Chapter 1

A Frequency Domain Approach to Controller Design for Linear, Time-Invariant, Infinite-Dimensional Systems

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Abstract – We present a new approach to frequency domain controller design for infinite dimensional systems. The chapter is organized around two examples: The first, an introductory example, is used to demonstrate how infinite dimensional systems arise in control theory and some of the features that distinguish them from finite dimensional systems. This example also provides an opportunity to introduce the basic algebraic structure on which our approach is based. The second example is used to motivate the bulk of the work in the chapter which specializes the algebraic approach to the function space H^∞. The advantage of this specialization is that it allows the introduction of powerful, general, analytic methods for constructing solutions to the algebraic equations that underly controller design. Finally we present the algebraic theory required to extend the design approach to multi-input, multi-output systems.

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1.1 AN INTRODUCTORY EXAMPLE

We begin with a presentation of an example which is chosen to serve a number of purposes: to show how infinite dimensional systems arise in control theory as a result of modeling systems by partial differential equations, to demonstrate how infinite dimensional systems can exhibit properties that do not occur in finite dimensional systems, and to establish a fundamental approach to frequency domain design in a simple setting.

The system is described by a one dimensional wave equation on a finite interval $[0, L]$

$$w_{tt} - \alpha^2 w_{xx} = 0, \qquad t > 0, \qquad 0 < x < L \tag{1.1}$$

Control and observation of the wave form $w(x,t)$ is achieved by monitoring and specifying boundary conditions at one end of the interval. The boundary conditions are:

$$\begin{aligned} w(0,t) &= 0 \\ -\alpha^2 w_x(L,t) &= u(t), \quad (\text{ the control input }), \end{aligned} \tag{1.2}$$

and the output equation is

$$y(t) = \begin{cases} w_t(L, t-\tau) & \text{if } t > \tau \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Initial conditions are specified by two functions:

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x). \tag{1.4}$$

Such a system might arise as a model for boundary control of torsional strain in a linearly elastic rod.

Taking Laplace transforms of (1.1) and (1.2) gives

$$-w_t(x,0) - zw(x,0) + z^2\widehat{w}(x,z) - \alpha^2\widehat{w}_{xx}(x,z) = 0, \tag{1.5}$$

an in-homogeneous ordinary differential equation that is valid on the right half plane, and has boundary conditions

$$\begin{aligned} \widehat{w}(0,z) &= 0 \\ -\alpha^2\widehat{w}_x(L,z) &= \widehat{u}(z), \end{aligned}$$

and an output equation

$$\widehat{y}(z) = -w(L,0) + ze^{-\tau z}\widehat{w}(L,z).$$

The ODE is solved by combining a homogeneous solution satisfying the boundary conditions with a particular solution to the forced equation with zero boundary conditions. The associated homogeneous equation has a solution $\widehat{w}(x,z) = A(z)e^{z\frac{x}{\alpha}} + B(z)e^{-z\frac{x}{\alpha}}$ with $A(z)$ and $B(z)$ determined by substituting the boundary conditions. Substituting the first boundary condition gives

$$\begin{aligned}
\widehat{w}(x,z) &= A(z)\sinh z\frac{x}{\alpha} \\
\widehat{w}_x(x,z) &= A(z)\frac{z}{\alpha}\cosh z\frac{x}{\alpha},
\end{aligned}$$

substituting the second gives

$$A(z) = \frac{\alpha}{z}\left(\cosh(zL/\alpha)\right)^{-1}\widehat{u}(z),$$

and the solution to the homogeneous part of (1.5) that satisfies the boundary conditions is

$$\widehat{w}(x,z) = \frac{\alpha}{z}\left(\cosh(zL/\alpha)\right)^{-1}\widehat{u}(z)\sinh(zx/\alpha).$$

If the forcing function in (1.5) is denoted by

$$f(x) = -w_t(x,0) - zw(x,0)$$

then a particular solution to (1.5) is a solution to

$$z^2\xi(x) - \alpha^2\xi_{xx}(x) = f(x) \tag{1.6}$$

that satisfies the boundary conditions $\xi(0) = \xi_x(L) = 0$. If the solution is an L^2 function on $[0,L]$, then it will have a Fourier series expansion

$$\xi(x) = \sum_{n=0}^{\infty} C_n \sin((2n+1)\frac{\pi x}{L}).$$

Differentiating twice with respect to x gives

$$\xi_{xx}(x) = \sum_{n=0}^{\infty} \frac{(2n+1)^2\pi^2}{4L^2} C_n \sin((2n+1)\frac{\pi x}{L})$$

and substitution into 1.6 gives

$$f(x) = \sum_{n=0}^{\infty}\left(z^2 + \alpha^2\frac{(2n+1)^2\pi^2}{4L^2}\right) C_n \sin((2n+1)\frac{\pi x}{L}).$$

The coefficients C_n can be determined from this equation by multiplying each side by $\sin((2n+1)x/L)$ and integrating over the interval $[0,L]$ to get

$$C_n = \frac{2}{L}\left(z^2 + \alpha^2\frac{(2n+1)^2\pi^2}{L^2}\right)^{-1}\int_0^L f(x)\sin(2n+1)\pi x/L\,dx.$$

The solution has been reduced to two parts, a periodic "transient" that is a linear function of the initial conditions, and a component that is a linear function of the boundary data. If the transient part of the solution is ignored, then the relationship between the input signal $u(t)$ and the output signal $y(t)$ may be written as a linear operator with transfer function $\widehat{y}(z) = F(z)\widehat{u}(z)$

$$F(z) = \alpha e^{-\tau z} \tanh(zL/\alpha)$$

The transfer function is analytic on the right half plane, but does not have bounded magnitude owing to the presence of poles on the imaginary axis at the points

$$z \in \{i(2n+1)\alpha\pi/(2L) : n \in \mathcal{Z}\}.$$

As a consequence, the operator is not a bounded linear mapping of L^2 into itself, and the transfer function does not represent a stable linear system. The control design problem is to find a feedback compensator that yields a stable closed loop system.

The design methodology that is presented in this chapter has its origins in standard frequency domain practice. A co-prime factorization for the transfer function is taken over a suitable ring, and the design problem is translated into an algebraic problem of choosing a factorization for the transfer function of a compensator that produces a suitable closed loop system. Suppose that a plant with transfer function $F(z)$ is controlled by a linear feedback compensator $G(z)$. Let $F(z) = F_2(z)^{-1}F_1(z)$ and $G(z) = G_1(z)G_2(z)^{-1}$ be co-prime factorizations of the plant and compensator transfer functions over a function ring $\mathcal{I}$. Then the transfer function for the closed-loop system has the following factorization over $\mathcal{I}$.

$$F_{\text{cl}}(z) = \frac{G_2(z)F_1(z)}{F_1(z)G_1(z) + F_2(z)G_2(z)}$$

The design synthesis problem is to choose the factors $G_1(z)$ and $G_2(z)$ so that the compensator transfer function $G(z)$ is realizable, and the closed loop transfer function $F_{\text{cl}}(z)$ possesses some desirable property such as robust stability.

In the case when $\mathcal{I}$ is a principal ideal domain a useful re-parameterization of the closed-loop transfer function is possible. A consequence of the principal ideal domain structure is the existence of solutions $X_1(z)$ and $X_2(z)$ to the Bezout equation

$$F_1(z)X_1(z) + F_2(z)X_2(z) = 1, \tag{1.7}$$

1 in this formula being the identity of $\mathcal{I}$. Given such a solution, it is possible to write a general solution to the linear Diophantine equation [1]

$$F_1(z)G_1(z) + F_2(z)G_2(z) = R(z).$$

[1]The term Diophantine equation is used to establish an analogy with Diophantine equations over the ring of integers.

The solution contains a free parameter $Q(z) \in \mathcal{I}$:

$$\begin{aligned} G_1(z) &= F_2(z)Q(z) + X_1(z)R(z) \\ G_2(z) &= -F_1(z)Q(z) + X_2(z)R(z). \end{aligned} \tag{1.8}$$

Identifying the solution $G_1(z)$ and $G_2(z)$ with the factorization of the compensator transfer function gives an alternative parameterization for the closed loop transfer function in terms of two free parameters $Q(z)$ and $R(z)$

$$F_{\mathrm{cl}}(z) = \frac{(-F_1(z)Q(z) + X_2(z)R(z))F_1(z)}{R(z)}. \tag{1.9}$$

The choice of the function ring $\mathcal{I}$ is critical both in determining the feasibility of solving (1.7), and in deciding how the design objectives should determine a choice of parameters in (1.9). Although existence of solutions to the Bezout equation is guaranteed when $\mathcal{I}$ is a commutative principal ideal domain, constructing solutions even in this case is generally a nontrivial problem. In the case of finite dimensional systems appropriate choices for $\mathcal{I}$ include the ring of polynomials over the real numbers, or the ring of rational H^∞ functions, and in both instances algebraic methods are available for solving the Bezout equation. In the case of the infinite dimensional systems considered in this paper the corresponding rings, rings of analytic functions, are not even principal ideal domains; additional conditions on the factorizations are needed before solutions to the Bezout equation are guaranteed, and the methods that are used to construct solutions in the finite dimensional cases, which rely on a Euclidean ring structure on $\mathcal{I}$, are no longer applicable. The main part of this chapter consists of the presentation of a method for constructing solutions to 1.7 over the ring H^∞. In this case the theorem stating necessary and sufficient conditions for a solution of the Bezout equation is called the Corona Theorem.

Returning to the example, the open loop transfer function is factored over the ring of exponential polynomials to give $F(z) = F_1(z)/F_2(z)$ with

$$\begin{aligned} F_1(z) &= \alpha(1 - e^{-2zL/\alpha}) \\ F_2(z) &= e^{\tau z}(1 + e^{-2zL/\alpha}) \end{aligned}$$

This choice for $\mathcal{I}$ allows the poles of the closed loop system to be directly selected through the choice of the parameter $R(z)$. Choosing

$$R(z) = e^{\tau z}(1 + e^{-2L(z+\beta)/\alpha})$$

and $Q(z) = 1$ in (1.9) ensures stability of the closed loop system by placing the closed loop poles at the points $\{-\beta + i(2n+1)\alpha\pi/(2L) : n \in \mathcal{Z}\}$, where β is an arbitrary positive real number. In this instance the Bezout equation has a simple solution

$$X_1(z) = \frac{1}{2\alpha} \qquad X_2(z) = \frac{1}{2}e^{-\tau z},$$

and the equations (1.8) may be used to calculate the following factorization for the compensator transfer function:

$$\begin{aligned} G_1(z) &= -\frac{1}{2\alpha}F_2(z) + X_1(z)R(z) \\ &= \frac{1}{2\alpha}e^{(\tau-2L/\alpha)z}(e^{-2L\beta/\alpha}-1) \\ G_2(z) &= \frac{1}{2\alpha}F_1(z) + X_2(z)R(z) \\ &= 1+\frac{1}{2}e^{-2Lz/\alpha}(e^{-2L\beta/\alpha}-1). \end{aligned}$$

This compensator may be realized by a simple system composed of two pure delays by using the network illustrated in Figure 1.1.

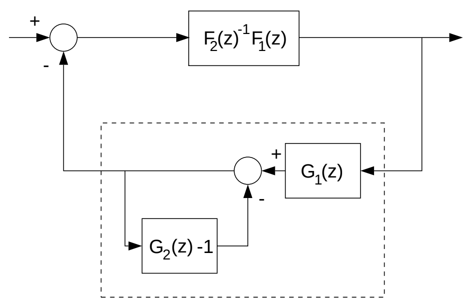

Figure 1.1. Compensator Structure.

The result obtained in the previous paragraph, exponential stability with arbitrary decay, seems too good to be true, and indeed it is; the compensated system is not robust, and is destabilized by arbitrarily small perturbations in the open loop system's parameters. If, for example, the value τ of the delay used in the controller is inaccurate, and the actual value of the delay in the plant is $\tau+\delta$, then the closed loop transfer function has poles at the zeros of the function $G_1(z)F_1(z)+G_2(z)F_2(z)$ which, when expanded, has the form

$$\begin{aligned} &e^{\tau z}(1+e^{-2Lz/\alpha}e^{-2L\beta/\alpha}) \quad + \\ &\qquad e^{\tau z}(e^{\delta z}-1)(1+e^{-2Lz/\alpha})(1+1/2e^{-2Lz/\alpha}(e^{-2L\beta/\alpha}-1)). \end{aligned}$$

The common factor $e^{\tau z}$ has no zeros on the complex plane and may be ignored leaving a two term expression $t_1(z)+t_2(z)$ with

$$\begin{aligned} t_1(z) &= 1+e^{-2Lz/\alpha}e^{-2L\beta/\alpha} \\ t_2(z) &= (e^{\delta z}-1)(1+e^{-2Lz/\alpha})(1+1/2e^{-2Lz/\alpha}(e^{-2L\beta/\alpha}-1)). \end{aligned}$$

From here it is a standard exercise to prove the existence of a right half plane zero by finding a bounded region of the right half complex plain D on which an application of the argument principle proves the existence of a zero of $g(z)$ in the interior of D, and on which an application of Rouché's theorem (or the small gain theorem) demonstrates that the sum $t_1(z)+t_2(z)$ has the same number of zeros in D as the function $t_2(z)$.

An important point to notice is that the compensator given in Figure 1.1 provides a closed loop system that is robust in the normal sense of H^∞ robust control, which is to say that an additive perturbation in the transfer function F with amplitude that is uniformly bounded over the right half plane will not destabilize the closed loop system provided the bound is small enough. An application of the small gain theorem shows that the controller in Figure 1.1 stabilizes a perturbed system provided that the magnitude of the perturbation Δ is bounded by

$$|\Delta(z)| < \alpha/2\,(1 - e^{-2L\beta/\alpha}).$$

The failure of the H^∞ design criterion to define an adequate measure of robustness is in this example is a result of discontinuity in the mapping from perturbations in the parameter τ to the space of perturbations to the closed loop transfer function denominator $F_1(z)G_1(z) + F_2(z)G_2(z)$ with the L^∞ norm.

1.2 A SECOND EXAMPLE

The example in the first section was introduced to demonstrate a simple, general, algebraic framework for controller design in the frequency domain. In order to apply this framework a suitable choice of ring for the factorization of the transfer function needs to be made. This choice is guided by two considerations: a method should exist for solving the Bezout equation within the ring chosen, and the ring should exhibit structure that allows the design objectives to be translated into a parameter choice in the parameterized equations for the controller. The example that was given was chosen so that the first requirement was trivially satisfied — general methods for solving Bezout equations in rings of exponential polynomials are given in [Berenstein and Yger, 1989]. The second requirement was satisfied by observing that the zeros of the parameter $R(z)$ determine the poles of the closed loop system so that a design objective that can be translated into a pole placement criterion can be easily satisfied. At this point we change tack by fixing the factorization ring to be H^∞, and the remainder of the chapter presents a way to deal with the associated problems of solving the Bezout equation in H^∞ and choosing a compensator that satisfies the design objectives. Again an example motivates the discussion. We use the example presented in [Enns *et al.* , 1992] to facilitate a comparison between the different methods of solution.

The plant in the example is the pitch-axis fast dynamics of an unstable aircraft, this is represented by the simplified model that is developed in [Enns *et al.* , 1992]. The model, which is given by the transfer function

$$F(z) = \frac{e^{-\tau z}}{\sigma z - 1}, \tag{1.10}$$

has an unstable pole at $z = 1/\sigma$, and a delay of τ seconds, as such it is amongst the simplest unstable infinite dimensional systems. The design objective is to produce a linear feedback controller with the configuration illustrated in Figure 1.2 that both robustly stabilizes the plant and maintains low low-frequency sensitivity. A factorization for the plant over the ring H^∞ is given by $F(z) = F_2(z)^{-1}F_1(z)$ with

$$\begin{aligned} F_1(z) &= e^{-\tau z}/(\sigma z + 1) \\ F_2(z) &= (\sigma z - 1)/(\sigma z + 1). \end{aligned}$$

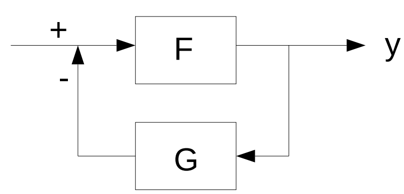

Figure 1.2. Feedback Controller

For this example there is no problem translating the control objective of robust stability into an H^∞ design requirement. If $G(z) = G_1(z)G_2(z)^{-1}$ is an H^∞ factorization of a stabilizing controller then the closed loop system has a transfer function

$$\frac{G_2(z)F_1(z)}{F_1(z)G_1(z) + F_2(z)G_2(z)}$$

Since the closed loop system is robustly stable in the H^∞ design sense, the denominator $F_1(z)G_1(z) + F_2(z)G_2(z)$ must be a unit in H^∞ and consequently has magnitude bounded away from zero on the imaginary axis; let ϵ be a bound that satisfies

$$0 < \epsilon < |F_1(iy)G_1(iy) + F_2(iy)G_2(iy)|, \qquad \forall y \in \mathbb{R}.$$

If the delay in the plant is perturbed by an amount δ then the closed loop transfer function for the perturbed plant with the same compensator is given by

$$\frac{G_2(z)F_1(z)}{(e^{-\delta z}-1)F_1(z)G_1(z)+F_1(z)G_1(z)+F_2(z)G_2(z)}.$$

The small gain theorem ensures stability provided that

$$|(e^{-\delta iy}-1)F_1(iy)G_1(iy)|<\epsilon, \qquad \forall y\in \mathbb{R}.$$

Substituting for $F_1(iy)$, and recalling that $G_1 \in \mathrm{H}^\infty$ gives the inequality

$$\left|\frac{(e^{-\delta iy}-1)}{i\sigma y+1}\right| < \epsilon/\|G_1\|_\infty, \qquad \forall y\in \mathbb{R} \tag{1.11}$$

which is satisfied by all δ lying in a sufficiently small open interval about zero. What this shows is that there is continuity in the mapping between the perturbations in the parameter τ and the corresponding perturbations in the H^∞ function $F_1(z)G_1(z)+F_2(z)G_2(z)$. This mapping determines the stability of the closed loop system, and it follows that robust stability in the H^∞ design sense implies robust stability with respect to variation in the parameter δ. Robust stability with respect to the parameter σ can be determined by a similar argument.

The choice of H^∞ for the factorization ring has two fortunate consequences: the Corona Theorem guarantees solutions to the Bezout equation provided that the factors $F_1(z)$ and $F_2(z)$ satisfy the inequality

$$\inf_{\mathrm{Re}\, z>0} |F_1(z)|+|F_2(z)|>0,$$

and the parameterization of all the stable, closed loop system transfer functions formed by linear, time-invariant feedback compensation has a very simple form. Substituting $R(z)=1$ in equation (1.8) gives a parameterization for all the stabilizing controllers

$$G(z)=\frac{X_1(z)+F_2(z)Q(z)}{X_2(z)-F_1(z)Q(z)}. \tag{1.12}$$

This is the Youla parameterization that was formulated for rational systems in [Youla *et al.* , 1976] and extended to irrational transfer functions by [Baras, 1980] who points out the importance of the Corona Theorem to controller design. The transfer function for the closed loop system that corresponds to a particular choice of the parameter Q is

$$F_{\mathrm{cl}}(z)=(-F_1(z)Q(z)+X_2(z))F_1(z). \tag{1.13}$$

The controller design problem is now reduced to the problem of finding a suitable choice for the parameter Q. Standard H^∞ design practice chooses the parameter Q to minimize a cost function that is expressed as the L^∞ norm of a weighted sum of appropriately chosen closed-loop transfer functions. The transfer functions that appear in the cost are chosen to reflect the imposed design criteria which, in this example, are robust stability and low low-frequency sensitivity. [Enns *et al.* , 1992] show that low sensitivity is achieved by choosing Q to minimize the norm of the transfer function

$$(1 + F(z)G(z))^{-1} = (-F_1(z)Q(z) + X_2(z))F_2(z),$$

and robust stability is achieved by choosing $Q(z)$ to minimize the norm of the transfer function

$$G(z)(1 + F(z)G(z))^{-1} = (F_2(z)Q(z) + X_1(z))F_2(z).$$

These two criteria place competing requirements on the choice of Q, which are resolved by minimizing a weighted combination of the transfer functions:

$$\sup_{y \in \mathbb{R}} \begin{pmatrix} |W_1(iy)(1 + F(iy)G(iy))^{-1}|^2 \\ + \quad |W_2(iy)G(iy)(1 + F(iy)G(iy))^{-1}|^2 \end{pmatrix}. \tag{1.14}$$

The weights that the authors choose are functions of frequency, a low pass filter is chosen for $W_1(z)$ to emphasize low-frequency sensitivity, and high pass filter is chosen for $W_2(z)$ reflecting the fact that system uncertainty is a worse problem at high frequencies. Minimizing the quantity (1.14) is equivalent to minimizing the H^∞ norm of the matrix valued function

$$\left\| \begin{pmatrix} W_1(z)(1 + F(z)G(z))^{-1} \\ W_2(z)G(z)(1 + F(z)G(z))^{-1} \end{pmatrix} \right\|_\infty$$

which, on substituting the parameterization for the transfer function in each entry, becomes

$$\left\| \begin{pmatrix} W_1(z)(-F_1(z)Q(z) + X_2(z))F_2(z) \\ W_2(z)(F_2(z)Q(z) + X_1(z))F_2(z) \end{pmatrix} \right\|_\infty. \tag{1.15}$$

Choosing Q to minimize this quantity is the Nehari interpolation problem.

The development so far has followed standard H^∞ design practice, and the problem of choosing an H^∞ function $Q(z)$ to minimize the norm in equation (1.15) is the Nehari problem of H^∞ control system design. Before the Nehari problem can be solved, a solution $X_1(z)$ and $X_2(z)$ to the Bezout equation needs to be calculated. The example presented is sufficiently simple that this solution may be easily given in closed form; [Enns *et al.* , 1992] give the solution

$$\begin{aligned} X_1(z) &= \frac{(\sigma z + 1) - 2e^{\tau/\sigma}e^{-\tau z}}{(\sigma z - 1)} \\ X_2(z) &= e^{\tau/\sigma}. \end{aligned}$$

Enns *et al.* then go on to apply the work from [Özbay *et al.* , 1993] to the design of a robust stabilizing controller for the system. It turns out that this example is sufficiently simple that the operator theory from Özbay *et al.* is able to produce a controller that achieves the minimum value for the norm in (1.15); the result is an optimal solution, and a good example against which other approaches can be measured.

The controller design that we present is based on a unified treatment of the Bezout equation (1.7) and the Nehari problem (1.15). If the 2×1 matrix in (1.15) is denoted by $P(z) = (P_1(z), P_2(z))^\top$, then the Nehari problem may be reformulated as the problem of finding a solution P, Q to the linear Diophantine equation

$$\begin{pmatrix} W_1(z)X_2(z)F_2(z) \\ W_2(z)X_1(z)F_2(z) \end{pmatrix} = 1 \begin{pmatrix} P_1(z) \\ P_2(z) \end{pmatrix} + Q(z) \begin{pmatrix} W_1(z)F_1(z)F_2(z) \\ -W_2(z)F_2(z)F_2(z) \end{pmatrix}$$

with the added requirement that the solution selected from the space of possible solutions should be one that minimizes the norm of P. Denoting the inhomogeneous term in (1.16) by $A(z)$ and the matrix function that multiplies $Q(z)$ by $B(z)$, equation (1.16) can be rewritten as

$$A(z) = 1P(z) + Q(z)B(z), \tag{1.16}$$

a form that makes the similarity between this equation and the Bezout equation (1.7) more obvious. The next section introduces the theory that is needed to solve these two problems, and the following section uses that theory to construct a stabilizing controller for the plant presented in equation (1.10).

1.3 SOLVING LINEAR DIOPHANTINE EQUATIONS IN H^∞

Let f_1, f_2 and h be three functions in H^∞. This section addresses the problem of finding solutions g_1 and g_2 in H^∞ for the equation

$$f_1 g_1 + f_2 g_2 = h \tag{1.17}$$

Equation (1.17) subsumes the Bezout equation (1.7), and with a little work it can be made to subsume (1.16), the equation that arises in the Nehari problem, as well. If f_1 and f_2 are outer functions, which is to say that they possess multiplicative inverses in H^∞, then the solution is easy. A family of solutions with parameter η an H^∞ function is formed by setting $g_1 = \eta h f_1^{-1}$ and $g_2 = (1-\eta) h f_2^{-1}$. When f_1 and f_2 have zeros in the right half plane, the inverses no longer exist and this method breaks down, but if the requirement that the solutions be in H^∞ is temporarily relaxed then bounded (but not analytic) solutions may be found as follows.

Let ϕ be a bounded function on the half plane with the property that the zero set of f_1 is bounded away from the support of ϕ, and the the zero set of f_2 is bounded away from the support of $1-\phi$. For such a function to exist some restriction needs to be placed on the functions f_1 and f_2, for instance, it is necessary that they should have no common zeros. Bounded solutions to equation 1.17 can be constructed in a piecewise fashion by taking $\widetilde{g}_1 = 0$ outside the support of ϕ, and $\widetilde{g}_1 = \phi h f_1^{-1}$ on the support of ϕ and for the second function $\widetilde{g}_2 = 0$ outside the support of $1-\phi$ and $\widetilde{g}_2 = (1-\phi) h f_2^{-1}$ on the support of $1-\phi$.

Observe that if e is a bounded function on the right half plane, then the two functions $-ef_1$ and ef_2 satisfy the relation

$$(-ef_1)f_2 + (ef_2)f_1 = 0.$$

With this in mind the step from the bounded solutions $\widetilde{g}_1$ and $\widetilde{g}_2$ to H^∞ solutions can be made if a bounded function e can be found such that the functions

$$\begin{aligned} g_1 &= \widetilde{g}_1 + ef_2 \\ g_2 &= \widetilde{g}_2 - ef_1 \end{aligned}$$

are are both in H^∞. In this section it is shown that if the function ϕ is chosen appropriately, a suitable function e can be calculated as a solution to a first order partial differential equation.

1.3.1 Algebraic Reformulation

The appropriate setting in which to make the introductory paragraphs of this section precise is the setting of homological algebra. This setting was first presented in [Hörmander, 1967] in conjunction with the corona problem, and has been used by [Berenstein and Struppa, 1986], [Berenstein and Taylor, 1980], [Struppa, 1983], and [Berenstein and Yger, 1989] for the investigation of linear Diophantine equations in algebras over rings of analytic functions of prescribed growth. In the simple cases of equations that arise from single-input single-output systems the algebraic formalism reduces precisely to the equations of the opening paragraph. The reason for introducing homological algebra is that it provides a concise, extensible framework in which the relationship between the algebraic and analytic aspects of the problem are made clear. A good introduction to the analysis used in this chapter is [Berenstein and Gay, 1991], and the survey [Berenstein and Struppa, 1993] indicates the role of the methods used in this paper in recent work on the analysis of linear operators.

Let R denote a ring of functions (or distributions) on the half plane $\mathcal{H}$. For any positive integer m let $\Lambda(R)$ denote the graded module[2] over

[2] A working definition of a module is given by the analogy that, a module is to a ring, as a vector space is to a field

R that consists of functions on $\mathbb{C}$ that take values in the exterior algebra of antisymmetric forms on an m-dimensional vector space, and let $\Lambda^k(R)$ denote the homogeneous elements in $\Lambda(R)$ of order k. $\Lambda(R)$ is a finite free module over R, and a basis element will be denoted by $e_{i_1,\ldots,i_k}$ with the indices ordered $1 \le i_1 \le \ldots \le i_k \le m$. For example, if $m = 2$ then $\Lambda(R)$ has the following matrix representation, $\Lambda(R)$ is the direct sum $\Lambda(R) = \Lambda^0(R) \oplus \Lambda^1(R) \oplus \Lambda^2(R)$ where: $\Lambda^0(R)$, the subspace of homogeneous elements of degree 0, is generated by a basis element that is the constant function 1 (this can be thought of as a 1×1 matrix); $\Lambda^1(R)$, the subspace of homogeneous elements of degree 1, is generated by the basis elements

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

and $\Lambda^3(R)$, the subspace of homogeneous elements of degree 2, is generated by the basis element

$$e_{12} = e_1 \wedge e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Although this example (m = 2) is all that will be needed for the equations that arise from single-input, single-output systems, developing the theory at a more abstract level will yield results that are applicable to the equations that arise from multi-input, multi-output systems.

For $j \in \{i_1, \ldots, i_k\}$ define the symbol $e_{i_1,\ldots,\widehat{j},\ldots,i_k}$ to be the basis element of Λ^{k-1} that is formed by deleting the index j from $e_{i_1,\ldots,i_k}$; if $j \notin \{i_1, \ldots, i_k\}$ then $e_{i_1,\ldots,\widehat{j},\ldots,i_k} = 0$. A derivation can be defined on $\Lambda(R)$ as follows. Suppose that $f = \{f_1, \ldots f_m\}$ is a finite subset of R, then the operator P_f defined on $\Lambda(R)$ acts on a basis element as follows

$$P_f(e_{i_1,\ldots,i_k}) = \sum_{j=1}^{k} (-1)^{j+1} f_{i_j} e_{i_1,\ldots,\widehat{i_j},\ldots,i_k}$$

The operator P_f forms an exact sequence over the homogeneous submodules Λ^k called the Koszul complex. In the case of $m = 2$ the Koszul complex is represented by the diagram

$$0 \longrightarrow \Lambda^2(R) \xrightarrow{P_f} \Lambda^1(R) \xrightarrow{P_f} \Lambda^0(R) \longrightarrow 0, \tag{1.18}$$

and when $R = \mathrm{H}^\infty$, this corresponds exactly to the concrete algebraic setting described in the opening paragraphs of this section. Finding a solution to Equation (1.17) is equivalent to inverting the operator P_f : $\Lambda^1(\mathrm{H}^\infty) \to \Lambda^0(\mathrm{H}^\infty)$. The approach to inverting the operator P_f that was outlined in the introduction was first to find an inverse in $\Lambda^1(R)$ a larger space than $\Lambda^1(\mathrm{H}^\infty)$, and then add an element from the image of P_f :

$\Lambda^2(R) \to \Lambda^1(R)$ that will return the solution to H^∞. The appropriate ring R in which to invert P_f is the ring of distributions that will be introduced with the following definitions from [Hörmander, 1967].

Denote the open right half plane by $\mathcal{H}$, its closure by $\bar{\mathcal{H}}$, and its boundary, the imaginary axis, by $\partial\mathcal{H}$. Let μ be a measure with support on $\mathcal{H}$, then a distribution u on $\mathcal{H}$ satisfies the Cauchy Riemann equation[3]

$$\frac{\partial u}{\partial \bar{z}} = \mu \tag{1.19}$$

if for any continuously differentiable test function ψ with support compactly contained in $\mathcal{H}$,

$$\begin{aligned}\left\langle \frac{\partial u}{\partial \bar{z}}, \psi \right\rangle &= -\int_{\mathcal{H}} u(z) \frac{\partial \psi(z)}{\partial \bar{z}}\, dx\, dy \\ &= \int_{\mathcal{H}} \psi(z)\, d\mu. \end{aligned} \tag{1.20}$$

The measure $dx\,dy$ in the first integral is the Lebesgue measure on $\mathbb{C}$. A distribution u that satisfies (1.20) is said to have boundary value ϕ, an L^∞ function on the imaginary axis, if there exists U, an extension of u to $\bar{\mathcal{H}}$, that satisfies:

$$\frac{\partial U}{\partial \bar{z}} = \mu - \phi dz/2i. \tag{1.21}$$

Each side of this formula is to be interpreted as a distribution acting on test functions supported in the closed half plane $\bar{\mathcal{H}}$ and the measure $\phi dz/2i$ is a measure on $\mathbb{C}$ with support on the imaginary axis. The motivation for this definition comes from Stokes' Theorem.

A measure μ in $\mathcal{H}$ is called a Carleson measure [Garnett, 1981] with Carleson constant C if

$$|\mu(S)| < C\ l(S)$$

for every square $S \subset \mathcal{H}$ with a side of length $l(S)$ lying on an interval on the imaginary axis. The space of Carleson measures is denoted by the symbol $\mathcal{C}$.

Let $\mathcal{B}$ denote the ring of distributions over $\mathcal{H}$ with boundary value in L^∞, and which satisfy

$$\frac{\partial b}{\partial \bar{z}} = \mu \tag{1.22}$$

for some Carleson measure μ in $\mathcal{H}$. This is the ring in which the operator P_f will be inverted. The differential operator $\partial/\partial\bar{z} : \mathcal{B} \to \mathcal{C}$ and the canonical injection $i : \mathrm{H}^\infty \to \mathcal{B}$ form an exact sequence

$$0 \longrightarrow \mathrm{H}^\infty \xrightarrow{i} \mathcal{B} \xrightarrow{\partial/\partial\bar{z}} \mathcal{C} \longrightarrow 0 \tag{1.23}$$

[3] When $\mathbb{C}$ is considered as a homeomorphic to $\mathbb{R}^2$ in the usual way $(x, y) \to x + iy$ Then the antiholomorphic derivative operator $\partial/\partial\bar{z}$ is expressed in local real coordinates as $\partial/\partial\bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$

which combines in the following way with the Koszul complex (1.18) to give a double complex.

Make three copies of the Koszul complex (1.18) and in each copy substitute the ring R with one of the three function spaces H^∞, $\mathcal{B}$, or $\mathcal{C}$. The operator P_f acts on modules defined over the same ring in accordance with the action that was specified for the Koszul complex. The operators i and $\partial/\partial\bar{z}$ operate between homogeneous modules of the same degree k, but over different function spaces, by acting on the coefficients in accordance with the sequence (1.23)

$$0 \longrightarrow \Lambda^k(\mathrm{H}^\infty) \xrightarrow{\ i\ } \Lambda^k(\mathcal{B}) \xrightarrow{\partial/\partial\bar{z}} \Lambda^k(\mathcal{C}) \longrightarrow 0.$$

For example, if $\alpha(z)e_{12} \in \Lambda^2(\mathcal{B})$ then

$$\frac{\partial}{\partial\bar{z}}(\alpha(z)e_{12}) = \left(\frac{\partial}{\partial\bar{z}}\alpha(z)\right) e_{12} \in \Lambda^2(\mathcal{C}).$$

The case $m = 2$ is represented by the commutative diagram of Figure 1.3. For notational convenience the modules $\Lambda^k(\mathcal{B})$ and $\Lambda^k(\mathcal{C})$ are denoted by Λ_1^k and Λ_2^k respectively. Readers familiar with complex manifold theory should recognize Equation (1.23) as a $\bar{\partial}$ co-homology sequence. Since the results of this paper are restricted to analytic functions defined on the complex half-plane, it is not difficult to avoid introducing the language of the co-homology of differential forms — the appropriate setting for analogous results about analytic functions of several complex variables.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Lambda^2(\mathrm{H}^\infty) & \xrightarrow{P_f} & \Lambda^1(\mathrm{H}^\infty) & \xrightarrow{P_f} & \Lambda^0(\mathrm{H}^\infty) & \xrightarrow{P_f} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Lambda_1^2 & \xrightarrow{P_f} & \Lambda_1^1 & \xrightarrow{P_f} & \Lambda_1^0 & \xrightarrow{P_f} & 0 \\
\downarrow{\scriptstyle \partial/\partial\bar{z}} & & \downarrow{\scriptstyle \partial/\partial\bar{z}} & & \downarrow{\scriptstyle \partial/\partial\bar{z}} & & \\
\Lambda_2^2 & \xrightarrow{P_f} & \Lambda_2^1 & \xrightarrow{P_f} & \Lambda_2^0 & \xrightarrow{P_f} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Figure 1.3. double complex for m = 2

The next theorem, which comes from [Hörmander, 1967], explains how the Koszul complex is used to provide solutions to the Diophantine equations. The construction of the solutions is given in the proof which is repeated here for the sake of completeness.

Theorem 1 [Hörmander]
Suppose that the following conditions are satisfied:

(i) Let s take the values 0 and 1, and r take the values 1 and 2. If $h \in \Lambda_r^s$ and $P_f h = 0$ then the equation $P_f g = h$ has a solution $g \in \Lambda_r^{s+1}$ with $\partial g/\partial \bar{z} \in \Lambda_{r+1}^{s+1}$ when $\partial h/\partial \bar{z} = 0$.

(ii) $\partial g/\partial \bar{z} = \mu$ has a solution $g \in \Lambda_1^2$ for every $\mu \in \Lambda_2^2$.

Then for every $h \in \Lambda_1^0$ with $\partial h/\partial \bar{z} = 0$ one can find $g \in \Lambda_1^1$ so that $\partial g/\partial \bar{z} = 0$ and $P_f g = h$.

Proof:
The result follows when premises (i) and (ii) are used to traverse the diagram in Figure 1.3 as follows.

Suppose that $h \in \Lambda_1^0$ is a holomorphic function with boundary value in L^∞; that is, $\partial h/\partial \bar{z} = 0$ on $\mathcal{H}$, and there exists a function $G(y) \in \mathrm{L}^\infty(\mathbb{R})$ such that for almost all $y \in \mathbb{R}$, $H(y) = \lim_{z \to iy} h(z)$ when the limit is non-tangential to the boundary. Then, by the first premise, there exists $g^1 \in \Lambda_1^1$ such that $P_f g^1 = h$ and $\partial g^1/\partial \bar{z} \in \Lambda_2^1$. Commutativity implies that $P_f \partial g^1/\partial \bar{z} = \partial/\partial \bar{z}\, P_f g^1 = 0$, so again by the first premise there exists $g^2 \in \Lambda_2^2$ such that $P_f g^2 = \partial g^1/\partial \bar{z}$. By the second premise the equation $\partial g^3/\partial \bar{z} = g^2$ has a solution $g^3 \in \Lambda_1^2$. Let $g = g^1 - P_f g^3$, then

$$\begin{aligned} \frac{\partial g}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}}(g^1 - P_f\, g^3) \\ &= \frac{\partial g^1}{\partial \bar{z}} - P_f\, \frac{\partial g^3}{\partial \bar{z}} \\ &= 0 \end{aligned}$$

and $P_f g = P_f(g^1 - P_f g^3) = P_f(g^1) - P_f P_f g^3 = P_f g^1 = h$ as required.

□

Before Theorem 1 can be used to construct solutions to Diophantine equations, explicit inversion formulas for the operators P_f and $\partial/\partial \bar{z}$ satisfying premises (i) and (ii) need to be presented — it was this need for inversion formulas which governed the definitions of the spaces $\mathcal{B}$ and $\mathcal{C}$. Formulae for the inversion of the operator P_f that are based on the work in [Hörmander, 1967] with only small modification are presented next. Only the case of $m = 2$, the case needed for single-input single-output systems, is considered here; a more general situation that will be used for multi-input multi-output systems is dealt with in Section 1.5. A constructive scheme for inverting the Cauchy Riemann operator $\partial/\partial \bar{z}$ is presented in the next section.

The inversion of the operator $P_f : \Lambda_1^1 \to \Lambda_1^0$ is dealt with first. It turns out that the requirement that is hardest to satisfy is the requirement that the antiholomorphic derivative of the inverse should be a bounded Carleson measure. To overcome this problem (which in fact presents a major obstacle in the proof of the Corona Theorem) the construction is based on an application of the following Lemma from [Hörmander, 1967]

Lemma 2 [Carleson - Hörmander] *Let* $f_j \in \mathrm{H}^\infty$, $j = 1, \ldots, m$, *and assume that for some* $c > 0$

$$|f_1(z)| + \ldots + |f_m(z)| \geq c. \tag{1.24}$$

Then for sufficiently small $\epsilon > 0$ *one can find a partition of unity* ϕ_j *subordinate to the covering of* $\mathcal{H}$ *by open sets* $\mathcal{H}_j = \{z : |f_j(z)| > \epsilon\}$ *such that* $\partial\phi_j/\partial\bar{z}$*,defined in the sense of distribution theory, is a Carleson measure for all* j.

This Lemma is a restatement of a result of Carleson's original paper [Carleson, 1962] in which he directly constructs the measure. A more recent account of the construction is given in [Garnett, 1981]. The difficult part of the lemma is the construction of a partition of the plane into two sets each of which contains the regions of the plane where one of the two functions f_1 or f_2 becomes very small. In general, the boundary between the two sets will be a complicated curve, however in practice, the functions f_1 and f_2 may possess some regularity that allows a boundary curve to be easily chosen. For example, when solving the Bezout equation that arises from the example presented in Section 1.2, f_1 and f_2 are the functions $F_1(z) = e^{-\tau z}/(1+z)$ and $F_2(z) = (1-z)/(1+z)$ so the only restriction on the partition is that it separate the point $z = 1$ where $F_2(z) = 0$ from the regions of the plane where $|z|$ is large and $F_1(z)$ tends to zero, and a simple geometry suffices.

In the case when $m = 2$ the partition of unity from Lemma 2 consists of two functions ϕ_1, and $\phi_2 = 1 - \phi_1$. The partition may be chosen so that the functions f_1 and f_2 have magnitude bounded below by $\epsilon/2 > 0$ on the support of $\partial\phi_2/\partial\bar{z}$. A left inverse for P_f on Λ_1^0 is constructed as follows: for $h \in \Lambda_1^0$, let

$$g_i = h\frac{\phi_i}{f_i}. \tag{1.25}$$

then $P_f g = f_1 g_1 + f_2 g_2 = h$. The right inverse g also satisfies the premise of Theorem 1, for if $\partial h/\partial\bar{z} = 0$, then $\partial g/\partial\bar{z} = h f_i^{-1}\partial\phi_j/\partial\bar{z}$ which by Lemma 2 is a Carleson measure.

A second inversion formula is needed to invert the operator $P_f : \Lambda_2^2 \to \Lambda_2^1$. In fact, for the purposes of Theorem 1 it suffices to invert P_f on the subspace consisting of measures $\partial g^1/\partial\bar{z}$ where g^1 is a solution of $P_f g^1 = h$

for some holomorphic function h. In this case the problem may be written down explicitly as a redundant set of equations for the coefficients of g^2

$$\begin{aligned} g^2_{12} f_2 &= h/f_1 \, \partial\phi_1/\partial\bar{z} \\ g^2_{21} f_1 &= h/f_2 \, \partial\phi_2/\partial\bar{z}, \end{aligned}$$

and, since both f_2 and f_1 are functions with magnitude bounded away from zero on the support of $\partial\phi_1/\partial\bar{z}$, a solution is given by

$$g_{12} = \frac{h}{f_1 f_2} \, \partial\phi_1/\partial\bar{z}.$$

1.3.2 Constructing Bounded Solutions to the Inhomogeneous Cauchy Riemann Equation

The development in the preceding section reduced the construction of solutions to the Bezout equation to two steps: the construction of the partition of unity ϕ_j, and the construction of bounded solutions to the Cauchy Riemann equation. This section describes a technique first published in [Jones, 1980] for solving the Cauchy Riemann equation; the presentation that follows is based on [Garnett, 1981].

The exact problem to be solved is: given μ, a Carleson measure on the right half plane, find a distribution b with bounded boundary values that satisfies

$$\partial b/\partial\bar{z} = \mu.$$

The solution, which is based on a Green's function argument, has three stages: the measure μ is approximated by a sequence of measures μ_j which converge weakly to μ, each μ_j being supported on a finite set of points; the supporting set of each measure μ_j is partitioned in such a way that the pseudo-hyperbolic distance[4] between any two points in the same set of the partition is bounded from below, and the measure μ_j is subdivided into a corresponding sum, $\sum \mu^k_j$, each μ^k_j having support on a distinct set in the partition; finally the Cauchy Riemann equation is boundedly solved for each μ^k_j and these solutions are summed to form the approximate solution b_j. The key to the method lies in the observation that controlling the spacing between every pair of points in the support of μ^k_j gives control over the bounds of the solutions to the corresponding Cauchy Riemann equations, and a uniform bound over the sequence b_j.

Before the solution is discussed in detail the fundamental solution to the Cauchy Riemann operator $\partial/\partial\bar{z}$ is introduced, and a result about interpolating Blaschke products is recounted. Let $D \subset \mathbb{C}$ be an open domain with

[4] The pseudo-hyperbolic distance between two points in the half-plane is defined as $\rho(z_1, z_2) = |z_1 - z_2|/|z_1 - \bar{z}_2|$

C^1 boundary that contains the origin $z = 0$. The fundamental solution to the operator $\partial/\partial\bar{z}$ on D is a distribution b that satisfies the identity

$$-\int_D b(z)\frac{\partial\phi(z)}{\partial\bar{z}}dxdy = \phi(0)$$

for any C^∞ function ϕ with support compactly contained in D [Hörmander, 1990]. In this formula the integral on the left hand side of the identity should be interpreted as the action of the distribution on a test function. The fundamental solution is computed as follows. Suppose that ϕ is an arbitrary C^∞ function with support compactly contained in D. Let $U \subset D$ have C^1 boundary and contain the support of ϕ in its interior. Consider the function $\phi(\zeta)/\zeta$, Stokes' theorem gives

$$\begin{aligned}\int_{\partial U}\frac{\phi(\zeta)}{\zeta}\,d\zeta - \int_{|\zeta|=\epsilon}\frac{\phi(\zeta)}{\zeta}\,d\zeta &= \int_{|\zeta|>\epsilon}\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\phi(\zeta)}{\zeta}\right)\,d\bar{\zeta}\wedge d\zeta \\ &= -2i\int_{|\zeta|>\epsilon}\frac{1}{\zeta}\frac{\partial\phi}{\partial\bar{\zeta}}\,dxdy.\end{aligned}$$

Because $\phi(z) = 0$ on the boundary of U, the first boundary integral is zero, and as $\epsilon \to 0$ the second integral approaches the limit $i2\pi\phi(0)$. Consequently a fundamental solution for $\partial/\partial\bar{z}$ is given by the distribution $b(z) = 1/(\pi z)$.

If $z = x + iy$ is a complex number, then the real conjugate of z is defined to be the number $\widetilde{z} = -x + iy$. The need for this usage results from considering Laplace transforms of system operators; the Laplace transform of a bounded causal system gives a transfer function which is analytic in the right half plane, so in places where a complex conjugate $\bar{z}$ occurs in the analysis of functions analytic in the upper half plane, it will be natural to substitute the real conjugate $\widetilde{z}$. For instance, given a set $\{\zeta_j = \xi_j + i\eta_j, \xi_j > 0\}$ that satisfies the condition

$$\sum \frac{\xi_n}{1+|\zeta_n|^2} < \infty,$$

a Blaschke product with zeros ζ_j is defined by the expression

$$B(z) = \left(\frac{z-1}{z+1}\right)^m \prod_{\zeta_j\neq 1}\frac{|\zeta_j - 1|}{\zeta_j - 1}\frac{z-\zeta_j}{z-\widetilde{\zeta}_j}.$$

The factors $|\zeta_j - 1|/(\zeta_j - 1)$ ensure that the product converges when the sequence $\{|\zeta_j|\}$ is infinite, and for finite zero sets they may be omitted. A finite number of zeros at $\zeta = 1$ may be introduced into the Blaschke product separately in the factor in front of the product sign.

Let $B_0(z)$ be a Blaschke product with a zero set $\{\zeta_j = \xi_j + i\eta_j\}$ that satisfies the condition

$$\prod_{j,j\neq k} \left| \frac{\zeta_k - \zeta_j}{\zeta_k - \widetilde{\zeta}_j} \right| \geq \delta > 0, \tag{1.26}$$

then the inverse $1/B_0(z)$ is an analytic function except on the zero set $\{\zeta_j\}$ and an expansion by partial fractions gives the expression

$$1/B_0(z) = 1 + \sum_j \frac{1/B_0'(\zeta_j)}{z - \zeta_j}.$$

If $1/B_0(z)$ is considered as a distribution on $\mathcal{H}$, then it follows from the discussion of the fundamental solution to the $\bar{\partial}$ operator that

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(1/B_0) &= \sum_j \frac{\pi}{B_0'(\zeta_j)} \delta_{\zeta_j} \\ &= \sum_j \beta_j \xi_j \delta_{\zeta_j}, \end{aligned} \tag{1.27}$$

where $1 \leq |\beta_j| \leq 1/\delta$.

The following theorem is from [Jones, 1983].

Theorem 3 [Jones]
Suppose $\{z_k\}$ is a sequence of points in the half plane that satisfies

$$\inf_j \prod_{k,k\neq j} \left| \frac{z_k - z_j}{z_k - \widetilde{z}_j} \right| \geq \delta > 0, \qquad k = 1, 2, \ldots.$$

Let $B(z)$ be the the Blaschke product with zeros at the points $\{z_k\}$ and $B_j(z)$ be the Blaschke product with zeros in the set formed by removing the point z_j from the set $\{z_k\}$. Let $E_j(z)$ be the function

$$E_j(z) = c_j B_j(z) \left(\frac{y_j}{z - \widetilde{z}_j} \right)^2 \exp\left\{ \frac{-i}{\log 2/\delta} \sum_{y_k \leq y_j} \frac{y_k}{z - \widetilde{z}_k} \right\}$$

where

$$c_j = -4(B_j(z_j))^{-1} \exp\left\{ \frac{i}{\log 2/\delta} \sum_{y_k \leq y_j} \frac{y_k}{z_j - \widetilde{z}_k} \right\}$$

Then $E_j(z_k) = \delta_{j,k}$ and

$$\sum_j |E_j(z)| \leq (C_0/\delta)\ \log(2/\delta) \tag{1.28}$$

for all $z \in \mathcal{H}$.

This theorem is an instance of Carleson's interpolation theorem that explicitly gives the form of the interpolating function. The bound on the norm of the interpolating function, $(C_0/\delta)\log(2/\delta)$ is optimal in δ up to the multiplicative factor C_0.

The application of Jones' interpolation formula requires the following lemma which is extracted from the proof of Carleson's interpolation theorem in Chapter 7 of [Garnett, 1981].

Lemma 4 *Let $\{z_j\}$ be a sequence in the right half plane, with points z_j well separated in the hyperbolic metric, i.e.*

$$\rho(z_k, z_j) = \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right| \geq a > 0, \qquad j \neq k,$$

and suppose that there exists a constant A such that for every square $Q = \{y_0 \leq y \leq y_0 + l(Q), 0 < x \leq l(Q)\}$

$$\sum_{z_j \in Q} x_j \leq A\, l(Q),$$

then

$$\inf_k \prod_{j, j \neq k} \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right| \geq \delta \geq \exp\left(-40A \left(1 + 2\log\frac{1}{a} \right) \right).$$

The bound that is given for δ in the lemma depends on the points $\{z_k\}$ having a minimum spacing a in the hyperbolic metric and on the measure $\sum x_j \delta_{z_j}$ being a Carleson measure with Carleson constant A. Unfortunately the generality of the theorem means that the bound derived will be conservative for many specific examples. This is particularly true of examples such as the one presented in this paper in which the measures have additional structure that is readily recognizable. Additional information about the distribution of the points $\{z_k\}$ could well be used to derive a less conservative estimate.

The next two lemmas contain the constructive solution to the Cauchy Riemann equation that is presented in Chapter 8 of [Garnett, 1981]. The proofs closely follow the work cited, but are given here because they contain the algorithms that are used to compute actual solutions. Jones' interpolation theorem and the discussion preceding it on fundamental solutions provide the basis for calculating solutions to the Cauchy Riemann equation in the following simple case.

Lemma 5 [Garnett, 1981]
Let z_j be a finite set of points satisfying (1.26) and let $\mu = \sum \alpha_j x_j \delta_{z_j}$ with $|\alpha_j| \leq 1$. Then the function

$$b(z) = E(z)/B_1(z)$$

satisfies $\partial b/\partial \bar{z} = \mu$ where $B_1(z)$ is a Blaschke product with zeros z_j, and $E(z)$ is a function that is analytic on the right half plane and has a bound that depends on the choice of μ only through the δ of Equation (1.26).

Proof:
Equation (1.27) states that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{B_1(z)} = \sum_j \beta_j x_j \delta_{z_j},$$

and that the coefficients β_j lie within the uniform bounds $1 \leq |\beta_j| \leq 1/\delta$. An application of Jones' interpolation theorem produces a function $E(z) = \sum \alpha_j/\beta_j E_j(z)$ that is analytic in the right half plane, interpolates the values α_j/β_j at the points z_j, and is bounded on the imaginary axis by

$$|E(z)| \leq (C_0/\delta)\ \log(2/\delta)$$

in which C_0 is an absolute constant. The result follows by taking $b(z) = E(z)/B_1(z)$.

□

The case of a general Carleson measure μ is tackled by constructing a sequence of approximating measures $\{\mu_n\}$ that converges (weakly) to μ; each measure in the sequence is supported on a finite set of points and has the form $\mu_n = \sum \alpha_j x_j \delta_{z_j}$. A uniformly bounded sequence of solutions $\{b_n\}$ to the equations $\partial b_n/\partial \bar{z} = \mu_n$ with the property that $\{b_n\}$ converges uniformly on compact sets is calculated, and it follows from weak compactness of the unit ball that $\{b_n\}$ converges in the weak-star topology to a bounded solution of $\partial b/\partial \bar{z} = \mu$. Lemma 5 is not quite enough to provide the sequence of solutions $\{b_n\}$; the difficulty is that the bound in Lemma 5 depends on the parameter δ which, through Lemma 4, is related to the spacing (in the pseudo-hyperbolic metric) of the points in the supporting set $\{z_j\}$, and if a general Carleson measure is going to be approximated by a sequence of measures with finite point support, then the spacing of the points in the support of the approximating measures will decrease to zero as the approximations converge. What is needed is a method for decomposing the approximating measures in such a way that the spacing between points of support for each element of the decomposition remains large, yet the sum of the Carleson constants of the elements in the decomposition remains constant. The next lemma uses this approach to produce a method for solving the equations $\partial b_n/\partial \bar{z} = \mu_n$ with a uniform bound on the sequence of solutions b_n.

Lemma 6 [Garnett, 1981]
Let $\mu = \sum_{j=1}^{M} \alpha_j x_j \delta_{z_j}$ be a measure supported on the finite set $\{z_j =$

$x_j + iy_j\}$, with masses $\alpha_j x_j$ at the points z_j, and with Carleson constant $N(\mu) \le C$. Then there exist an integer N, functions $b_p(z)$, and a function

$$b(z) = \frac{1}{N}\sum_{p=1}^{2N} b_p(z)$$

such that each $b_p(z)$ is a function of the type produced in Lemma 5, $\partial b/\partial \bar{z} = \widehat{\mu}$ for a measure $\widehat{\mu}$ that is arbitrarily close to μ, and $|b(it)| < KC$ for $t \in \mathbb{R}$ and K a constant independent of μ.

Proof:
First it is shown that μ may be approximated arbitrarily closely by a new measure $\widehat{\mu}$ of the form $C/N \sum x_j \delta_{z_j}$. The support of $\widehat{\mu}$ is the same as the support of μ, but each point mass $x_j \delta_{z_j}$ may be repeated a finite, and possibly large number of times in the new sum. If N is chosen to be a sufficiently large positive integer, the coefficients α_j in the finite sum μ may be uniformly approximated to arbitrary accuracy by $\alpha_j \approx n_j/NC$ in which n_j are positive integers and C is the Carleson constant of μ. If each term in the sum $\sum \alpha_j x_j \delta_{z_j}$ is expanded as

$$\alpha_j x_j \delta_{z_j} \approx \frac{C}{N}(x_j \delta z_j + \ldots (n_j \ \textit{times}) \ldots + x_j \delta z_j)$$

then a renumbering of the terms in the summation gives the approximation

$$\mu \approx \widehat{\mu} = \frac{C}{N}\sum_j x_j \delta_{z_j}$$

From here on no distinction will be made between the measure μ and the approximation $\widehat{\mu}$.

In the second part of the proof a systematic method of decomposing the measure μ is established. The point masses $x_j \delta_{z_j}$ are distributed amongst a finite number of sets in such a way that the distance between any two points in the same set is large in the hyperbolic metric.

Choose a square Q_0 with $\operatorname{supp} \mu \subset Q_0 \subset \mathcal{H}$ and a side of length $l(Q_0)$ lying on the imaginary axis. This square may be subdivided to form a dyadic sequence of squares of uniform hyperbolic size as follows (Figure 1.4 illustrates the construction). Let Q_1, Q_2 be the two adjacent squares that comprise the left half of the square Q_0. Each have sides of length $l(Q)/2$, and each have one side on the imaginary axis; continue this subdivision process inductively on each square Q_i until the squares $Q_{2^n}, Q_{2^{n+1}-1}$ are outside the support of μ for some n (the process is guaranteed to stop because the support of μ, which is finite, is compactly contained in $\mathcal{H}$). Since the Carleson constant of μ is fixed to the constant C, a simple count shows that the right hand section of any dyadic square Q can contain at

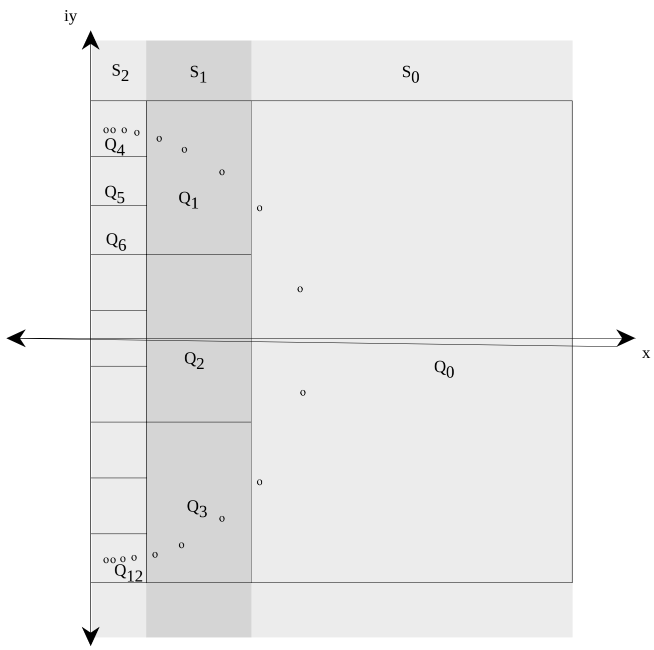

Figure 1.4. Dyadic Subdivision of the Half Plane

most $2N$ points z_j. This allows the points $\{z_j\}$ to be partitioned into $2N$ sets $\{S_n\}$ in such a way that the spacing between any two points in the same set is uniformly bounded from below by $a = 1/3$.

The sets S_n are explicitly defined as follows. For every n, let $S_n = \{z_j : l(Q_0)2^{-n-1} \leq x_j \leq l(Q_0)2^{-n}\}$ and order the elements of each S_n so that $S_n = \{x_{k,n} + iy_{k,n}\}$ with

$$y_{k-1,n} \leq y_{k,n} \leq y_{k+1,n}.$$

Then the set $\{z_j\}$ may be split into $2N$ sequences $Y_1, \ldots, Y_{2N}$ such that the points in each S_n are evenly distributed between the Y_r, i.e. if $z_j = x_{k,n} + iy_{k,n} \in S_n$ then $z_j \in Y_r$ if $r = k \bmod 2N$. Now suppose that $P \subset \mathcal{H}$ is a fixed square of arbitrary size with one side lying on the imaginary axis, let $M_n(P)$ be the number of points in $S_n \cap P$, then each set $Y_r \cap S_n \cap P$ must contain fewer than $1 + M_n(P)/(2N)$ points z_j, and

$$\sum_{Y_r \cap P} x_j \quad \leq \quad \sum_{n: S_n \cap P \neq \phi} \left(1 + \frac{M_n(P)}{2N}\right) 2^{-n}\, l(Q_0)$$

$$
\begin{aligned}
&\le 2l(P)\sum_{n=0}^{\infty} 2^{-n} + \frac{1}{mN}\sum_{z_j\in P} mx_j \\
&\le 4l(P) + \mu(P) \\
&\le 5l(P)
\end{aligned}
\tag{1.29}
$$

Consider the sets $\{X_p\}$ defined by

$$X_r = Y_r \cap \bigcup_{n \text{ even}} S_n \tag{1.30}$$

and

$$X_{2r+1} = Y_r \cap \bigcup_{n \text{ odd}} S_n. \tag{1.31}$$

then the measures $\mu_p = \sum_{z_j\in X_p} x_i\delta_{z_j}$ satisfy $\mu = C/N\sum_p \mu_p$ and up to the factor C/N provide a decomposition of μ into measures with well spaced support and Carleson constant uniformly bounded by 5.

A bound on the separation between points of support is arrived at by the following argument. If $z_i \in S_n$ and $z_j \in S_{n-2}$ then $\rho(z_i, z_j) > 1/3$ by the definition of the sets S_n. On the other hand, if z_i and z_j are in the same set S_n then it follows, from the fact that the top half of any Q_j contains at most $2N$ points, and the way in which the set Y_r that corresponds to X_p was constructed, that z_i and z_j must be separated by at least 1 square of length $l(Q_0)2^{-n}$. Consequently, the distance between z_i and z_j must be bounded below by

$$
\begin{aligned}
\rho(z_i, z_j) &\ge \frac{2^{-n}}{\sqrt{2^{-2n} + 8^{-2n}}} \\
&\ge 1/3.
\end{aligned}
\tag{1.32}
$$

Lemma 5 can now be applied to the measures μ_p to produce functions b_p that satisfy

$$\frac{\partial b_p}{\partial \bar{z}} = \mu_p.$$

The constant δ in Lemma 5 which determines the bounds on the norms $\|b_p\|$ is estimated by using Lemma 4 and the inequalities (1.29) and (1.32). This gives an estimate on the norms $\|b_p\|$ of

$$\|b_p\| \le K,$$

in which K is an absolute constant that is independent of the measure μ_p. Let

$$b(z) = \frac{C}{N}\sum_{p=1}^{2N} b_p(z).$$

Then $\partial b/\partial \bar{z} = C/N \sum z_j \delta_{z_j} = \mu$ and

$$\|b\| \leq 2CK, \tag{1.33}$$

which completes the proof.

□

The case for a general Carleson measure μ is now straightforward, but since the proof that the approximations converge to a distribution with a bounded L^∞ function as boundary value does not add anything to the construction of the approximations, the reader is referred to Chapter 8 of [Garnett, 1981] for the details, the essential result is stated here as a theorem

Theorem 7 *Let μ be a Carleson measure with Carleson constant $N(\mu) \leq 1$. Then there is a distribution $b(z)$ with L^∞ boundary value, supported on $\bar{\mathcal{H}}$ such that*

$$\frac{\partial b}{\partial \bar{z}} = \mu,$$

and the boundary value satisfies $\|b\|_\infty < C$ for C a positive constant independent of the choice of μ. Further, there exists a sequence of measures μ_k that satisfy the criteria of Lemma 6, and which converge in a weak-star sense to $\{\mu\}$, and the corresponding sequence of solutions $\{b_k(z)\}$ converge in a distributional sense on $\mathcal{H}$ to $b(z)$, and have boundary values $b_k(iy)$ that converge weak-star to $b(iy)$ on the imaginary axis.

The reason for presenting Lemma 6 in such detail is that the construction in the proof provides a key part of the algorithm that is used to compute solutions to the Diophantine equations arising from the control problems. In this application a bound on the norm of the solution to the equation $\partial b/\partial \bar{z} = \mu$ has physical significance, and consequently a tight a priori estimate of this bound would be valuable. Unfortunately, the generality of the methods presented means that the estimates on the norms that can be obtained from Lemmas 4, 5 and 6 are too conservative to be of practical use.

1.4 A SOLUTION TO EXAMPLE 2

The theory in the previous section provides a practical way to design linear compensators for a general class of linear time invariant systems. The remainder of this chapter provides examples that illustrate the use of the theory. This section continues the example that was started in Section 1.2 by showing how a compensator is calculated for the plant that was given in the example. The next section shows how the theory applies to multi-input, multi-output problems.

The plant from the example in Section 1.2 is described by the transfer function in Equation (1.10),

$$F(z) = \frac{e^{-\tau z}}{\sigma z - 1}.$$

Values are ascribed to the parameters for the numerical calculations: σ takes the value $\sigma = 1$, making the open loop system unstable, and τ takes the values $\tau = 0.06$ or $\tau = 0.37$, the first is a small delay which has little influence on the behavior of the open loop system, and the second is a large value for the delay that makes the problem of robust stabilization significantly more challenging. The transfer function of a stabilizing controller is given in terms of an H^∞ parameter Q by the Youla parameterization (1.12). Evaluating this expression requires a solution $X_1(z)$, $X_2(z)$ to the Bezout equation 1.7 as well as the parameter $Q(z)$ which is a solution to the equation (1.16) that is associated with the Nehari problem (1.15); the Bezout equation is dealt with first. The notation of Section 1.3.1 is used to rewrite the Bezout equation in the form $P_f h = g$ in which g is the constant function $g(z) = 1$, f is the pair of co-prime factors $f_1(z) = X_1(z)$ and $f_2 = X_2(z)$, and h, the solution, is a function in $\Lambda^1(R)$ with H^∞ components $h_1 = X_1(z)$ and $h_2 = X_2(z)$. From Theorem 1 the solution may be written as $h = h^1 - P_f h^3$ in which h^1 satisfies the equations $P_f h^1 = g$ and $\partial h^1/\partial\bar{z} = P_f h^2$ for h^2 a Carleson measure on $\mathbb{C}$, and h^3 satisfies $\partial h^3/\partial\bar{z} = h^2$ for the same measure h^2. Substituting the notation of the problem gives the following set of equations:

$$\begin{aligned} X_1(z) &= \widetilde{X}_1(z) - b(z)F_2(z) \\ X_2(z) &= \widetilde{X}_2(z) + b(z)F_1(z) \end{aligned} \tag{1.34}$$

where $\widetilde{X}_1$ and $\widetilde{X}_2$ satisfy the equation

$$\widetilde{X}_1(z)F_1(z) + \widetilde{X}_2(z)F_2(z) = 1, \tag{1.35}$$

The function $b(z)$ is a distributional solution of

$$\frac{\partial b}{\partial \bar{z}} = \mu \tag{1.36}$$

for a Carleson measure μ that is supported on the half plane and that satisfies the equivalent equations

$$\begin{aligned} \frac{\partial \widetilde{X}_1}{\partial \bar{z}} &= \mu F_2 \\ \frac{\partial \widetilde{X}_2}{\partial \bar{z}} &= -\mu F_1. \end{aligned} \tag{1.37}$$

Lemma 2 guarantees the existence of a choice for $\widetilde{X}_1$, $\widetilde{X}_2$ and μ that satisfy Equations (1.35) and (1.37). In fact, since the function $F_1(z) = e^{-hz}/(z+1)$ is bounded away from zero on any set compactly contained in the right half plane, and the function $F_2(z) = (1-z)/(1+z)$ has a single zero at the point $z = 1$ and is bounded away from zero on any set that excludes a neighborhood of that point, a partition of unity that satisfies the condition in Lemma 2 is the following:

$$\phi_1(z) = \begin{cases} 1, & |z-1| < r \\ 0, & |z-1| \geq r \end{cases} \qquad \phi_2(z) = \begin{cases} 1, & |z-1| \geq r \\ 0, & |z-1| < r \end{cases}$$

Substituting this choice of partition into equation (1.25) gives $\widetilde{X}_1(z) = 1/F_1(z)\ \phi_1(z)$ and $\widetilde{X}_2(z) = 1/F_2(z)\ \phi_2(z)$ as a solution to(1.35). Taking antiholomorphic derivatives, and substituting into the first of the equations (1.37) yields $\mu = 1/(F_1(z)F_2(z))\,\partial\phi_1/\partial\bar{z}$.

The meaning of the expression $\partial\phi_1/\partial\bar{z}$ is elucidated by mollification. Let Ω denote the support of ϕ_1, and let ψ_k be a sequence of positive C^∞ functions supported on connected neighborhoods of the origin, and with the property that diameter(supp ψ_k) $\to \{0\}$ as $k \to \infty$. For each k the C^∞ function $\widetilde{\phi}_k = \phi_1 * \psi_k$ is a mollification of ϕ_1; for sufficiently large k it is supported on a region slightly larger than Ω, and takes the constant value 1 on a region slightly smaller than Ω. Let D_k denote the support of the C^∞ function $\eta_k = \partial\widetilde{\phi}_k/\partial\bar{z}$. It follows that D_k is a tubular neighborhood of the boundary of the support of ϕ_k, and that $\widetilde{\phi}_k$ takes the value 1 on the interior part of ∂D (the boundary of D), and the value 0 on the exterior part of ∂D. The function η_k may be interpreted as a complex valued measure on $\mathbb{C}$ in the following sense. Let χ be a compactly supported C^∞ function, then an application of Stokes theorem gives

$$\begin{aligned} \int \chi\, d\eta_k &= \int_{D_k} \chi \frac{\partial\widetilde{\phi}_k}{\partial\bar{z}}\, dxdy \\ &= \frac{i}{2}\int_{D_k} \chi \frac{\partial\widetilde{\phi}_k}{\partial\bar{z}}\, dz \wedge d\bar{z} \\ &= \frac{i}{2}\int_{\partial D_k} \widetilde{\phi}_k \chi\, dz - \frac{i}{2}\int_{D_k} \widetilde{\phi}_k \frac{\partial\chi}{\partial\bar{z}}\, dz \wedge d\bar{z} \end{aligned} \tag{1.38}$$

As $k \to 0$ the sequence $\widetilde{\phi}_k$ converges to ϕ_1 in the topology of the space of distributions, the area of the region D_k converges to 0, and the boundary ∂D_k converges to the set $\partial\Omega$. Since the function $\partial\chi/\partial\bar{z}$ is uniformly bounded, the second integral in (1.38) converges to 0, and the first integral, which only has a contribution from the interior part of ∂D_k, converges to $-i/2\int_{\partial\Omega}\chi\, dz$. The negative sign on the contour integral is a consequence of the orientation of the boundary ∂D_k. The expression $\partial\phi_1/\partial\bar{z}$ is interpreted as a measure supported on the set $\partial\Omega$, which acts on a C^∞ function χ by the formula

$$\int \chi \, d\left(\frac{\partial\phi}{\partial\bar{z}}\right) = -\frac{i}{2}\int_{\partial\Omega} \chi \, dz.$$

The shape of the region Ω is arbitrary provided that the partition of unity that it determines satisfies the condition in Lemma 2. For the actual computation of the solutions to the Bezout equation, Ω was chosen to be a circle for the pragmatic reasons that it is a simple curve to describe and that it seems to give reasonable results. The radius for the circle was chosen with a view to keeping the Carleson constant of the measure μ small, $r = 0.7$ was found to be a suitable value when the delay in the plant takes either of the values $\tau = 0.06$ or $\tau = 0.37$.

The only remaining step in determining a solution X_1, X_2 to equation (1.7) is the calculation of a solution to the Cauchy Riemann equation (1.36). The algorithm presented in the proof of Lemma 6 provides a way to numerically calculate an approximation to a solution of this equation. It follows from Theorem 7 that the measure μ may be approximated by a sequence of discretizations $\{\mu_k\}$, and as the discretization for the approximating measure μ_k is refined, the solutions b_k computed in Lemma 6 converge to an exact solution b in a weak-star sense. The topology associated with weak-star convergence is a natural one for a space of transfer functions in the sense that the objects of real interest are the signals in the signal space that the system modeled by the transfer function is acting on. Suppose that this signal space consists of functions in $L^1 \cap L^2$ with derivatives in L^2, then the signals have Laplace transforms in L^1. Let B be a stable, linear, time-invariant system, then the system may be identified with its impulse response, and the output that results from the system B operating on the input ν may be written as the convolution $B * \nu$. (Mathematically this is the kernel theorem for linear operators [Hörmander, 1990].) Convergence of a sequence of systems may be defined in terms of the signal space as follows: A sequence $\{B_k\}$ of systems is said to converge to a system B if, for all signals ν

$$\lim_{k\to\infty} \|B_k * \nu - B * \nu\|_2 \to 0.$$

Weak-star convergence of the corresponding sequence of transfer functions is a sufficient condition for convergence of systems defined in this way.

The numerical solution to the Cauchy Riemann equation is computed by a pair of processes running compiled C code.

Process 1 This process computes a discretization μ_k of the measure μ. The inputs to the process are a parameterization of the supporting set for μ and the weighting function associated with the measure. It discretizes the measure into a finite set of point masses, makes the weighting on the point masses uniform in size by replication, and then partitions the set of supports into subsets of well spaced points. The output from the process is a partitioned set of point masses $\Pi = \{\pi_l\}$, each partition being a set of ordered pairs $\pi_l = \{(\zeta_j, W_j)\}$ in which ζ_j are points of support, and W_j are the masses associated with them.

Process 2 This process evaluates the numerical approximation b_k to the solution b of the Cauchy Riemann equation on a given set of points. The inputs to the process are μ_k, the partitioned discrete approximation to the measure μ from Process 1, and a set of points $\{z_i\}$ on which the solution is to be evaluated. For each point z_i the process computes the value

$$b_k(z_i) = \sum_{\pi_l \in \Pi} \sum_{\zeta_j \in \pi} E(z_j; \pi_l, \zeta_j) W_j .$$

The dependence of the kernel function E on the interpolating sets is indicated explicitly by including the partition and ζ_j in the argument. The process outputs a list of values for the function b_k at the points specified in the input $\{z_i\}$.

One crucial step remains in the computation of a compensator transfer function and that is the computation of a value for the parameter Q in the Youla parameterization (1.12). This is carried out by solving the Diophantine equation (1.16),

$$A(z) = 1P(z) + Q(z)B(z),$$

The same methods that were used for the Bezout equation are used for this equation, only here there is an added requirement of optimality that comes from the Nehari Problem. An optimal solution being one with minimum norm for $P(z)$. The data in the Nehari Problem are

$$A = \begin{pmatrix} F_2 X_1 W_2 \\ F_2 X_2 W_1 \end{pmatrix} \qquad B = \begin{pmatrix} F_2^2 W_2 \\ -F_2 F_1 W_1 \end{pmatrix} .$$

The weighting functions W_1 and W_2 are those used in [Enns *et al.* , 1992],

$$\begin{aligned} W_1 &= 2\frac{1+z}{1+10z} \\ W_2 &= 0.2 . \end{aligned}$$

Equation (1.16) is rewritten in the notation of Section 1.3.1 as $P_f h = g$. The right hand side, $g = A$, is a vector with two H^∞ components, The two parts that make up f are the constant H^∞ function $f_1 = 1$ and the two component H^∞ vector $f_2 = B$, and the solution h, consists of the two component H^∞ vector $h_1 = P$, and the H^∞ function $h_2 = Q$. The same formalism that was used for the Bezout equation now yields the set of equations:

$$\begin{aligned} P &= \widetilde{P} - b(z)B \\ Q &= \widetilde{Q} + b(z) \end{aligned} \tag{1.39}$$

where $\widetilde{P}$ and $\widetilde{Q}$ satisfy

$$\widetilde{P} + \widetilde{Q}B = A \tag{1.40}$$

and $b(z)$ is a distributional solution of

$$\frac{\partial b}{\partial \bar{z}} = \mu$$

for a Carleson measure μ that satisfies either of the equivalent equations

$$\begin{aligned} \frac{\partial \widetilde{P}}{\partial \bar{z}} &= \mu B \\ \frac{\partial \widetilde{Q}}{\partial \bar{z}} &= -\mu. \end{aligned}$$

The solutions $\widetilde{P}$ and $\widetilde{Q}$ are not uniquely determined by Equation 1.40 and the choice that is made for the bounded solutions here will affect the norms of the final H^∞ solutions P and Q. The objective is to find a solution that minimizes the norm of P, one way to approach this is by trying to minimize the norm of each term on the right hand side of (1.39); that is to say, $\widetilde{P}$ and $\widetilde{Q}$ are chosen to minimize the norm of $\widetilde{P}$, and the Carleson constant for the measure μ. This strategy does not guarantee an optimal solution, indeed, although the results from Section 1.3 give an a priori bound on how far the solution will be from optimal, this bound will not generally be tight. To pursue the strategy outlined, choose $\widetilde{P}$ to be the part of A that is orthogonal to B, and choosing $\widetilde{Q} = A\cdot B/B\cdot B$. This gives a bounded analytic solution to (1.40) away form the points where $B\cdot B = 0$ If $B\cdot B$ where bounded away from zero we would have an optimal solution to the Nehari problem, as things are, these neighborhoods need to be treated separately. Choose a region Ω in the complex plane that contains neighborhoods of all the zeros of $B\cdot B$, and is bounded by an absolutely continuous closed curve, and let $\phi_1(z) = 1$ when $z \in \Omega$ and $\phi_1(z) = 0$ when $z \notin \Omega$. Modified choices for $\widetilde{Q}$ and $\widetilde{P}$ that are bounded (but no longer analytic) on all $\mathcal{H}$ are:

$$\begin{aligned} \widetilde{Q} &= \frac{A\cdot B}{B\cdot B}\phi_1 \\ \widetilde{P} &= A - \widetilde{Q}B \\ &= A - \frac{A\cdot B}{B\cdot B}B\phi_1. \end{aligned}$$

Substituting in the values for A and B gives

$$\begin{aligned}\widetilde{Q} &= \frac{W_2^2F_2X_1 - W_1^2F_1X_2}{W_2^2F_2^2 + W_1^2F_1^2}\,\phi_1 \\ \widetilde{P_1} &= F_2X_1W_2(1-\phi_1) + \frac{F_2F_1W_1^2}{W_2F_2^2 + W_1W_2}\,\phi_1 \\ \widetilde{P_2} &= F_2X_2W_1(1-\phi_1) + \frac{F_2^2W_1}{F_2^2 + W_1}\,\phi_1,\end{aligned}$$

and for the measure μ acting on a C^∞ function χ,

$$\begin{aligned}\mu(\chi) &= -\frac{i}{2}\int_{\partial\Omega}\left(\frac{A\cdot B}{B\cdot B}\right)\chi\,dz \\ &= -\frac{i}{2}\int_{\partial\Omega}\frac{W_2^2F_2X_1 - W_1^2F_1X_2}{W_2^2F_2^2 + W_1^2F_1^2}\chi\,dz.\end{aligned}$$

The criteria for choosing the region Ω are that it should include the zeros of the function $W_2^2F_2^2 + W_1^2F_1^2$ and that the Carleson constant of the measure μ should be minimized. For the results presented, the region Ω was chosen by plotting the weighting function $(W_2^2F_2X_1 - W_1^2F_1X_2)/(W_2^2F_2^2 + W_1^2F_1^2)$ and choosing by inspection a contour that includes the singularities of the function, yet keeps the Carleson constant of μ small. A more rigorous approach may lead to better results, but would certainly require greater computational effort and more precise information about the functions involved. The chosen contour is illustrated in Figure 1.5. With all the pieces now assembled the parameter Q, and consequently the controller transfer function, are calculated using the same algorithms that were used for computing solutions to the Bezout equation. The results are given in Figures 1.6 to 1.9.

Figure 1.6 contains two graphs that describe the solution for the H^∞ parameter Q in terms of a transfer function and the time domain response to a square input pulse of unit magnitude and 1 second duration. The value for the delay chosen was $\tau = 0.37$. Figure 1.7 contains Nyquist plots of the open loop transfer functions of the combined system including Plant and feedback compensator for two values of the delay $\tau = 0.06$ and $\tau = 0.37$. The frequency variable that parameterizes the curves is y. A comparison with the corresponding Nyquist plot from [Enns *et al.* , 1992], Figure 7, illustrates well the differences in the controllers that are produced by the two different approaches. The controller of Enns *et al.* achieves better stability and better low frequency sensitivity, by placing a non minimum-phase zero in the controller transfer function. This advantage is to be expected since the methods that the authors use allow them to produce an optimal solution to the Nehari problem associated with the controller design. However, in order to apply the theory from [Özbay *et al.* , 1993] some requirements

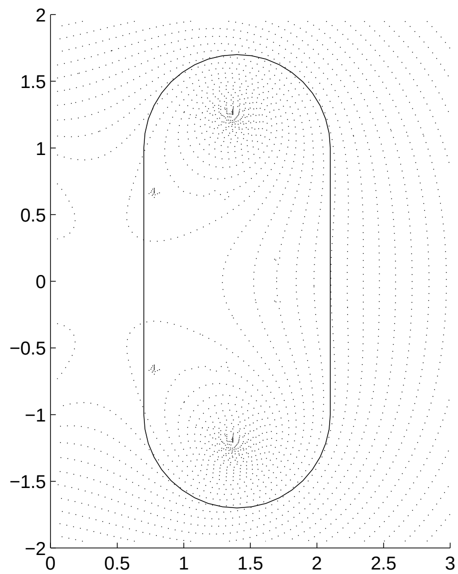

Figure 1.5. Contour for Enns' Solution

need to be placed on the systems that they consider. The plant transfer function must be factored as a product of an H^∞ function and a rational function with inverse in H^∞, and each factor needs to be decomposed by an inner outer factorization. Although a large number of interesting systems satisfy this first requirement, the requirement itself is restrictive. The second requirement poses a computational problem, and the recent work [Flamm and Crow, 1994] which addresses the problem of computing numerical approximations to inner outer factorizations should extend the applicability of the results in Özbay *et al* to more complicated examples. The method presented in this paper avoids both of these restrictions by avoiding the operator theoretic techniques that Özbay *et al* use. A second problem that Özbay *et al.* face is that they need a solution to the Bezout equation in a factored form that they can use. In the example that Enns *et al.* choose the Bezout equation has a simple solution, but in general, finding a solution is a difficult problem.

Figures 1.8 and 1.9 plot the transfer functions that determine the closed loop sensitivity and the robustness. Comparison with Figures 9 and 10 of [Enns *et al.* , 1992], provides confirmation of the comments made in the previous paragraph.

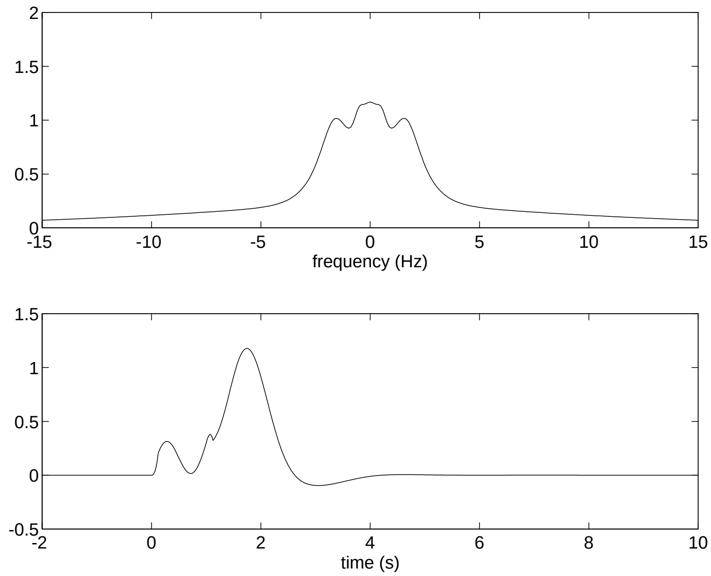

Figure 1.6. Transfer function and pulse response for $Q(z)$ with $\tau = 0.37$.

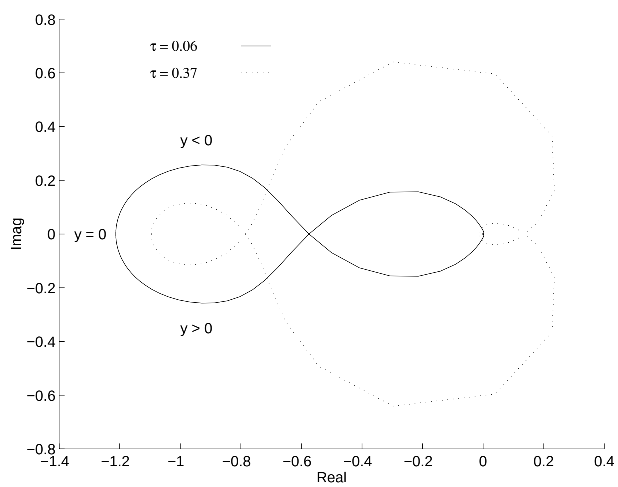

Figure 1.7. Nyquist plot of loop gain for controlled system.

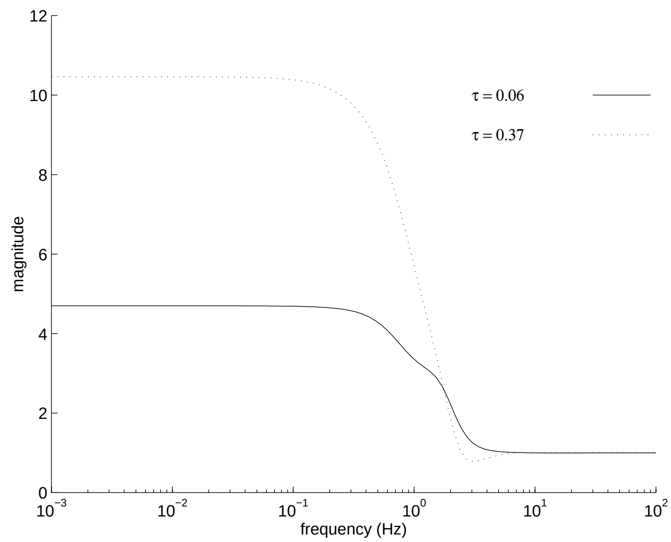

Figure 1.8. $(1 + P(iy)C(iy))^{-1}$.

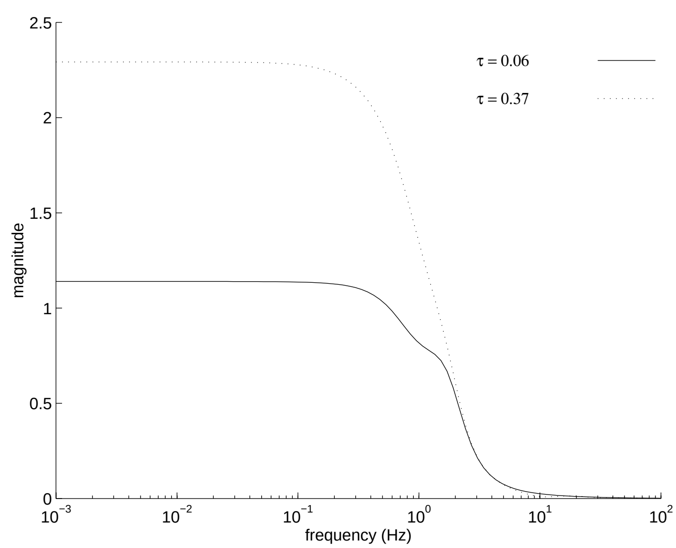

Figure 1.9. $W_2(iy)C(iy)(1 + P(iy)C(iy))^{-1} \leq 1$.

1.5 GENERAL UNDERDETERMINED SYSTEMS OF LINEAR DIOPHANTINE EQUATIONS

This section extends the computational technique that provided a stabilizing controller for a single-input single-output system, and shows how this extension is used to compute a stabilizing controller for a multiple-input multiple-output system. Multiple-input multiple-output systems are treated along the lines of the "standard problem" formulated in [Francis, 1987] The framework that Francis used for systems described by rational transfer function matrices applies also to systems described by matrices of irrational transfer function matrices.

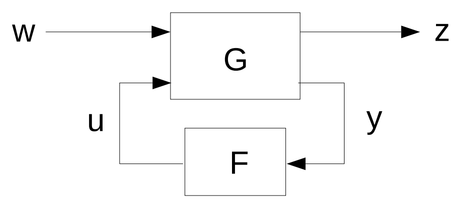

Figure 1.10. Configuration of multiple-input multiple-output controller

Figure 1.10 depicts a feedback controller for a general class of multiple-input multiple-output robust control problems The plant has a block transfer function matrix

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

and the equations for the closed loop system are:

$$\begin{aligned} z &= F_{11}w + F_{12}u \\ y &= F_{21}w + F_{22}u \\ u &= Gy \end{aligned}$$

The robust stabilization problem is the problem of choosing a controller with transfer function matrix G in such a way as to minimize the operator norm of the matrix transfer function that maps the input signal w on to the output signal z. For a stable linear system the operator norm is the L^∞ norm of the largest singular value of the transfer function matrix which, in the single-input single-output case is just the H^∞ norm of the transfer function. The following theorem combines a number of results from Chapter 4 of [Francis, 1987] including the Youla parameterization of stabilizing controllers.

Theorem 8 [Francis, 1987]
Assume that F is stabilizable, then:

(i) G stabilizes F if and only if G stabilizes F_{22}.

(ii) Suppose $F_{22} = F_1 F_2^{-1} = \widetilde{F}_2^{-1}\widetilde{F}_1$ are co-prime factorizations of F_{22}, then there exist X_1, X_2, and $\widetilde{X}_1$, $\widetilde{X}_2$ such that[5]

$$\begin{bmatrix} \widetilde{X}_2 & -\widetilde{X}_1 \\ -\widetilde{F}_1 & \widetilde{F}_2 \end{bmatrix} \begin{bmatrix} F_2 & X_1 \\ X_1 & X_2 \end{bmatrix} = \mathbf{1}. \tag{1.41}$$

and the set of all G stabilizing F_{22} is parameterized by the formulae

$$\begin{aligned} G &= (X_1 - F_2 Q)(X_2 - F_1 Q)^{-1} \\ &= (\widetilde{X}_2 - Q\widetilde{F}_1)^{-1}(\widetilde{X}_1 - Q\widetilde{F}_2) \\ Q &\in \mathrm{H}^\infty. \end{aligned}$$

(iii) With G given by the parameterization in (ii), and with the transfer functions T_1, T_2, T_3 given by

$$\begin{aligned} T_1 &= F_{11} + F_{12}F_2\widetilde{X}_1 F_{21} \\ T_2 &= F_{12}F_2 \\ T_3 &= \widetilde{F}_2 F_{21}, \end{aligned}$$

the transfer function from w to z in Figure 1.10 equals $T_1 - T_2 Q T_3$.

The applicability of the theorem hinges on the definition of co-prime factorization. Given a factorization ring for a class of transfer functions, an appropriate notion of co-primeness is one that implies the existence of a doubly co-prime factorization that satisfies equation (1.41). When the factorization is taken over H^∞, co-primeness will be closely related to the condition in the premise of Lemma 2. Malcolm Smith proves in [Smith, 1989] that if a plant has a transfer function matrix that is factorizable over H^∞, and is feedback stabilizable, then it has a doubly co-prime factorization that satisfies (1.41). The approach to controller design presented in Section 1.2 is generalized to multiple-input multiple-output systems by Theorem 8. The problems here that correspond to the problems of solving the Bezout equation, are the problem of finding a doubly co-prime factorization, and finding the H^∞ matrix that minimizes the norm of the transfer function that maps w to z

$$P = T_1 - T_2 Q T_3. \tag{1.42}$$

The next paragraphs demonstrate how equations (1.41) and (1.42) may be recast as underdetermined systems of equations over the ring H^∞ of the form

$$Ax = b, \tag{1.43}$$

[5] $\mathbf{1}$ is used to denote the identity matrix which, in this context, is a matrix of constant H^∞ functions

and the remainder of this section shows how the theory from section 1.3 can be applied to the solution of these systems.

Finding a doubly co-prime factorization for the transfer function matrix F that satisfies equation (1.41) is equivalent to finding a left factorization $F = \widetilde{F}_2^{-1}\widetilde{F}_1$ a right factorization $F = F_1 F_2^{-1}$, and H^∞ matrices X_1, X_2, $\widetilde{X}_1$ and $\widetilde{X}_2$ that satisfy the four equations

$$\begin{aligned} \widetilde{X}_2 F_2 - \widetilde{X}_1 F_1 &= 1 && (1.44)\\ \widetilde{F}_2 X_2 - \widetilde{F}_1 X_1 &= 1 && (1.45)\\ \widetilde{F}_2 F_1 - \widetilde{F}_1 F_2 &= 0 && (1.46)\\ \widetilde{X}_2 X_1 - \widetilde{X}_1 X_2 &= 0. && (1.47) \end{aligned}$$

Equations (1.44) and (1.45) are matrix Bezout equations, and equation (1.46) is automatically satisfied since the left and right factorizations are factorizations of the same transfer function matrix. Given left and right co-prime factorizations of F, and arbitrary solutions $\widetilde{Y}_1$ and $\widetilde{Y}_2$ to equation (1.45), and Y_1, Y_2 to equation (1.44), a little algebraic manipulation yields the following parameterization of all doubly co-prime factorizations:

$$\begin{aligned} X_1 &= Y_1 + F_2 A\\ X_2 &= Y_2 + F_1 A\\ \widetilde{X}_1 &= \widetilde{Y}_1 + (A - \widetilde{Y}_1 Y_2 + \widetilde{Y}_1 Y_2)\widetilde{F}_2\\ \widetilde{X}_2 &= \widetilde{Y}_2 + (A - \widetilde{Y}_1 Y_2 + \widetilde{Y}_1 Y_2)\widetilde{F}_2. \end{aligned}$$

The parameter A is a matrix with entries in H^∞. With this result the computation of a doubly co-prime factorization reduces to the solution of the two matrix Bezout equations (1.44) and (1.45). The form of these equations is similar to the form of (1.42), and with suitable substitutions the solution of each of the three equations is subsumed by the following problem: given A^1, B^1, A^2, B^2, and C, find X^1 and X^2 that solve

$$A^1 X^1 B^1 + A^2 X^2 B^2 = C. \tag{1.48}$$

This equation is the matrix analog of (1.17) for multiple-input multiple-output systems.

Equation (1.48) has the form of a general linear equation in the entries of the matrices X^1 and X^2. The next step in the computation of solutions is to recast this equation in the form of (1.43), $Ax = b$. This is done by stacking the columns of the matrices X^1 and X^2 to form a long vector, and replacing the left and right multiplying matrices by one left multiplying matrix. Fix the dimension of A to be $m \times n$ with $m < n$, then A represents a module homomorphism with domain $\mathrm{H}^\infty \times \overset{n}{\cdots} \times \mathrm{H}^\infty$, and image $\mathrm{H}^\infty \times \overset{m}{\cdots} \times \mathrm{H}^\infty$.

As in Section 1.3.1 the solution to (1.43) is based on Theorem 1, but the definition of the spaces and the operators in (1.18) and Figure 1.3 need to be changed. Define the following modules over a ring R

$$\begin{aligned}\Lambda^0(R) &= R\times \overset{m}{\ldots} \times R\\ \Lambda^1(R) &= R\times \overset{n}{\ldots} \times R\\ \Lambda^2(R) &= \wedge^{n-m-1}(R\times \overset{m}{\ldots} \times R).\end{aligned}$$

The three rings of interest are the same as those in Section 1.3, H^∞, $\mathcal{B}$, and $\mathcal{C}$. Denote the rows of the matrix A by $A_1 \ldots A_m$ and the columns by $a_1 \ldots a_n$, then $A_i \in \Lambda^1(\mathrm{H}^\infty)$, and $a_j \in \Lambda^0(\mathrm{H}^\infty)$. Define the homomorphism $P_A : \Lambda^1(R) \to \Lambda^0(R)$ by $P_A x = \sum x_i a_i$, then Equation (1.43) can be written $P_A x = b$. let $\{e_\beta\}$ be a basis for $\wedge^{n-m-1}\mathbb{C}^n$, and let $y = y_\beta e_\beta$ be an element of $\Lambda^2(R)$. Define a second homomorphism $P_A : \Lambda^2(R) \to \Lambda^1(R)$ by

$$(P_A y) = \sum_\beta y_\beta(\star(A_1 \wedge \ldots \wedge A_m \wedge e_\beta)) \tag{1.49}$$

in which the star homomorphism[6] is taken relative to the normal Euclidean scalar product on $\mathbb{C}^n$. With these definitions the sequence (1.18) may be rewritten as the sequence

$$\Lambda^2(R) \xrightarrow{P_A} \Lambda^1(R) \xrightarrow{P_A} \Lambda^0(R) \longrightarrow 0, \tag{1.50}$$

which is exact at $\Lambda^1(R)$. In fact the sequence in (1.50) may be extended leftward to form a complete sequence, but the definitions given are already enough for an application of Theorem 1.

Theorem 2 provides an algorithm that produces a solution to (1.43) as follows. First find $x^0 \in \Lambda^1(\mathcal{B})$ that solves the equation

$$P_A x^0 = b. \tag{1.51}$$

The ring $\mathcal{B}$ is the ring of distributions with boundary values in L^∞ that was introduced in Section 1.3.1. The solution needs to be chosen so that $\partial x^0/\partial \bar{z} \in \Lambda^1(\mathcal{C})$, where $\mathcal{C}$ is the ring of Carleson measures that have support on the right half plane. With this in mind choose $x^1 \in \Lambda^2(\mathcal{C})$ to be a solution of

$$P_A x^1 = \frac{\partial x^0}{\partial \bar{z}} \tag{1.52}$$

and $x^2 \in \Lambda^2(\mathcal{B})$ to be a solution of

$$\frac{\partial x^2}{\partial \bar{z}} = x^1. \tag{1.53}$$

It follows from Theorem 1 that a solution to (1.43) is given by

$$x = x^0 - P_A x^2. \tag{1.54}$$

[6]The star homomorphism is determined by its action on homogeneous forms. On these it satisfies the equation $\star(e_{i_1} \wedge \ldots \wedge e_{i_k}) \wedge (e_{i_1} \wedge \ldots \wedge e_{i_k}) = e_1 \wedge \ldots \wedge e_n$

As in the scalar case, the real computational problems lie in inverting the three operators $\partial/\partial\bar{z} : \Lambda^2(\mathcal{B}) \to \Lambda^2(\mathcal{C})$, $P_A : \Lambda^1(\mathcal{B}) \to \Lambda^0(\mathcal{B})$ and $P_A : \Lambda^2(\mathcal{C}) \to \Lambda^1(\mathcal{C})$. Fortunately though, the same approach that was used in the scalar case may be applied to systems of equations with some extra algebra. The first operator, $\partial/\partial\bar{z}$ is the easiest to deal with, Equation (1.53) may be solved by applying the method of Section 1.3.2 to calculate each component of x^2 from the corresponding component of x^1. The remaining operators are inverted by using a method due to [Rao, 1983] to construct a left inverse, and some algebraic constructions that are similar to those presented in [Berenstein and Struppa, 1986] and [Struppa, 1983].

Consider first equation (1.51). Denote the rank m minors of A by A_γ, then the index γ can take one of $n!/(n-m)!m!$ values that correspond to the choices of m columns from the n columns of A. Provided that the functions A_γ satisfy the condition of Lemma 2, there exists a partition of the plane into sets Ω_γ such that if ϕ_γ is the characteristic function of the set Ω_γ, that is, A_γ is bounded away from zero outside the set Ω_γ, and

$$\phi_\gamma(z) = \begin{cases} 0 & z \in \Omega_\gamma \\ 1 & z \notin \Omega_\gamma \end{cases} ,$$

then the distributional derivatives $\partial\phi_\gamma/\partial\bar{z}$ are Carleson measures supported on the boundaries $\partial\Omega_\gamma$. Choose $G_\gamma = \phi_\gamma/A_\gamma$, then each G_γ is a bounded analytic function on the interior of Ω_γ, is identically zero outside Ω_γ, and has a distributional derivative $\partial G_\gamma/\partial\bar{z}$ that is a Carleson measure supported on the boundary $\partial\Omega_\gamma$. Further, the functions G_γ solve the equation

$$\sum_\gamma A_\gamma G_\gamma = 1.$$

[Rao, 1983] uses the Cauchy Binet theorem to show that if

$$g_{jk} = \sum_\gamma G_\gamma \frac{\partial A_\gamma}{\partial a_{kj}}$$

then the matrix $G = [g_{jk}]$ is a right inverse of A with rank m minors G_γ. It follows that a solution to equation (1.51) is given by

$$\begin{aligned} x_j^0 &= \sum_k \sum_\gamma G_\gamma \frac{\partial A_\gamma}{\partial a_{kj}} b_k \\ &= \sum_k \sum_\gamma \frac{\phi_\gamma}{A_\gamma} \frac{\partial A_\gamma}{\partial a_{kj}} b_k \end{aligned} \tag{1.55}$$

The final equation that needs to be solved is equation (1.52) $P_A x^1 = \partial x^0/\partial\bar{z}$. Let x^1 have components y_β with respect to the canonical basis for $\Lambda^2(\mathcal{C})$. When the solution from (1.55) is substituted for x^0, and the expression for the operator P_A from (1.49) is expanded in coordinates, the j'th component of equation (1.52) becomes

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \sum_{\substack{\gamma \\ j\in\gamma}} \sum_k \frac{\partial}{\partial \bar{z}} \left(G_\gamma \left(\frac{\partial A_\gamma}{\partial a_{kj}} \right) b_k \right)$$

The summation on the left hand side in this formula is taken over all multi-indices α and β such that $j \notin \alpha \cup \beta$, $\alpha \cap \beta = \emptyset$, and $|\alpha| = m$. Substituting the solution for G_γ gives

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \sum_{\substack{\gamma \\ j\in\gamma}} \sum_k \frac{b_k}{A_\gamma} \frac{\partial A_\gamma}{\partial a_{kj}} \frac{\partial \phi_\gamma}{\partial \bar{z}}.$$

It follows from the choice of ϕ that the sum on the right hand side is supported entirely on the curve segments $\partial\Omega_{\gamma_p} \cap \partial\Omega_{\gamma_p}$. So the components y_β of the solution x^2 are measures supported on the boundaries $\partial\Omega_{\gamma_p}$, and at any point on these boundaries there are n equations for the $n!/m!(n-m-1)!$ variables y_β of the form

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \begin{cases} 0 & j \notin \gamma_p \cup \gamma_q \\ \sum_k \pm \frac{b_k}{A_{\gamma_p}} \frac{\partial A_{\gamma_p}}{\partial a_{jk}} & j \in \gamma_p - \gamma_q \\ \sum_k \left(\pm \frac{1}{A_{\gamma_p}} \frac{\partial A_{\gamma_p}}{\partial a_{jk}} \pm \frac{1}{A_{\gamma_q}} \frac{\partial A_{\gamma_q}}{\partial a_{jk}} \right) & j \in \gamma_p \cap \gamma_q \end{cases}$$

The arbitrary signs are determined by the sense of integration inherent in the measures $\partial\psi_p/\partial\bar{z}$ and $\partial\psi_q/\partial\bar{z}$

Although the algebra associated with the inversion of the operators $P_A : \Lambda^1(\mathcal{B}) \to \Lambda^0(\mathcal{B})$ and $P_A : \Lambda^2(\mathcal{C}) \to \Lambda^1(\mathcal{C})$ seems complicated, the real computational difficulties are the same as those experienced with the single-input single-output system, namely, choosing a partition Ω_γ and computing minimal norm solutions of $\partial b/\partial\bar{z} = \mu$ for a Carleson measure μ. The requirement that Lemma 2 places on the minors A_γ of the matrix A induces the appropriate co-primeness conditions on the left and right factorizations of the transfer function matrix F for the multiple-input multiple-output system.

1.6 CONCLUSION

In this chapter we have presented a new computational method for H^∞ controller design. The method places two requirements on the systems to which it applies: an explicitly computable co-prime factorization of the system over H^∞ functions should exist, and sufficient information about the location of the zeros of the factors is needed to construct the partition of unity in Lemma 2. These requirements are very close to necessary conditions for a linear plant to be stabilizable, a fact that indicates that the techniques presented are potentially widely applicable.

The method has been demonstrated on a simple example drawn from the literature. For this example a procedure already exists to design a controller that is optimal in the sense of H^∞ control, and as such it provides a good standard against which to measure the results of our methods.

Two areas provide obvious avenues for further research. The first is the selection of the partition of unity that is postulated in in Lemma 2. The particular selection made for a given problem affects the quality of the solution through the norm of the inverse in equation (1.25), and through the Carleson constant associated with the Blaschke product in the inequality (1.26). The intricate construction that is required in the Carleson's proof of the Corona Theorem would indicate that in the most general case choosing an optimal selection is a difficult problem. however, many of the situations that are of interest in engineering are described by boundary value problems and delay differential equations of the type presented in this paper; in these cases the additional structure provided by the problem description can be exploited to provide partitions of unity without recourse to elaborate constructions.

The second avenue for future research is the problem of finding a better interpolating function for use in the inversion of the $\partial/\partial\bar{z}$ operator. The function given in Theorem 3 is not unique, and although the bound given in equation (1.28) is optimal in the sense that the constant C_0 is independent of the measure, a particular choice of interpolating function tailored to a particular measure could produce a lower bound.

Finally, a note on the calculations. The method described is computationally intensive, however, with careful programming, the computations are not prohibitive. The computations for the controller presented in section 1.4 took minutes, rather than hours, when the algorithms were coded in C and run on a SPARC Station 10 workstation. The predominant computation involves evaluating a small set of functions over a large set of data points with no interdependencies in the evaluations; performing this type of calculation on an SIMD architecture would lead to a large increase in speed.

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