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Discrete Random Sets: An Inverse Problem, plus tools for the Statistical Inference of the Discrete Boolean Model *

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Abstract

We consider digital binary images as realizations of a bounded discrete random set, a mathematical object which can be defined directly on a finite lattice. In this setting, we show that it is possible to move between two equivalent probabilistic model specifications. We formulate a restricted version of the discrete-case analog of a Boolean random set model, obtain its probability mass function, and employ some methods of Morphological image analysis to derive tools for its statistical inference.

1 Introduction

The *Boolean random set* is arguably the most important random set model to date. Its importance stems from two principal considerations: its analytical tractability and its power in modeling many interesting phenomena. Typical applications include but are not limited to: random clumping of dust, or powder particles, or blood cells; modeling of geological structures, patterns in photographic emulsion, colloids in gel form, and structural inhomogeneities in amorphous matter [20, p.68, and references therein]. Other potential applications include particle counting and size analysis in images of cell cultures, and modeling of clutter in infrared imaging. Informally, a Boolean random set is constructed by centering a simple random shape (set), such as a disc of random size, at each point of a Poisson field of points in the plane, and then taking the union of the resulting sets. Random shapes centered at different points of the Poisson field are assumed to be independent and statistically equivalent. The points of the Poisson field associated with a Boolean random set are known as the "germs", and the Poisson field itself is sometimes called the germ process. The random shapes are known as the "primary grains".

The transition from continuous domain random sets to discrete domain random sets is a troublesome one [19]. In practice, one usually deals with discrete, finite data. Therefore, we have chosen to define discrete random sets directly on the appropriate spaces, and base subsequent developments on this axiomatic definition. This idea has been concurrently and independently developed in [10, 9]. There exist certain differences between the two formulations, and the results evolve in different directions. First we present a general theorem, then we consider a restricted version of the discrete case analog of a Boolean random set model, and develop some tools for its statistical inference. Statistical inference techniques are very scarce in random set theory. Most of the existing literature is specific to the Boolean model, and is concerned with parameter estimation, based on some variant of the method of moments [18, 4, 3, 6, 1, 19, 7].

***Keywords:** Image Analysis, Bayesian Decision Theory, Discrete Random Sets, Boolean Model, Mathematical Morphology, Skeleton

2 Discrete random set fundamentals

Definition 1 Let B be a bounded subset of \mathbb{Z}^2 . Assume that B contains the origin. Let $\Sigma(\Omega)$ denote the σ -algebra on Ω . Let $\Sigma(B)$ denote the power set (i.e. the set of all subsets) of B , and let $\Sigma(\Sigma(B))$ denote the power set of $\Sigma(B)$. A **Discrete Random Set (DRS)**, X , on B , is a measurable mapping of a probability space $(\Omega, \Sigma(\Omega), P)$ into the measurable space $(\Sigma(B), \Sigma(\Sigma(B)))$. A DRS X , on B , induces a unique probability measure, P_X , on $\Sigma(\Sigma(B))$.

Definition 2 The functional

$$Q_X(K) = P_X(X \cap K = \emptyset)$$

is known as the **generating functional** of the DRS X .

Lemma 1 (Variant of Mobius inversion for Boolean algebras) Let v be a function on $\Sigma(B)$. Then v can be represented as

$$v(A) = \sum_{S \subseteq A^c} u(S) \quad \text{"external decomposition"}$$

The function u is uniquely determined by v , namely

$$u(S) = \sum_{C \subseteq S} (-1)^{|C|} v(S^c \cup C)$$

where c denotes complement with respect to B .

Proof:

Uniqueness: Assume that the external decomposition formula holds. Look at the right hand side of the inversion formula.

$$\begin{aligned} \sum_{C \subseteq S} (-1)^{|C|} v(S^c \cup C) &= \sum_{C \subseteq S} (-1)^{|C|} \sum_{D \subseteq S \cap C^c} u(D) = \\ \sum_{C \subseteq S} (-1)^{|C|} \sum_{D \subseteq S \setminus C} u(D) &= \sum_{C \subseteq S} \sum_{D \subseteq S \setminus C} (-1)^{|C|} u(D) = \\ \sum_{D \subseteq S} \sum_{C \subseteq S \setminus D} (-1)^{|C|} u(D) &= \sum_{D \subseteq S} u(D) \sum_{C \subseteq S \setminus D} (-1)^{|C|} = u(S) \end{aligned}$$

Since

$$\sum_{C \subseteq S} (-1)^{|C|} = \begin{cases} 0 & , S \neq \emptyset \\ 1 & , S = \emptyset \end{cases}$$

Existence: Assume that the inversion formula holds, and look at the right hand side of the external decomposition formula.

$$\begin{aligned} \sum_{S \subseteq A^c} u(S) &= \sum_{S \subseteq A^c} \sum_{C \subseteq S} (-1)^{|C|} v(S^c \cup C) = \\ \sum_{S \subseteq A^c} \sum_{C \subseteq S} (-1)^{|C|} v((S \setminus C)^c) &= \sum_{D \subseteq A^c} \sum_{C \subseteq A^c \setminus D} (-1)^{|C|} v(D^c) = \\ \sum_{D \subseteq A^c} v(D^c) \sum_{C \subseteq A^c \setminus D} (-1)^{|C|} &= v((A^c)^c) = v(A) \end{aligned}$$

As for the uniqueness part. \square

Theorem 1 Given $Q_X(K')$, $\forall K' \in \Sigma(B)$, $P_X(A)$, $\forall A \in \Sigma(\Sigma(B))$ is uniquely determined, and, in fact, can be recovered via the measure reconstruction formulas

$$P_X(A) = \sum_{K \in A} P_X(X = K)$$

with

$$P_X(X = K) = \sum_{K' \subseteq K} (-1)^{|K'|} Q_X(K^c \cup K')$$

Proof:

The reconstruction formula for the functional $P_X(X = K)$ in terms of the functional Q_X is a direct consequence of Lemma 1 and the fact that Q_X can be expressed in terms of P_X as

$$Q_X(K) = \sum_{K' \subseteq K^c} P_X(X = K')$$

□

The *uniqueness* part of this Theorem is originally due to Choquet [2], and it has been independently introduced in the context of continuous-domain random set theory by Kendall [11] and Matheron [14, 15]. Related results can also be found in Ripley [17]. However, the measure reconstruction formulas are essentially only applicable within our formulation. In the continuous case, the uniqueness result relies heavily on Kolmogorov's extension theorem, which is non-constructive.

3 A DRS analog of the Boolean model

Let H be a convex¹ subset of B' , $|B'| \ll |B|$, which contains the origin. In the terminology of Mathematical Morphology², H is a *convex structuring element*. Let X_h denote the translate of X by the vector h , and

$$H^s = \{-h \mid h \in H\}$$

Definition 3 : The erosion, $X \ominus H^s$, of a bounded set $X \subset \mathbb{Z}^2$ by a structuring element H , is defined as

$$X \ominus H^s = \bigcap_{h \in H} X_{-h} = \{z \in \mathbb{Z}^2 \mid H_z \subseteq X\}$$

Definition 4 : The dilation, $X \oplus H^s$, of a bounded set $X \subset \mathbb{Z}^2$ by a structuring element H , is defined as

$$X \oplus H^s = \bigcup_{h \in H} X_{-h} = \{z \in \mathbb{Z}^2 \mid H_z \cap X \neq \emptyset\}$$

Definition 5 : The opening, $X \circ H$, of a bounded set $X \subset \mathbb{Z}^2$ by a structuring element H , is defined as

$$X \circ H = (X \ominus H^s) \oplus H$$

¹In digital topology [12, 19, 9], the *convex hull* of a bounded set, $H \subset \mathbb{Z}^2$, is defined as the intersection of the convex hull of H in the topology of \mathbb{R}^2 , with \mathbb{Z}^2 . A bounded set, $H \subset \mathbb{Z}^2$, is *convex* if it is identical to its convex hull.

²Refer to [19] for a thorough introduction to the principles of Mathematical Morphology. Here we merely reproduce some basic definitions.

In the discrete case the notion of size can be formalized via the operation of set dilation

$$rH = \begin{cases} \{\bar{0}\} \oplus H \oplus H \oplus \dots \oplus H, & (r \text{ times}), r = 1, 2, \dots \\ \{\bar{0}\} & , r = 0 \end{cases}$$

Definition 6 Let Ψ be a generalized Bernoulli lattice process on B , constructively defined in the following manner: each point $z \in B$ is contained in Ψ with probability $\lambda_s(z)$, independently of all others. Let $\{G_1, G_2, \dots\}$ be a set of nonempty, convex i.i.d. DRS's on B' , each given by $G_i = R_i H$, where $\{R_1, R_2, \dots\}$ form an i.i.d. sequence of Z_+ -valued r.v.'s which is independent of Ψ , and each R_i is distributed according to a pmf $f_R(r)$, which is compactly supported on $\{0, 1, \dots, \bar{R}\}$. Define

$$X = \bigcup_{i=1,2,\dots} G_i \oplus \{y_i\}$$

where $\Psi = \{y_1, y_2, \dots\}$. Then X will be called a **Discrete Radial Boolean Random Set (DRBRS)**, with parameters (λ_s, H, f_R) , and will be denoted by (λ_s, H, f_R) -DRBRS. The points $\{y_1, y_2, \dots\}$ will be called the germs, and the DRS's $\{G_1, G_2, \dots\}$ will be called the primary grains of the DRBRS X .

Remark: For brevity, we assume from this point on that the result of a \oplus operation is automatically restricted to B . Also, c stands for complement with respect to B .

We now proceed to compute the generating functional of a (λ_s, H, f_R) -DRBRS. Define

$$d^H(z, K) = \min_{k \in K} \|z - k\|_H$$

where

$$\|z - k\|_H = \min\{n \geq 0 \mid (\{z\} \oplus nH) \cap \{k\} \neq \emptyset\}$$

Observe that for $z \in K$, $d^H(z, K) = 0$, since H contains the origin. We remark that $d^H(z, K)$, as defined above, is a digital uniform step metric, which is a generalization of the digital Hausdorff metric. With this notation in place, and employing some simple geometric arguments, it can be shown that

$$Q_X(K) = \prod_{z \in K \oplus \bar{R}H} [(1 - \lambda_s(z)) + \lambda_s(z) F_R(d^H(z, K) - 1)]$$

where

$$F_R(m) = \sum_{l=0}^m f_R(l)$$

and $F_R(-1) = 0$, by convention.

We can now use Theorem 1 to compute $P_X(X = K)$ in terms of Q_X .

$$P_X(X = K) = \sum_{K' \subseteq K} (-1)^{|K'|} \prod_{z \in (K^c \cup K') \oplus \bar{R}H} [(1 - \lambda_s(z)) + \lambda_s(z) F_R(d^H(z, K^c \cup K') - 1)]$$

4 Statistical inference of the DRBRS model

The computational complexity associated with a brute-force application of the last formula of the previous section is exponential in $|K|$. One would therefore be interested in obtaining tight bounds on $P_X(X = K)$. In order to be useful, these bounds must be reasonably well behaved, and relatively easy to compute. For the simple case of

a DRBRS model with constant intensity, $\lambda_s(z) = p = 1 - q, \forall z \in B$, and primary grains of fixed size (one, by convention), the generating functional is simply given by

$$Q_X(K) = q^{|K \oplus H^s|}$$

We have the following result for this model.

Proposition 1 For all $q \in [0, 1]$, and all realizable³ $K \in \Sigma(B)$, $K \neq \emptyset, B$

$$L_q(K) \leq P_X(X = K) \leq U_q(K)$$

with

$$L_q(K) = q^{|K^c \oplus H^s|} (1 - q)^{|(K^c \oplus H^s)^c|}$$

and

$$U_q(K) = \frac{1}{2} q^{|K^c|} \left[(1 + q)^{|K|} + (1 - q)^{|K|} \right] - 2^{|K|-1} q^{|K^c \oplus H^s| + |K \oplus H^s|}$$

Both bounds are polynomials in q , they are equal to zero at the endpoints $q = 0, 1$, strictly positive for all $q \in (0, 1)$, and unimodal in $(0, 1)$. The mode of the lower bound is located at

$$\hat{q}(K) = \frac{|K^c \oplus H^s|}{|B|}$$

Remark: The mode of the upper bound has to be determined numerically.

Proof: Upper bound:

$$\begin{aligned} P_X(X = K) &= \sum_{K' \subseteq K} (-1)^{|K'|} q^{(K^c \cup K') \oplus H^s|} \\ &= \sum_{K' \subseteq K, |K'|=\text{even}} q^{(K^c \cup K') \oplus H^s|} - \sum_{K' \subseteq K, |K'|=\text{odd}} q^{(K^c \cup K') \oplus H^s|} \end{aligned}$$

Observe that, by distributivity of dilation over union, and using the union bound

$$|(K^c \cup K') \oplus H^s| = |(K^c \oplus H^s) \cup (K' \oplus H^s)| \leq |K^c \oplus H^s| + |K' \oplus H^s|$$

Furthermore, since H is assumed to contain the origin

$$|(K^c \cup K') \oplus H^s| \geq |K^c \cup K'| = |K^c| + |K'|$$

Therefore, since q is a probability

$$\begin{aligned} P_X(X = K) &\leq \sum_{K' \subseteq K, |K'|=\text{even}} q^{|K^c| + |K'|} - \sum_{K' \subseteq K, |K'|=\text{odd}} q^{|K^c \oplus H^s| + |K' \oplus H^s|} \\ &= q^{|K^c|} \sum_{K' \subseteq K, |K'|=\text{even}} q^{|K'|} - q^{|K^c \oplus H^s|} \sum_{K' \subseteq K, |K'|=\text{odd}} q^{|K' \oplus H^s|} \end{aligned}$$

³Meaning that K can be written as $K = L \oplus H$, for some $L \in \Sigma(B)$. If K can not be written this way, then it is not a realization of the DRBRS model under consideration, and, therefore, its probability is zero.

$$\begin{aligned} &\leq q^{|K^c|} \sum_{K' \subseteq K, |K'|=\text{even}} q^{|K'|} - q^{|K^c \oplus H^*|} \sum_{K' \subseteq K, |K'|=\text{odd}} q^{|K \oplus H^*|} \\ &= q^{|K^c|} \sum_{K' \subseteq K, |K'|=\text{even}} q^{|K'|} - q^{|K^c \oplus H^*| + |K \oplus H^*|} \sum_{K' \subseteq K, |K'|=\text{odd}} 1 \end{aligned}$$

Thus

$$P_X(X = K) \leq q^{|K^c|} \left(\sum_{i:\text{even}} \binom{|K|}{i} q^i \right) - q^{|K^c \oplus H^*| + |K \oplus H^*|} \left(\sum_{i:\text{odd}} \binom{|K|}{i} \right)$$

Using the fundamental identity

$$\sum_{i=0}^{|K|} \binom{|K|}{i} z^i = (1+z)^{|K|}, \quad \forall z \in \mathcal{C}$$

and successively setting $z = -1, 1$, we obtain

$$\sum_{i:\text{odd}} \binom{|K|}{i} = 2^{|K|-1}$$

Similarly, replacing z by qz and then setting $z = -1, 1$, we obtain

$$\sum_{i:\text{even}} \binom{|K|}{i} q^i = \frac{1}{2} [(1+q)^{|K|} + (1-q)^{|K|}]$$

From which, we finally obtain the expression for the upper bound.

We will need the following Lemma.

Lemma 2 (Descartes's rule of signs [16, pp. 36-43]) *Let $p(x)$ be a polynomial of a real variable, with real coefficients.*

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n$$

Let C denote the number of changes of sign of the sequence of its coefficients (for each $m \geq 1$, if $\alpha_{m-1} \alpha_m < 0$, then (α_{m-1}, α_m) constitute a change of sign). Let Z be the number of positive real zeros of $p(x)$ (a zero of multiplicity k is counted as k zeros). Then:

$$C - Z \geq 0$$

and $C - Z$ is an even number.

Using once more the identity

$$\sum_{i=0}^{|K|} \binom{|K|}{i} z^i = (1+z)^{|K|}, \quad \forall z \in \mathcal{C}$$

it can be seen that $U_q(K)$ can be written as

$$U_q(K) = \sum_{i:\text{even}} \binom{|K|}{i} q^{|K^c|+i} - 2^{|K|-1} q^{|K^c \oplus H^*| + |K \oplus H^*|}$$

Since all the coefficients of this polynomial are strictly positive, except for the coefficient of the highest degree which is strictly negative, by employing Descartes's rule of signs, we conclude that $U_q(K)$ has at most one zero in $(0, \infty)$. But $U_1(K) = 0$, and, therefore, this is the unique zero in $(0, \infty)$. Hence, $U_q(K) > 0, \forall q \in (0, 1)$.

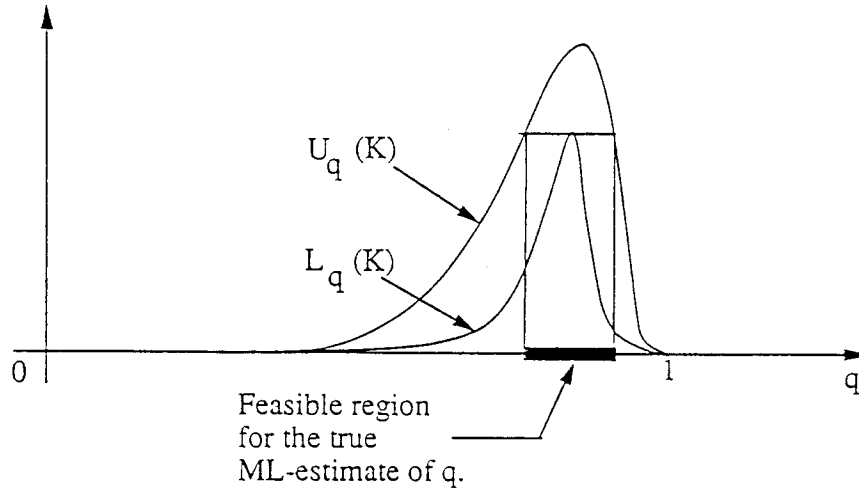


Figure 1: The concept of the feasible region.

Next, consider the derivative of the upper bound, with respect to q . After some algebraic manipulation, it can be written as

$$\frac{d}{dq}U_q(K) = q^{|K^c|-1} \left[|K^c| + \sum_{i=2, i: \text{even}}^{|K|} \left[\binom{|K|}{i} (|K^c| + i) \right] q^i - 2^{|K|-1} [|K^c \oplus H^s| + |K \oplus H^s|] q^{|K^c \oplus H^s| + |K \oplus H^s| - |K^c|} \right]$$

Again, since all the coefficients of this polynomial are strictly positive, except for the coefficient of the highest degree which is strictly negative, by employing Descartes's rule of signs, we conclude that $\frac{d}{dq}U_q(K)$ has at most one zero in $(0, \infty)$. But

$$\left. \frac{d}{dq}U_q(K) \right|_{q=0} = 0$$

$$\left. \frac{d}{dq}U_q(K) \right|_{q=0^+} > 0$$

and

$$\left. \frac{d}{dq}U_q(K) \right|_{q=1} = |K^c|2^{|K|-1} + |K|2^{|K|-2} - 2^{|K|-1} [|K^c \oplus H^s| + |K \oplus H^s|] < 0, \forall K \neq \emptyset$$

Therefore, by continuity, we conclude that $\frac{d}{dq}U_q(K)$ has at least one zero in $(0, 1)$, which must also be unique. Hence, since its derivative has only one zero crossing in $(0, 1)$, $U_q(K)$ must be *unimodal* in $(0, 1)$.

Lower bound: It can be easily seen that *one* possible germ configuration which can give rise to the observation, K , is given by the set of points $(K^c \oplus H^s)^c$. In particular, let L denote the germ point process (which is itself a DRS). Then X can be written as $X = L \oplus H$. By simple geometric arguments $(X^c \oplus H^s)^c \oplus H = X$, and $L \subseteq (X^c \oplus H^s)^c$, i.e. $X^c \oplus H^s \subseteq L^c$. Hence

$$P_X(X = K) \geq q^{|K^c \oplus H^s|} (1 - q)^{|(K^c \oplus H^s)^c|}$$

The lower bound is strictly positive for all $q \in (0, 1)$ (by inspection). One can easily show that it is unimodal. The mode location is obtained by simple differentiation of the logarithm of the lower bound. \square

Remark: Bounds of this type can be used to obtain the *feasible region* of the true Maximum Likelihood (ML) estimate of q , given the observation K . Since these bounds are typically very high degree polynomials, their (unique) modes are very sharp, leading to very accurate localization of the true ML estimate of q . Specifically, since the

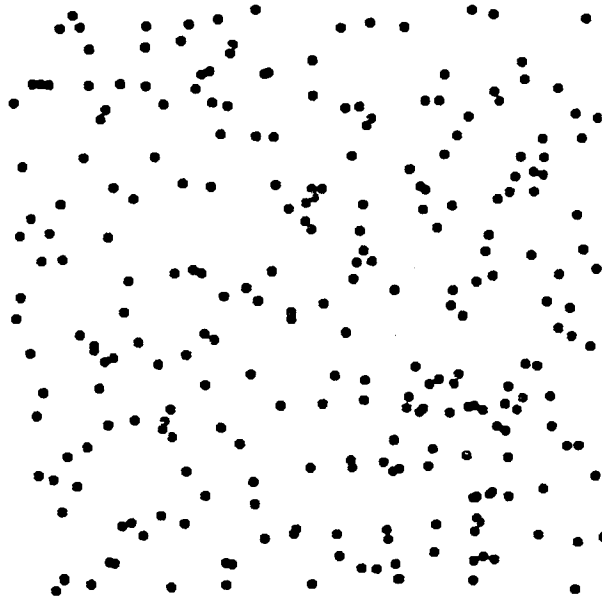


Figure 2: Realization of a Boolean model of constant intensity and fixed primary grain.

bounds are *unimodal polynomials*, the true maximum likelihood estimate of q must be within the closed interval of q -values delimited by the two q -values at which the upper bound is equal to the peak of the lower bound. This situation is illustrated in figure 1. This technique can also be used for binary hypothesis testing between two values of q . If these two values are sufficiently far apart, the bounds will indicate that the corresponding likelihoods are in disjoint regions, in which case a decision procedure based on these bounds is as good as one based on the exact likelihood, but tremendously faster.

The mode, $\hat{q}(K)$, of the lower bound, $L_q(K)$, *underestimates* q , i.e. $\hat{q}(X)$ is a *biased* estimator of q on the basis of the observation, X . This can be seen as follows.

$$E\hat{q}(X) = \frac{E|X^c \oplus H^s|}{|B|}$$

Let L denote the germ point process. Then $X^c \oplus H^s \subseteq L^c$. Thus

$$|X^c \oplus H^s| \leq |L^c|$$

and

$$E|X^c \oplus H^s| \leq E|L^c|$$

So

$$E\hat{q}(X) \leq \frac{E|L^c|}{|B|} = \frac{q|B|}{|B|} = q$$

A simple estimator of q on the basis of X is

$$\hat{q}(X) = \frac{|B| - CC(X)}{|B|}$$

where $CC(X)$ is the number of connected components of X . Obviously, this estimator overestimates q , because the number of germs (points) of any particular realization of L is always greater than or equal to the number of connected components of the corresponding realization of $X = L \oplus H$. We also remark that the ML estimator of q on the basis of X is not guaranteed to be unbiased. In practice, for "typical" observations, all these estimates are "close" to each other. As an example, figure 2 depicts a realization of a DRBRS of constant intensity and fixed primary grain.

For this example, $q = 0.999$, and the computed estimates are $\hat{q} = 0.998$, $\hat{q} = 0.99918$, whereas the feasible region is $[0.9978, 0.9992]$.

Similar bounds can be worked out for a DRBRS X of constant intensity, $\lambda_s(z) = p = 1 - q \ll 1$, $\forall z \in B$, and primary grains of random size. It can be shown that

$$Q_X(K) \cong q^{E|K \oplus RH'|}$$

where the expectation is taken with respect to the pmf f_R of the radii. Using this approximation, which is asymptotically good as q goes to 1, and Theorem 1, one can obtain the same upper bound, but this time on the *approximate (instead of the actual) probability*. In this case, H in the expression for the upper bound is replaced by $\bar{R}H$, where \bar{R} is the maximum possible radius. A unimodal lower bound can be obtained by employing the *Morphological Skeleton Transform*.

4.1 Morphological Skeletonization as a method of obtaining a consistent realization of the underlying marked point process

Let us consider the following simple vs. simple hypothesis testing problem

$$\begin{aligned} H_0: X &\sim (\lambda_s^{(0)}, H, f_R^{(0)})\text{-DRBRS} \\ \text{vs. } H_1: X &\sim (\lambda_s^{(1)}, H, f_R^{(1)})\text{-DRBRS} \end{aligned}$$

In principle, given any observation $K \in \Sigma(B)$, the probability of this observation under each one of the two hypotheses can be computed using Theorem 1, and the Bayesian rule of choice can be implemented. In practice, the computational cost associated with this brute-force method limits its applicability. We therefore pursue an alternative approach. The key idea is the following. Suppose that instead of the DRBRS realization, K , we were given the realizations of the germ point process $\{y_1, y_2, \dots\}$ and the associated radii $\{R_1, R_2, \dots\}$ that produced K . Let these data be represented by an ordered list of collections of sites $\{L_1, \dots, L_N\}$, corresponding to radii $\{r_1, \dots, r_N\}$ respectively, where $\bar{R} \geq r_1 > r_2 > \dots > r_N \geq 0$, $r_i \in \mathcal{Z}_+$, $N \leq \bar{R}$. The log-likelihood ratio is

$$\log \frac{Pr_1\{(L_1, r_1), \dots, (L_N, r_N)\}}{Pr_0\{(L_1, r_1), \dots, (L_N, r_N)\}} = \sum_{x \in B \mid x \notin \cup_{i=1}^N L_i} \log \left(\frac{1 - \lambda_s^{(1)}(x)}{1 - \lambda_s^{(0)}(x)} \right) + \sum_{i=1}^N |L_i| \log \left(\frac{f_R^{(1)}(r_i)}{f_R^{(0)}(r_i)} \right) + \sum_{i=1}^N \sum_{x \in L_i} \log \frac{\lambda_s^{(1)}(x)}{\lambda_s^{(0)}(x)}$$

Therefore, we can easily classify the observation, according to Bayesian decision theory. However, the recovery of these data from the observation K is an ill-posed problem.

Simply put, the *Morphological Skeleton*⁴ [13, 8] of a binary shape, K , with respect to a structuring element, H , is the locus of the centers of all *maximal inscribable replicas* of H in K . A *replica* of H is a scaled and sifted version of H . A replica of H is *maximal* in K iff it cannot be properly contained in any other replica of H which can be inscribed in K . The *Morphological Skeleton Function (MSF)* of K with respect to H is the function whose support is the Morphological Skeleton of K with respect to H , and its value at each skeleton point is equal to the radius of the corresponding maximal inscribable replica of H . The Morphological Skeleton is explicitly given by

$$SK(K) = \bigcup_{n=0}^N S_n(K) = \bigcup_{n=0}^N [(K \ominus nH^s) - (K \ominus nH^s) \circ H]$$

⁴Many other related notions of a skeleton exist. However, the given definition is sufficient for our purposes.



Figure 3: Realization of a DRBRS and its skeleton

where

$$N = \max \{n \mid K \ominus nH^s \neq \emptyset\}$$

The set $S_n(K)$ is the locus of centers of maximal inscribable replicas of size n , and it is called the n^{th} skeleton subset of K . Given all the skeleton subsets, the MSF is uniquely determined.

The MSF provides *one* realization of the germ points (the support set of the MSF), along with their associated radii (the values of the MSF), which can give rise to K . Therefore, we can obtain a lower bound on $P_X(X = K)$ by simply computing the probability of this realization of the germ points and their associated radii. We propose the use of the log-likelihood ratio test applied to these data as a decision rule for the simple hypothesis testing problem under consideration⁵. If the grains of K are disconnected (in the chessboard-block sense) and contained in B , then the true (unique) realization of the underlying marked point process is actually recovered, and the proposed decision rule is exact Maximum Likelihood. The overall procedure can be efficiently implemented (in polynomial time), thanks to the existence of fast Morphological Skeletonization algorithms [13]. Figure 3 depicts a realization of a DRBRS and its skeleton. Simulation results have been very encouraging, even when the primary grains overlap substantially. These simulations suggest that, for the purposes of hypothesis testing between two DRBRS models of different intensities, the size of the skeleton is an important statistic, in the sense of possessing high discriminatory power. This prompted us to investigate whether it is possible to make Maximum Likelihood decisions between DRBRS models of different intensities (but otherwise identical), based solely on the size of the skeleton. As it turns out this is not entirely true. In fact, the important statistic is the size of a superset of the skeleton. This is the subject of the following theorem.

Theorem 2 Consider the simple vs. simple hypothesis testing problem:

$$\begin{aligned} H_0: X &\sim (p_0, H, f_R)\text{-DRBRS} \\ \text{vs. } H_1: X &\sim (p_1, H, f_R)\text{-DRBRS} \end{aligned}$$

where p_0, p_1 are constants, both in $(0, 1)$, $p_1 > p_0$, and $f_R(r)$ (the common size distribution) is zero outside $\{\underline{R}, \dots, \bar{R}\}$, where $\underline{R} \geq 0$. Define

$$\gamma(X) \triangleq \frac{|(X^c \oplus \underline{R}H^s)^c|}{|B|}$$

⁵This idea has been concurrently and independently developed in [10], as a means of performing shape/size analysis and synthesis of a different class of DRS models.

Let K be the observation, and let $P_0(X = K)$, $P_1(X = K)$ denote the probability of the observation under the null and alternative hypothesis, respectively. If

$$\gamma(K) < l(p_0, p_1) \triangleq \frac{\log(1 - p_1) - \log(1 - p_0)}{\log(p_0(1 - p_1)) - \log(p_1(1 - p_0))}$$

then $P_0(X = K) > P_1(X = K)$.

Proof:

Let L be a realization of the germ points which can give rise to the observation, K . The probability of this realization under p_0 is $Pr_0(L) = p_0^{|L|}(1 - p_0)^{|B| - |L|}$, whereas under p_1 it is $Pr_1(L) = p_1^{|L|}(1 - p_1)^{|B| - |L|}$. It is easy to see that

$$Pr_0(L) > Pr_1(L) \iff \frac{|L|}{|B|} < l(p_0, p_1)$$

But, any L which can give rise to K necessarily satisfies

$$L \subseteq (K^c \oplus \underline{RH}^*)^c$$

thus

$$\frac{|L|}{|B|} \leq \gamma(K) < l(p_0, p_1), \text{ by assumption}$$

Therefore, $Pr_0(L) > Pr_1(L)$, uniformly over all L which can give rise to K . Hence, since the two models have the same primary grain and size distribution, we conclude that $P_0(X = K) > P_1(X = K)$. \square

This theorem may find application in the automated screening of cell samples, where the alternative hypothesis corresponds to an abnormally high average number of cells per unit area. Then, most of the observed samples can be classified with minimal effort, whereas the few samples which do not meet the criterion of theorem 2 can be examined in greater detail, by either a machine, or a human expert.

5 Conclusions

In this paper, we have taken the approach of modeling discrete and binary image data as realizations of a bounded Discrete Random Set. We have argued for the merits of such an approach, most notably the ability to recover the associated probability measure given knowledge of the generating functional, by means of a Mobius-type transformation. Based on this result, we have developed a discrete analog of the Boolean random set, and provided various tools for its statistical inference. Although, in reality, binary image data are sampled versions of an underlying physical process, which lives in a continuum, the data *per se* can only assume a finite number of realizations. This is the case in many applications, in which there exist physical barriers that limit the available resolution. Although a Discrete Random Set approach may ignore the "fine letter" of the underlying physical structure, it provides a useful, and, most importantly, *tractable* idealization, which, as demonstrated, can lead to practical inference procedures.

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