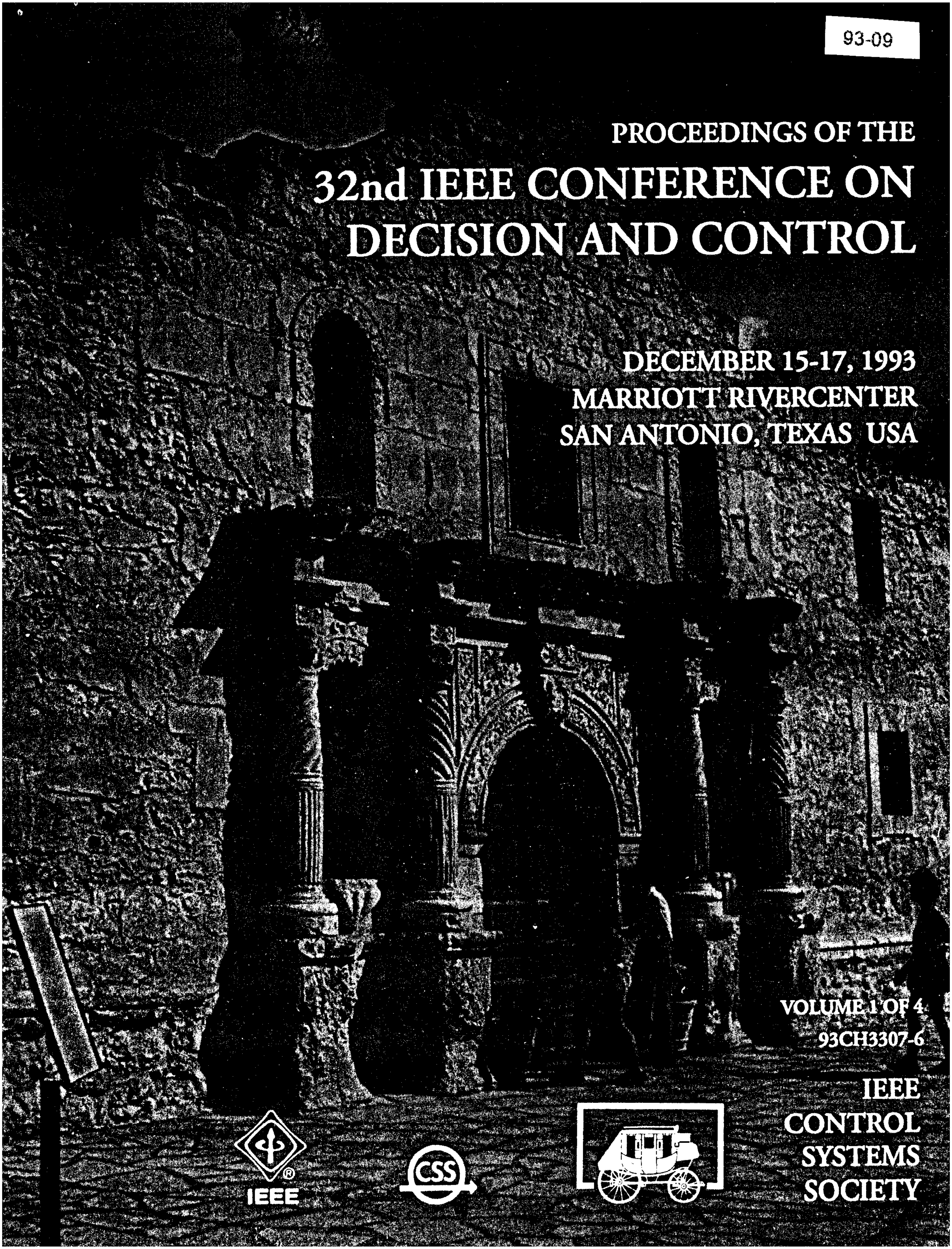

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ROBUST OUTPUT FEEDBACK CONTROL FOR DISCRETE-TIME NONLINEAR SYSTEMS: THE FINITE-TIME CASE*

J.S. Baras[1] M.R. James[2]

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[1] Martin Marietta Chair in Systems Engineering, Department of Electrical Engineering and Institute for Systems Research, University of Maryland, College Park, MD 20742, USA

[2] Department of Systems Engineering and Cooperative Research Centre for Robust and Adaptive Systems, Australian National University, Canberra, ACT 0200, Australia

Abstract

In this paper we present a new approach to the solution of the output feedback robust control problem. We employ the recently developed concept of *information state* for output feedback dynamic games, and obtain necessary and sufficient conditions for the solution to the robust control problem expressed in terms of the information state. The resulting controller is an information state feedback controller, and is intrinsically *infinite dimensional.*

1. Introduction

In this paper we present a new approach to the solution of the output feedback robust control problem. We consider discrete-time nonlinear systems (plants) Σ described by the state space equations of the general form

$$\begin{aligned} x_{k+1} &= b(x_k, u_k, w_k), \\ z_{k+1} &= l(x_k, u_k, w_k), \\ y_{k+1} &= h(x_k, u_k, w_k). \end{aligned} \tag{1.1}$$

Here, $x_k \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an ouput quantity $y_k \in \mathbf{R}^p$ is observed. The additional output quantity $z_k \in \mathbf{R}^q$ is the regulated variable, depending on the particular problem at hand. The control input is $u_k \in U \subset \mathbf{R}^m$, and $w_k \in \mathbf{R}^r$ is a disturbance input. For instance, w could be due to modelling errors, sensor noise, etc. It is assumed that the origin is an equilibrium for the system (1.1): $b(0,0,0) = 0$, $l(0,0,0) = 0$, and $h(0,0,0) = 0$.

The (finite time) *output feedback robust control problem* is: given $\gamma > 0$ and a finite time interval $[0, M]$, find a controller $u = u(y(\cdot))$, responsive only to the observed output y, such that the resulting closed loop system Σ^u achieves the following goal;

> Σ^u is *finite gain*, i.e., for each initial condition $x_0 \in \mathbf{R}^n$ the input-output map $\Sigma^u_{x_0}$ relating w to z is finite gain, which means that there exists a finite quantity $\beta^u(x_0)$ such that
>
> $$\begin{cases} \sum_{i=0}^{M-1} |z_{i+1}|^2 \le \gamma^2 \sum_{i=0}^{M-1} |w_i|^2 + \beta^u(x_0) \\ \text{for all } w \in \ell_2([0, M-1], \mathbf{R}^r). \end{cases} \tag{1.2}$$
>
> Since $x_0 = 0$ is an equilibrium, we also require that $\beta^u(0) = 0$.

Of course, β will also depend on γ and M.

Note that we have specified the robust control problem in terms of the family of initialized input-output maps $\{\Sigma^u_{x_0}\}_{x_0 \in \mathbf{R}^n}$, whereas the conventional problem statement for linear systems [5] refers only to the single map Σ^u_0. This is often expressed in terms of the "H_∞ norm"(i.e. L_2 gain) of Σ^u_0:

$$\| \Sigma^u_0 \|_{H_\infty[0,M]} \triangleq \sup_{w \in \ell_2([0,M-1],\mathbf{R}^r),\, w \neq 0} \frac{\|z\|_{\ell_2([1,M],\mathbf{R}^q)}}{\|w\|_{\ell_2([0,M-1],\mathbf{R}^r)}}.$$

For linear systems, the linear structure means that the solvability of the finite time robust control problem is equivalent to the solvability of a pair of Riccati difference equations (and a coupling condition), and so implicitly all the maps $\Sigma^u_{x_0}$ are considered. For nonlinear systems, our formulation seems natural and appropriate, since otherwise if we were to follow the linear systems formulation, one would need assumptions relating non-zero initial states x_0 to the equilibrium state 0 (such as reachability in the infinite-time case). Indeed, a solution $u = u^*$ to our problem yields

$$\| \Sigma_0^u \|_{H_\infty[0,M]} \leq \gamma, \tag{1.3}$$

as is the case for linear systems.

The solution to the output feedback robust control problem for linear systems [5] has the structure of an observer and a controller, and involves filter and control type Riccati equations. The standard LQG (or H_2) controller obtains as $\gamma \to \infty$. The nonlinear problem has been considered recently by a number of authors, for example Ball *et al* [1], [2], Isidori-Astolfi [6]. However, a complete solution is not yet available.

Our approach to this problem was motivated by ideas from stochastic control and large deviations theory. In our earlier paper [7], we explored the connection between a partially observed risk-sensitive stochastic control problem and a partially observed dynamic game, and we introduced the use of an "information state"for solving such games. The information states for each of these two problems are solutions of nonlinear infinite dimensional dynamical systems, and contain observable information that is relevant to the control objectives. They do not necessarily attempt to estimate the state of the system being controlled. In other words, estimating the value of the state is not the exclusive objective of the information state system; instead, the control objective is taken into consideration and so the resulting state estimate is suboptimal, but nonetheless more suitable to achieving the control objective than an observer designed with the exclusive aim of state estimation. Thus the information state is the optimal trade-off between estimation and control, and is determined naturally by the problem. The original output feedback problem is replaced by an equivalent one with "complete"state information, namely the information state. The concept of information state is well known in stochastic control theory, and for the risk-neutral (or H_2) problem, it is the conditional density, possibly unnormalized, and is concerned only with state estimation (Kumar-Varaiya [9]).

In this paper we apply the concept of information state to the output feedback robust control problem, and obtain both necessary and sufficient conditions expressed in terms of dynamic programming equations involving the information state. Our solution is analogous to the filter and control type Riccati equations arising in the case of linear systems. In particular, our results imply that if the robust control problem is at all solvable by an output feedback controller, then it is solvable by an *information state feedback controller.* The information state feedback controller we obtain has an observer/controller structure. The "observer"is the (infinite dimensional) dynamical system for the information state:

$$p_k = F(p_{k-1}, u_{k-1}, y_k).$$

The control

$$u_k = \bar{u}_k^*(p_k)$$

is determined by a dynamic programming equation, and the value function solving it is a function of the information state. This dynamic programming equation is an infinite dimensional recursion defined for an infinite dimensional control problem, namely that of controlling the information state. Our solution is therefore an *infinite dimensional dynamic compensator*, Figure 1.

The infinite time output feedback robust control problem will be considered in a separate paper. The continuous-time problem is also solvable using our framework, at least formally [8].

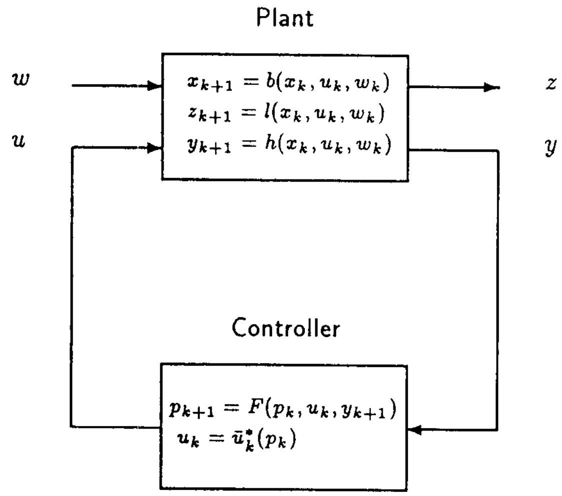

Figure 1

2. Dynamic Game

Our aim in this section is to express the output feedback robust control problem in terms of a dynamic game.

Let $\mathcal{O}_{k,l}$ denote the set of output feedback controllers defined on the time interval $[k, l]$, so $u \in \mathcal{O}_{k,l}$ means that for each $j \in [k, l]$ there exists a function $\bar{u}_j : \mathbf{R}^{(j-k+1)p} \to U$ such that $u_j = \bar{u}_j(y_{k+1,j})$. For $u \in \mathcal{O}_{0,M-1}$, Σ^u denotes the closed loop system (1.1). We introduce the function space

$$\mathcal{E} = \{p : \mathbf{R}^n \to \mathbf{R}^*\},$$

and define for each $x \in \mathbf{R}^n$ a function $\delta_x \in \mathcal{E}$ by

$$\delta_x(\xi) \triangleq \begin{cases} 0 & \text{if } \xi = x, \\ -\infty & \text{if } \xi \neq x. \end{cases}$$

For $u \in \mathcal{O}_{0,M-1}$ and $p \in \mathcal{E}$ define the functional $J_p(u)$

for the system (1.1) by

$$J_p(u) \triangleq \sup_{w\in\ell_2([0,M-1],\mathbf{R}^r)} \sup_{x_0\in\mathbf{R}^n} \{p(x_0)+ \sum_{i=0}^{M-1} |z_{i+1}|^2 - \gamma^2|w_i|^2\Big\}. \quad (2.4)$$

Remark 2.1 The quantity $p \in \mathcal{E}$ in (2.4) can be chosen in a way which reflects knowledge of any *a priori* information concerning the initial state x_0 of Σ^u. □

The finite gain property of Σ^u can be expressed in terms of J as follows.

Lemma 2.2 *$\Sigma^u_{x_0}$ is finite gain if and only if there exists a finite quantity $\beta^u(x_0)$ such that*

$$J_{\delta_{x_0}}(u) \leq \beta^u(x_0), \quad (2.5)$$

and $\beta^u(0) = 0$.

It is of interest to know when $J_p(u)$ is finite. For a finite gain system Σ^u, we write

$$\text{dom}J_p(u) = \{p \in \mathcal{E} : (p, \beta^u),\ (p,0) \text{ finite}\},$$

where we use the pairing [7]

$$(p, q) = \sup_{x\in\mathbf{R}^n} \{p(x) + q(x)\}. \quad (2.6)$$

Lemma 2.3 *If each map $\Sigma^u_{x_0}$ is finite gain, then*

$$(p,0) \leq J_p(u) \leq (p, \beta^u), \quad (2.7)$$

and so $J_p(u)$ is finite for $p \in \text{dom}J_p(u)$.

Proof. Set $w \equiv 0$ to deduce $(p,0) \leq J_p(u)$. Next, select $w \in \ell_2([0, M-1], \mathbf{R}^r)$ and $x_0 \in \mathbf{R}^n$. Then (1.2) implies

$$p(x_0) + \sum_{i=0}^{M-1} |z_{i+1}|^2 - \gamma^2|w_i|^2 \leq p(x_0) + \beta^u(x_0) \leq (p, \beta^u).$$

This implies (2.7). □

The (finite time) *output feedback dynamic game* is to find a control policy $u \in \mathcal{O}_{0,M-1}$ which minimizes each functional $J_{\delta_{x_0}}$. The idea then is that a solution to this game problem will solve the output feedback robust control problem.

3. Information State Formulation

To solve the game problem, we borrow an idea from stochastic control theory (see, e.g., [4], [9]) and replace the original problem with a new one expressed in terms of a new state variable, viz., an information state [7].

For fixed $y_{1,k} \in \ell_2([1,k], \mathbf{R}^p)$ we define the *information state* $p_k \in \mathcal{E}$ by

$$p_k(x) \triangleq \sup_{w\in\ell_2([0,k-1],\mathbf{R}^r)} \sup_{x_0\in\mathbf{R}^n} \{p_0(x_0)+ \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2|w_i|^2 : x_k = x, h(x_i, u_i, w_i) = y_{i+1}, 0 \leq i \leq k-1\}. \quad (3.8)$$

If Σ^u is finite gain, then

$$-\infty \leq p_k(x) \leq (p_0, \beta^u) < +\infty.$$

A finite lower bound depends on possible degeneracies of the system (1.1).

In order to write the dynamical equation for p_k, we define $F(p, u, y) \in \mathcal{E}$ by

$$F(p,u,y)(x) = \sup_{\xi\in\mathbf{R}^n} \{p(\xi) + B(\xi, x, u, y)\}, \quad (3.9)$$

where the extended real valued function B is defined by

$$B(\xi, x, u, y) = \sup_{w\in\mathbf{R}^r} \{|l(\xi, u, w)|^2 - \gamma^2|w|^2 : b(\xi, u, w) = x,\ h(\xi, u, w) = y\}. \quad (3.10)$$

Here, we use the convention that the supremum over an empty set equals $-\infty$.

Lemma 3.4 *The information state is the solution of the following recursion:*

$$\begin{aligned} p_k &= F(p_{k-1}, u_{k-1}, y_k), \\ p_0 &\in \mathcal{E}. \end{aligned} \quad (3.11)$$

Proof. The result is proven by induction. Assume the assertion is true for $0, \dots, k-1$; we must show that p_k defined by (3.8) equals $F(p_{k-1}, u_{k-1}, y_k)$ defined by (3.9). Now

$$\begin{aligned} &F(p_{k-1}, u_{k-1}, y_k)(x) \\ &= \sup_\xi \{p_{k-1}(\xi) + B(\xi, x, u_{k-1}, y_k)\} \\ &= \sup_\xi \{p_{k-1}(\xi) + \sup_{w_{k-1}} (|l(\xi, u_{k-1}, y_k)|^2 - \gamma^2|w_{k-1}|^2 : b(\xi, u_{k-1}, w_{k-1}) = x,\ h(\xi, u_{k-1}, w_{k-1}) = y_k)\} \\ &= p_k(x) \end{aligned}$$

using the definition (3.8) for p_{k-1} and p_k. □

Remark 3.5 Note that we can write

$$p_k(x) \triangleq \sup_{\xi\in\ell_2([0,k],\mathbf{R}^n)} \{p_0(\xi_0) + \sum_{i=0}^{k-1} B(\xi_i,\xi_{i+1},u_i,y_{i+1}) : \xi_k = x\}. \quad (3.12)$$

□

We now state the following representation result:

Theorem 3.6 *For $u \in \mathcal{O}_{0,M-1}$, $p \in \mathcal{E}$, such that $J_p(u)$ is finite, we have the representation*

$$J_p(u) = \sup_{y_{1,M}\in\ell_2([1,M],\mathbf{R}^p)} \{(p_M, 0) : p_0 = p\}. \quad (3.13)$$

PROOF. We have

$$\sup_{y_{1,M}\in\ell_2([1,M],\mathbf{R}^p)} \{(p_k, 0) : p_0 = p\}$$

$$= \sup_y \sup_\xi \left\{p(\xi_0) + \sum_{i=0}^{M-1} B(\xi_i,\xi_{i+1},u_i,y_{i+1})\right\}$$

$$= \sup_w \sup_{x_0} \left\{p(x_0) + \sum_{i=0}^{M-1} |z_{i+1}|^2 - \gamma^2|w_i|^2\right\}$$

$$= J_p(u).$$

□

Theorem 3.6 enables us to express the finite gain property of Σ^u in terms of the information state p, as the following corollary shows:

Corollary 3.7 *For any output feedback controller $u \in \mathcal{O}_{0,M-1}$, the closed loop system Σ^u is finite gain if and only if the information state p_k satisfies*

$$\sup_{y_{1,k}\in\ell_2([1,k],\mathbf{R}^p)} \{(p_k, 0) : p_0 = \delta_{x_0}\} \le \beta^u(x_0), \quad (3.14)$$

for all $k \in [0, M]$, for some finite $\beta^u(x_0)$ with $\beta^u(0) =$ 0.

Remark 3.8 In view of the above, the name "information state" for p is justified. Indeed, p_k contains all the information relevant to the key finite gain property of Σ^u that is available in the observations $y_{1,k}$. □

Remark 3.9 We now regard the information state dynamics (3.11) as a new (infinite dimensional) control system Ξ, with control u and disturbance y. The state p_k and disturbance y_k are available to the controller, so the original output feedback dynamic game is equivalent to a new one with *full information*. The cost is now the RHS of (3.13). The analogue in stochastic control theory is the dynamical equation for the conditional density (or variant), and y becomes white noise under a reference probability measure [7], [9]. □

We say that Ξ^u is *finite gain* if and only if (3.14) holds for some finite $\beta^u(x_0)$ with $\beta^u(0) = 0$.

Now that we have introduced the new state variable p, we need an appropriate class $\mathcal{I}_{k,l}$ of controllers which feedback this new state variable. A control u belongs to $\mathcal{I}_{k,l}$ if for each $j \in [k,l]$ there exists a map $\bar{u}_j$ from a subset of $\mathcal{E}^{j-k+1}$ into U. such that $u_j = \bar{u}_j(p_{k,j})$. Note that since p_k depends only on the observable information $y_{1,k}$, $\mathcal{I}_{0,M-1} \subset \mathcal{O}_{0,M-1}$.

4. Solution to the Robust Control Problem

In this section we use dynamic programming to obtain necessary and sufficient conditions for the solution of the output feedback robust control problem. We make use of the dynamic programming approach used in [7] to solve the output feedback dynamic game problem.

For a function $W : \mathcal{E} \to \mathbf{R}^*$, we write

$$\text{dom}W = \{p \in \mathcal{E} : W(p) \text{ finite }\}.$$

Theorem 4.10 (Necessity) *Assume that a controller $u^o \in \mathcal{O}_{0,M-1}$ solves the output feedback robust control problem. Then there exists a solution W to the dynamic programming equation*

$$W_k(p) = \inf_{u\in U} \sup_{y\in\mathbf{R}^p} \{W_{k+1}(F(p,u,y))\},$$

$$W_M(p) = (p,0), \quad (4.15)$$

such that $\text{dom}J_p(u^o) \subset \text{dom}W_k$, $W_k(\delta_0) = 0$, $W_k(p) \ge (p,0)$, $k \in [0,M]$.

PROOF. For $p \in \text{dom}J_p(u^o)$, define

$$W_k(p) = \inf_u \sup_y \{(p_M, 0) : p_k = p\}. \quad (4.16)$$

Note the alternative expression for $W_k(p)$:

$$W_k(p) =$$

$$\inf_u \sup_w \sup_{x_k} \left\{p(x_k) + \sum_{i=k}^{M-1} |z_{i+1}|^2 - \gamma^2|w_i|^2\right\}. \quad (4.17)$$

Define $u \in \mathcal{O}_{k,M-1}$ by setting $u(j) = u^o(j-k)$. Using this control the finite gain property implies

$$\sum_{i=k}^{M-1} |z_{i+1}|^2 \le \gamma^2 \sum_{i=k}^{M-1} |w_i|^2 + \beta^{u^o}(x_k),$$

for all $w \in \ell_2([k, M-1], \mathbf{R}^r)$, and thus

$$W_k(p)$$

$$\le \sup_w \sup_{x_k} \left\{p(x_k) + \sum_{i=k}^{M-1} |z_{i+1}|^2 - \gamma^2|w_i|^2\right\}$$

$$\le (p, \beta^{u^o}).$$

Thus $\mathrm{dom} J_p(u^o) \subset \mathrm{dom} W_k$. Also, we have

$$W_k(p) \geq (p, 0).$$

Since $\beta^{u^o}(0) = 0$, $(\delta_0, 0) = 0$, we have $W_k(\delta_0) = 0$. Finally, the proof of Theorem 4.4, **[7]** shows that W_k is the unique solution of the dynamic programming equation (4.15). □

Theorem 4.11 (Sufficiency) *Assume there exists a solution W to the dynamic programming equation (4.15) such that $\delta_x \in \mathrm{dom} W_k$, for all $x \in \mathbf{R}^n$, $W_k(\delta_0) = 0$, $k \in [0, M]$. Let $u^* \in \mathcal{I}_{0,M-1}$ be a policy such that $u_k^* = \bar{u}_k^*(p_k)$, where $\bar{u}_k^*(p)$ achieves the minimum in (4.15). Then u^* solves the output feedback robust control problem.*

PROOF. Following the proof of Theorem 4.6 of **[7]**, we see that

$$W_0(p) = J_p(u^*) \leq J_p(u)$$

for all $u \in \mathcal{O}_{0,M-1}$, $p \in \mathrm{dom} W_0$. Now

$$\sup_y \{ (p_M, 0) \; : \; p_0 = \delta_{x_0}, \, u = u^* \} \leq W_0(\delta_{x_0}),$$

which implies by Corollary 3.7 that Σ^{u^*} is finite gain with $\beta^{u^*}(x_0) \triangleq W_0(\delta_{x_0})$, and hence u^* solves the output feedback robust control problem. □

Remark 4.12 Note that the controller obtained is an *information state feedback* controller. □

Corollary 4.13 *If the output feedback robust control problem is solvable by an output feedback controller $u^o \in \mathcal{O}_{0,M-1}$, then it is also solvable by an information state feedback controller $u^* \in \mathcal{I}_{0,M-1}$.*

Remark 4.14 The necessity result is important. A common approach **[1]**, **[2]**, **[6]** to solving the output feedback robust control problem is to specify *a priori* a finite dimensional observer structure, say,

$$\xi_{k+1} = A(\xi_k, u_k, y_{k+1}),$$

$$u_k = B(\xi_k),$$

for some A, B, chosen so that ξ_k tracks x_k and the closed loop system is finite gain, to yield a controller $u^\dagger \in \mathcal{O}_{0,M-1}$. Our results imply that the sufficient conditions required for the success of such approaches are not in general also necessary. Consequently, the nonlinear output feedback robust control problem is intrinsically *infinite dimensional.* □

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