x}(w)dwds\end{aligned} \tag{6.3}$$

where $\mathbf{r}_0 = \big(\boldsymbol{\phi}(0) + A\int_{-\tau}^{0}\boldsymbol{\phi}(s)ds\big)$ which is equivalent to Eq. (3.3). Then we combine (6.3) and (6.2) in a similar manner to (3.3) and (3.2) and obtain

$$\begin{aligned}\mathbf{x}(t) = e^{-D\tau}e^{-L(t-\tau)}\mathbf{r}_0 &+ \int_{t-2\tau}^{t-\tau}\Big(e^{-D(t-\tau-s)} - e^{-D\tau}\Big)A\mathbf{x}(s)ds + \\ &+ e^{-D\tau}\int_0^{t-\tau} e^{-L(t-\tau-s)}LA\int_{s-\tau}^{s}\mathbf{x}(w)dwds\end{aligned} \tag{6.4}$$

For the uniques solution $\mathbf{x} = \tilde{\mathbf{x}}$ of the system in $[-\tau, \tau]$ we define the space

$$\mathbb{M}^* = \big\{\mathbf{y} \in C^0([-\tau,\infty),\mathbb{R}^2) : \mathbf{y} = \tilde{\mathbf{x}}|_{[-\tau,\tau]}, \sup_{t\geq\tau} e^{\gamma t}|\mathbf{y}(t) - \mathbb{1}k| < \infty\big\}$$

for $|\mathbf{y}| = \max_i |y_i|$ the $p = 1$ norm and $\mathbb{1}k = (k, k)^T$. This is the space of continuous functions that take value in $\mathbb{R}^2$ and converge to a common value k at least as slow as $e^{-\gamma t}$. Together with the metric

$$\rho(\mathbf{y}_1, \mathbf{y}_2) = \sup_{t\geq\tau} e^{\gamma t}\big|\mathbf{y}_1(t) - \mathbf{y}_2(t)\big|$$

the pair $(\mathbb{M}^*, \rho)$ constitutes a complete metric space. We define the solution operator

$$(\mathcal{S}\mathbf{x})(t) = \begin{cases}\tilde{\mathbf{x}}, t \in [-\tau, \tau] \\ \mathbf{x}_{(6.4)}(t), t \geq \tau\end{cases}$$

At first we prove that $\mathcal{S} : \mathbb{M}^* \to \mathbb{M}^*$. This is true if $0 < \gamma < \min\{a, b\}$ for $\min\{a, b\} > 0$, else $\gamma < a$ and

$$k = \frac{b\phi_1(0) + a\phi_2(0) + ab\int_{-\tau}^{0}(\phi_1(s) + \phi_2(s))ds}{a + b + 2ba\tau}$$

Next we show that $\mathcal{S}$ can be a contraction for γ small enough: For and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^*$ we take the difference

$$e^{\gamma t}|(\mathcal{S}\mathbf{x}_1)(t) - (\mathcal{S}\mathbf{x}_2)(t)|$$

which is upper bounded by

$$\int_{t-2\tau}^{t-\tau}\left|\left(e^{-D(t-\tau-s)} - e^{-D\tau}\right)A\mathbf{x}_{12}(s)\right|ds + e^{-D\tau}\int_{0}^{t-\tau}\left|e^{-L(t-\tau-s)}LA\int_{s-\tau}^{s}\mathbf{x}_{12}(w)dw\right|ds$$

where $\mathbf{x}_{12} := \mathbf{x}_1 - \mathbf{x}_2$. With careful algebra (see also Remark (8)) we get that the first row is bounded by

$$ae^{\gamma\tau}\frac{1 - e^{-a\tau}}{a - \gamma} - e^{-a\tau}\frac{e^{\gamma\tau} - 1}{\gamma} + e^{-a\tau}\frac{e^{\gamma\tau} - 1}{\gamma}\frac{b(a + b)}{a + b - \gamma}e^{\gamma\tau}$$

and the second row is bounded by

$$be^{\gamma\tau}\frac{1 - e^{-b\tau}}{b - \gamma} - e^{-b\tau}\frac{e^{\gamma\tau} - 1}{\gamma} + e^{-b\tau}\frac{e^{\gamma\tau} - 1}{\gamma}\frac{a(a + b)}{a + b - \gamma}e^{\gamma\tau}$$

modulo $\rho(\mathbf{x}_1, \mathbf{x}_2)$. Then a small γ to make $\mathcal{S}$ a contraction in $\mathbb{M}^*$ can always be found since $\gamma \downarrow 0$ drives these bounds down to

$$1 - e^{-a\tau} \text{ and } 1 - e^{-b\tau}$$

respectively.

7. Discussion and Concluding Remarks

In this paper we studied the stability of solutions for the general class of linear differential equations with multiple time-varying delays that exhibit asymptotically constant solutions, purely by means of fixed point theory. We approached the problem with two different strategies:

I. In §3 and §4 our results are based on the existence of multiple forms of solutions which we combined into a new one. In §3, the existence of an integral of motion (Eq. (1.11)) allows for combining two forms of the solution to obtain a new one which will serve as a solution operator that is a self-mapping contraction in a suitable metric space. Then the Contraction Mapping Principle is applied to ensure the existence and uniqueness of a point in the metric space with the desired stability properties. Our contraction condition is general and it includes both positive and negative a_i's whereas the asymptotic constant has a closed form that depends on the parameters and the initial datum. In particular, we show that whenever a_i's are non-negative, delay-independent result are obtained and the rate of convergence is exponential with an estimate to be explicitly determined from a_i's and τ_i's.

   The approach we followed in §4 is a generalization of that in §3 but concerns the dynamic behavior of $\dot{x}$ and the rate at which it vanishes, since x converges to a real constant provided $\int^{\infty}\dot{x}(s)ds$ exists. We showed that $\dot{x}$, satisfies both an integrodifferential equation together with an integral of motion. Consequently the new solution expression is obtained with the use of resolvent functions from the theory of integrodifferential equations [25]. Contrary to §3 and the existing literature the results of this section are not clearly delay-independent when simply $a_i(t) \geq 0$ as the imposed conditions rather ask for $a_i > 0$ and $\dot{a}_i \leq 0$. It is still an open problem to improve the first method to the point of delay independent results whenever the weights $a_i(t)$ are non-negative. We suspect that a different combination of the solutions may be the answer.

II. In view of this drawback in §5 we studied the same problem following a different approach. We begun with the study of the sub-system of positive $a_i(t)$'s and recover the same rate estimates [16] by means of fixed point theory. Based on this result we implemented again fixed point theory methods for the stability analysis of the overall system, after deriving a solution operator derived from the variation of parameters formula for functional differential equations [4]. Contrary to §4, Schauder's first fixed point theorem was applied.

As a supplementary remark on the second method we note that a standard advantage of fixed point theory over the Lyapunov is that, instead of the functionals of Lyapunov theory, fixed point methods rely on non-trivial expressions of the solutions (usually obtained through variation of constants formulae) to be used as operators. Hence, mild sufficient conditions for stability may be obtained as it is only necessary to regard the average behavior of the system's parameters, contrary to the stricter pointwise conditions, the Lyapunov methods impose. This advantage can be easily illustrated in systems where the un-delayed part dominates the delayed one, as for example in Eq. (1.2). In this work, we examined a system that does not meet this convenient characteristic. Yet separating the dissipative from the non-dissipative dynamics (as in Eq. (5.2)) we were able to create a similar argument by means of stability in variation.

Conclusively, each approach attains their advantages and disadvantages. For the first one, we see that in the simple case of time-invariant dynamics the results are remarkably satisfying. However, in the case of time-varying dynamics this method is yet to be improved. On the other hand, the second method, provides delay-independent results with estimates already obtained in the literature, but these rates are far from optimal. In the examples sections we showed that for delay bounds where the first method applies, it's estimates are much better than those of the second method. Additionally, we see that the second method is not applicable to multi-dimensional systems as Eq. (1.10).

7.1. **Extensions and future work.** As an extension of the linear case one may consider systems of the type

$$\dot{x}(t) = -ax(t) + ax(t-\tau) - g(x(t)) + g(x(t-\tau))$$

with g being is a Lipschitzian function. The way to attack such problems are a combination of the methods developed in this paper and these in [20]. Another direction of extending the models of this work can be the case of neutral delayed functional equations as discussed in [8]. Finally, another approach of interest is to study the effects of state-dependent delays and weights.

In any case we emphasize that the major challenge is to be able to improve this combination of solution forms to derive delay-independent stability results for the time-varying version of the model, as well.

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