PROCEEDINGS OF THE

31st IEEE CONFERENCE ON DECISION AND CONTROL

DECEMBER 16-18, 1992
WESTIN LA PALOMA
TUCSON, ARIZONA, USA

IEEE
Control
Systems
Society

VOLUME 4 OF 4

92CH3229-2

Proceedings of the 31st Conference
on Decision and Control
Tucson, Arizona • December 1992

An Adaptive Quasi Linear Representation - A Generalization of Multiscale Edge Representation*

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Abstract

The analysis of the discrete multiscale edge representation is considered. A general signal description, called an inherently bounded Adaptive Quasi Linear Representation (AQLR), motivated by two important examples: the wavelet maxima representation and the wavelet zero-crossings representation, is introduced. This paper addresses the questions of uniqueness, stability, and reconstruction. It is shown, that the dyadic wavelet maxima (zero-crossings) representation is, in general, non unique. Nevertheless, these representations are always stable. A reconstruction algorithm, based on the minimization of an appropriate cost function, is proposed. The convergence of the algorithm is guaranteed for all inherently bounded AQLR. In the case, of the wavelet transform, this method yields an efficient, parallel algorithm, especially promising in an analog-hardware implementation.

1. Introduction

Traditionally, multiscale edges are determined either as extrema of Gaussian-filtered signals [10] or as zero-crossings of signals convolved with the Laplacian of a Gaussian (see e.g. [5] for a comprehensive review). S. Mallat in a series of papers [6,7,8] introduced zero-crossings and extrema of the wavelet transform as a multiscale edge representation. Two important advantages of this method are low algorithmic complexity and flexibility in choosing the basic filter. Moreover, [6] and [7] propose reconstruction procedures and show accurate numerical reconstruction results from zero-crossings and maxima representations. From the theoretical point of view, there are still important open problems, e.g. stability, uniqueness, and structure of a reconstruction set (a family of signals having the same representation).

Our objective is to analyze these theoretical questions using a model of an actual implementation. The main assumption is that the data are discrete and finite. Since reconstruction sets of both maxima and zero-crossings representations have a similar structure, a general form called the Adaptive Quasi Linear Representation (AQLR) is introduced. This paper uses the idea of the AQLR to investigate rigorously three fundamental questions: uniqueness, stability, and reconstruction.

We refer to [3,4] for proofs and details.

2. The Multiscale Maxima Representation

Consider $\mathcal{L}$, a linear space of real, finite sequences. Let X and Y denote operators on $\mathcal{L}$ which provide the sets of local maxima and minima, respectively, of a sequence $f \in \mathcal{L}$. In this work, in order to avoid boundary problems, an N-periodic extension of finite sequences is assumed.

Let $W_1, W_2, \ldots, W_J, S_J$ be linear operators on $\mathcal{L}$. The sets XW_jf, YW_jf are local maxima

*Research supported in part by NSF grant NSFD CDR8803012

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and minima points of the sequence W_jf. The values of W_jf at extreme points are denoted by $\{W_jf(k)\}_{k \in XW_jf \cup YW_jf}$. The multiscale local extrema representation, R_mf is defined as:

$$\left\{ \left\{ XW_jf, YW_jf, \{W_jf(k)\}_{k \in XW_jf \cup YW_jf} \right\}_{j=1}^{J}, S_Jf \right\}.$$

Following [7] R_mf, will be called the maxima representation. In the particular case, when $W_1, W_2, \ldots, W_J, S_J$ correspond to the wavelet transform, R_mf will be called the wavelet maxima representation. Generally speaking, R_m is a nonlinear operator. Our approach is to separate linear and non-linear components. This observation is the motivation to consider R_mf as consisting of two parts: the sampling information and the maxima information. The sampling information is the sequence S_Jf and the values of W_jf at the points $XW_jf \cup YW_jf$ (j=1,2,...,J). The maxima information consists of the sets XW_jf, YW_jf and the fact that they contain local maxima and minima of W_jf.

Let T_{mf} denote the linear operator associated with the sampling information.

Now, R_mf is written in an alternative way as:

$$R_mf = \left\{ \{XW_jf, YW_jf\}_{j=1}^{J}, T_{mf}f \right\}.$$

For a given representation Rf, a reconstruction set $\Gamma(Rf)$ is defined as a set of all sequences satisfying this representation, i.e.

$$\Gamma(Rf) \triangleq \{\gamma \in \mathcal{L} : R\gamma = Rf\}.$$

It is clear that in order to satisfy a given maxima representation, a sequence $h \in \mathcal{L}$, in addition to obeying the sampling information $T_{mf}h = T_{mf}f$, needs to meet the requirement that W_jh has local extrema at points of XW_jf and YW_jf.

For all $k \in (XW_jf \cup YW_jf)^r$, the type of k, $t(k)$ is defined by:

$$t(k) \triangleq \begin{cases} -1 & \text{if } k \in XW_jf \\ 1 & \text{otherwise.} \end{cases}$$

In view of these considerations, the following theorem is easily verified.

Theorem 1 *R_mf is a given maxima representation. $h \in \Gamma(R_mf)$ if and only if*

$$T_{mf}h = T_{mf}f$$
$$t(k) \cdot (W_jf(k+1) - W_jf(k)) > 0$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for all

$$k \in \overline{XW_jf \cup YW_jf} \cup (XW_jf \cup YW_jf)^r.$$

The maxima representations can be cast into the form $Rf = \{Vf, Tf\}$ where Vf is a set of points and T is a linear operator which may depend on Vf. However, the key feature of the maxima representation is the fact that the set Vf yields additional linear inequalities.

Definition 1 *$Rf = \{Vf, Tf\}$ is called an Adaptive Quasi Linear Representation (AQLR) if there exists a linear operator A and a sequence a such that:*

$$x \in \Gamma(Rf) \Leftrightarrow Tx = Tf \text{ and } Ax > a.$$

A, a may depend on Vf, but they must be independent of Tf.

Definition 2 *An AQLR is called inherently bounded if there exists a real $K > 0$ such that*

$$x \in \Gamma(Rf) \Rightarrow \|x\| \leq K\|Tf\|.$$

The coefficient K can depend on the parameters of the representation e.g. $N, J, W_1, \ldots, W_J, S_J$ but it must be independent of Vf and Tf.

Clearly, the following is true.

Proposition 1 *The wavelet maxima representation is an AQLR.*

Proposition 2 *The wavelet maxima representation is inherently bounded AQLR.*

3. The Multiscale Zero-Crossings Representation

Let Z be an operator which provides a set of zero-crossings points of a given sequence $f \in \mathcal{L}$. The sequence of sums of $h(n)$ between consecutive zero-crossings points of f at level j is, $U_j^{zf}h$. As in the maxima representation case, for fixed sets ZW_jf, the remaining data $U_j^{zf}f$ and S_Jf

are obtained by the linear sampling operator, denoted T_{zf}. The zero crossings representation becomes:

$$R_z f = \left\{\{ZW_j f\}_{j=1}^{J}, T_{zf} f\right\}.$$

In order to have $h \in \Gamma(R_z f)$, in addition to obeying

$$T_{zf} h = T_{zf} f,$$

$W_j h$ has to satisfy sign constraints yielding zero-crossings exactly at $ZW_j f$ points. We have

Theorem 2 *Let $R_z f$ be a given zero-crossings representation. $h \in \Gamma(R_z f)$ if and only if*

$$T_{zf} h = T_{zf} f$$
$$sgn\left(U_j^{zf} f(k)\right) \cdot W_j h(i)) > 0.$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for all $i \in \overline{ZW_j f} \cup (ZW_j f)^r$ where k satisfies $i \in P_j^{zf}(k)$.

As an immediate consequence of the theorem we have:

Proposition 3 *The multiscale zero-crossings representation is an AQLR.*

Theorem 3 *The wavelet zero-crossings representation is an inherently bounded AQLR.*

Notice, that for the wavelet zero-crossings representation, $K = 1$, regardless the values of N and J.

4. Basic Properties of AQLR's

A representation $Rf = \{Vf, Tf\}$ is said to be unique, if the reconstruction set $\Gamma(Rf)$ consists of exactly one element. We have the following uniqueness characterization for AQLR's.

Lemma 1 *Let $Rf = \{Vf, Tf\}$ be an AQLR. Then Rf is unique if and only if the kernel of the operator T is trivial, i.e. $\mathcal{N}T = \{0\}$.*

Perhaps the most important consequence of Lemma 1 is the fact that uniqueness of the representation Rf is equivalent to uniqueness of the underlying irregular sampling Tf. On the other hand, from the signal compression, understanding and interpretation point of view, it seems to be desirable that little information would be specified explicitly by Tf and as much as possible information about a signal should be described implicitly by $Af > a$. Therefore, the most important and interesting features of AQLRs appear in the non unique case.

The closure of the reconstruction set,[1] Γ^c is a convex polyhedron.

$$\Gamma^c = \{x : Tx = Tf, \quad Ax \geq a\}.$$

Without loss of generality, we can assume that

$$\Gamma^c = \{x : Bx \geq b\}$$

for a $p \times N$ matrix B and a $p \times 1$ vector b. For inherently bounded AQLR's, the associated Γ^c is bounded. Thereforeas a special case of the theorem of Krein and Milman, the following holds.

Theorem 4 *For an inherently bounded AQLR, the closure of the reconstruction set is the convex hull of its finitely many vertices.*

Let $\{x : Bx \geq b\}$ be a polyhedron and v^i its vertex. Then, there exist N rows of B, which constitute a regular matrix $[B]^i$ such that:

$$v^i = \left([B]^i\right)^{-1} \cdot [b]^i$$

where $[b]^i$ is a subvector of b corresponding to these N rows. By inserting zero columns to the matrix $\left([B]^i\right)^{-1}$, the matrix D^i is obtained:

$$v^i = D^i b.$$

Since the closure of the reconstruction set is the convex hull of its vertices, the above equation can characterize the changes in the reconstruction set due to perturbations in either the matrix B or the vector b. Accordingly it will be used to prove the stability results.

5. The theory of non-uniqueness

The section aims to construct that, in general, the discrete dyadic wavelet maxima (zero-crossings) representation is not unique. The results are consequences of Lemma 1, which relates uniqueness of the representation to the set $\mathcal{N}T$, the kernel of the sampling information. The main idea is to show a sequence f such that the set $\mathcal{N}T$ corresponding to the representation Rf cannot be $\{0\}$.

[1]The abbreviated notation Γ is used instead of $\Gamma(Rf)$.

Theorem 5 *A discrete dyadic wavelet maxima (zero-crossings) representation based on a discrete low pass filter $H(w)$ is given. If $H(\pi) = 0$, $J \geq 3$, and N is a multiple of 2^J then there exists a sequence f which has a non-unique maxima (zero-crossings) representation.*

Let us point out that although the hypothesis of the theorem may seem to be demanding, it is just a technical condition. All filters used by Mallat, Zhong and many others fulfill these conditions.

The construction of the counter example is based on the set $\mathcal{B}$, defined as follows:

$$\mathcal{B} = \left\{ \{c_r\}_{r=1}^{2^{J-1}}, \{s_r\}_{r=1}^{2^{J-1}-1} \right\}$$

where

$$c_r(n) = \cos(\frac{2\pi r n}{2^J}) \quad n = 0, 1, \ldots, N-1$$

$$s_r(n) = \sin(\frac{2\pi r n}{2^J}) \quad n = 0, 1, \ldots, N-1.$$

Proposition 4 *The set $\mathcal{B}$ is included in $\mathcal{N}S_J$, the kernel of the operator S_J.*

Notice, that $s_{2^{J-1}}$ does not appear in the set $\mathcal{B}$. The reason is that $s_{2^{J-1}} \equiv 0$ and in the next proposition the independence of the set $\mathcal{B}$ is asserted.

Proposition 5 *The set $\mathcal{B}$ is linearly independent.*

As a universal counter example of non-uniqueness the following sequence is proposed.

$$f(n) = \cos(2\pi \frac{n}{2^J}) \quad n = 0, 1, \ldots, N-1.$$

Observe that the same sequence is proposed for all dyadic wavelet transforms and for both the maxima representation and the zero-crossings representation.

The representation $R_m f$ ($R_z f$) is unique if and only if $\mathcal{N}T_{mf} = \{0\}$ ($\mathcal{N}T_{zf} = \{0\}$). Consequently, the non-uniqueness of $R_m f$ ($R_m f$) is easily deduced from the following proposition.

Proposition 6 *The equation*

$$T_{mf} h = 0 \quad (T_{zf} h = 0) \quad h \in span(\mathcal{B})$$

has a non trivial solution.

Some remarks need to be made at this point. It turns out, that it is relatively easy to produce more examples of non unique dyadic wavelet maxima (zero-crossings) representations using 2^p-periodic signals, where p is an integer. Hummel with Moniot [5], Mallat [6], and Mallat with Zhong [7] have reported that high frequency errors may occur in the discrete maxima (zero-crossings) representation. For these 2^J-periodic signals, components of the reconstruction error can appear as 2^p-periodic signals for $p = 1, 2, \ldots, J$. Most of them cannot be related as high frequency errors. For more details the reader is referred to [1,2].

From our simulations and from Mallat's results it turns out that for the vast majority of signals, the representation is unique. We even conjecture that the wavelet maxima (zero-crossings) representation is unique for a generic family of signals.

6. Stability

Addressing the stability issue, the standard approach is to introduce the notion of perturbations: of the representation and of the reconstruction set. In addition, measures for a distance between distinct representations and for a distance between different reconstruction sets should be defined. Recall that Vf, Tf may have different sizes for different representations. Fortunately, for inherently bounded representations, the following characterization of BIBO (bounded input, bounded output) stability is easily verified.

Proposition 7 *Let $Rf_i = \{Vf_i, T_i f_i\}$ $i = 1, 2$ be inherently bounded AQLR's. Then for all $K_I > 0$ there exist K_O such that:*

$$\|T_i f_i\| \leq K_I \ \ (i = 1, 2) \Rightarrow \|x_1 - x_2\| \leq K_O \ \ \forall x_i \in \Gamma_i$$

In many applications, the reasons for a perturbation in a representation are arithmetic or quantization errors in a reconstruction algorithm. This kind of perturbations may change the continuous values of Tf but it preserves the discrete values of Vf. Therefore the perturbed representation, $(Rf)_p$, can be written as:

$$(Rf)_p = \{Vf, Tf + \Delta(Tf)\}.$$

Let Γ_p be the corresponding reconstruction set. The distance between two reconstruction sets, Γ

and Γ_p, is defined by:

$$d(\Gamma, \Gamma_p) \triangleq \sup\{||\gamma - \gamma_p|| : \gamma \in \Gamma, \gamma_p \in \Gamma_p\}.$$

Observe, that for inherently bounded AQLR's, $d(\Gamma, \Gamma_p)$ is always finite. The measure of the perturbation in the reconstruction set is the difference between $d(\Gamma, \Gamma_p)$ and the size of Γ which is defined as follows:

$$s(\Gamma) \triangleq d(\Gamma, \Gamma) = \sup\{||\gamma_1 - \gamma_2|| : \gamma_1, \gamma_2 \in \Gamma\}.$$

$s(\Gamma)$ and $d(\Gamma, \Gamma_p)$ describe the largest possible L_2 norm of a reconstruction error, from the original representation and from a perturbed one.

Theorem 6 *For all inherently bounded AQLR, there exists $K > 0$ such that:*

$$d(\Gamma, \Gamma_p) \leq K \cdot ||\Delta(Tf)|| + s(\Gamma).$$

Observe that the above result is global in the sense that as long as $\Delta(Tf)$ gives rise to a non empty reconstruction set, the theorem holds regardless the size of $\Delta(Tf)$.

7. A Reconstruction Scheme

In a non unique case, there are several ways to define a reconstruction algorithm. In this work, it is defined as a procedure to find any element x belonging to the closure of the reconstruction set, Γ^c. The proposed reconstruction algorithm is based on an appropriate potential function $v(x)$ satisfying:

$$v(x) = 0 \quad \forall x \in \Gamma^c$$
$$v(x) > 0 \quad \forall x \in \overline{\Gamma^c}.$$

where $\overline{\Gamma^c}$ denotes the complement of Γ^c in $\mathcal{L}$. Furthermore, it will be shown that the proposed $v(x)$ does not have any local extremum outside Γ^c, i.e.

$$||\nabla v(x)|| > 0 \quad \forall x \in \overline{\Gamma}.$$

With such a potential function, the reconstruction is achieved by any minimization algorithm operating on $v(x)$. We will focus on the reconstruction algorithm based on the differential equation:

$$\dot{x}(t) = -\nabla\left(v\left(x(t)\right)\right)$$

whose analog hardware implementation give rise to a very fast algorithm.

In this section, a general inherently bounded Adaptive Quasi Linear Representation (AQLR) is considered. As mentioned in Section 4, the closure of the reconstruction set, Γ^c, can be written as:

$$\Gamma^c = \{x : Bx \geq b\}$$

for a given $p \times N$ matrix B and a p-dimensional vector b. The function $v(x)$ is derived from this representation in the subsequent way.

$$v(x) \triangleq \sum_{i=1}^{p} f(Bx - b)_i$$

where $(Bx - b)_i$ denotes the i-th component of the vector $Bx - b$. The function $f(\cdot)$ is defined by:

$$f(\xi) \triangleq \begin{cases} \xi^2 & \text{if } \xi < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Observe that $f(\xi)$ is continuously differentiable. Therefore $v(x)$ is continuous and continuous differentiable. The gradient of $v(x)$ is given by:

$$\nabla v(x) = 2B'Z(Bx - b)$$

where Z is a $p \times p$ diagonal matrix satisfying:

$$Z(i,i) = \begin{cases} 1 & \text{if } (Bx - b)_i < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Naturally, B' denotes the transpose of the matrix B.

The following theorem states that $v(x)$ does not have local extrema outside the set Γ^c.

Theorem 7 *Let Γ be non empty. Then $\nabla v(x) = 0$ if and only if $x \in \Gamma^c$.*

Theorem 8 *Exactly one of the two alternatives holds:*

1. *$\exists x$ s.t. $ZBx \geq Zb$.*

2. *$\exists b_o$ such that $(ZB)'b_o = 0$ $b_o \geq 0$ $(Zb)'b_o > 0$.*

In view of these considerations, a reconstruction scheme can be implemented as:

$$\arg\min \{v(x) : x \in \mathcal{L}\}.$$

The minimization is significantly facilitated by the property that local extrema of $v(x)$ appear only in Γ^c. We are going to focus on the algorithm based on the differential equation. The desired property is that for all $x(0)$, $x(t)$ will approach the set Γ^c as $t \rightarrow \infty$. The convergence result is based on La Salle's Theorem.

Theorem 9 *Let Γ^c be the closure of the reconstruction set for of the inherently bounded AQLR. Then for all $x(0)$, the solution of*

$$\dot{x}(t) = -\nabla(v(x(t))).$$

will approach Γ^c as $t \rightarrow \infty$.

The idea to minimize a cost function in order to reconstruct a signal from the multiscale edge representation has appeared in many works. The comparison reveals the following advantages of the proposed algorithm.

- This algorithm is based on continuously differentiable cost function.
- It does not apply approximations.
- It is adapted for both unique and non unique cases.
- Its validity and convergence is guaranteed

Conclusions

Summarizing, the results described about uniqueness and stability are new theoretical results. In our opinion, the most significant contribution of this work is to create a framework to define, analyze, and reconstruct a wide family of representations. Important examples are generalizations of a basic maxima representation obtained by using only a subset of local extreme points. This is the subject of the undergoing research.

Acknowledgments We would like to thank S. Mallat for drawing our attention to the importance of the stability problem. We are grateful to A. Tits for giving the idea for the proof of Theorem 6.

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