Title: Complex Variables and Infinite Dimensional Linear Systems

Abstract: Complex function algebras play a fundamental role in the analysis and synthesis of linear infinite dimensional systems. In this talk we present concrete applications of complex variable methods to such problems as: stabilization, parametrization of stable controllers, spectral factorization, controller design. The mathematical topics discussed include: Corona theorems, $\bar{\partial}$ theory, algebraic functions, H^∞ theory.

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Complex Variables and Infinite Dimensional Linear Systems

D_r must be column reduced
D_e " " row reduced

Division theorem: Given N, D, we can find
Q, R s.t.

$$N(s) = Q(s)\, D(s) + R(s) \quad \text{with } R(s)\, D^{-1}(s) \text{ strictly proper}$$

if N is $p \times m$, Q $m \times m$ nonsingular $\exists$ unique

if further Q is column reduced
uniqueness is ensured if columns of R have degrees <
corresp. columns of $D(s)$.

p. 8 bottom ν is the highest row degree = obs. index of given system
this is also the highest column degree

p. 9 $$\det \Delta(s) = s^{m(\nu-1)} + \beta_1(s)\, s^{(m-1)(\nu-1)} + \cdots + \beta_m(s)$$
arbitrary

p. 9, mention poles of closed loop are
precisely zeros of $\det \Delta(s) \det D_c(s)$

$$T_0(z) = T_1 z^{-1} + T_2 z^{-2} + \cdots$$
$$= \frac{n_0 + n_1 z + \cdots}{d_0 + d_1 z + \cdots} = \frac{n_0(z)}{d_0(z)}$$

$D(s) \in H^\infty(\Pi^-)$ inner

look at D_{ij} j fixed.

what is column <u>degree?</u>

matrix of highest column <u>degrees</u> <u>is.</u>

constructed as <u>follows</u>

$d_{ij}(s)$ ← polynomials.

s^{k_j}

$\lim_{s\to\infty} s^{-k_j}$

$(s^{-k_j})\, a_0 s^{k_j} + a_1 s^{k_j - 1} + a_2 s^{k_j - 2} \cdots$

$a_0 + a_1 s^{-1} + \cdots$

$$\lim_{s\to\infty} \Big(\underbrace{\begin{bmatrix} d^1 & d^2 & \cdots & d^m \end{bmatrix}}_{m} \Big)_m \begin{bmatrix} s^{-k_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} s^{-k_2} \end{bmatrix} \begin{bmatrix} d^1 \end{bmatrix}$$

$$= D_{hc}$$

$$D_{hc} = \lim_{s\to\infty} D(s)\, \delta^{-1}(s)$$

$$T_0 = \underset{p\times m}{N_r}\,\underset{m\times m}{D_r^{-1}} = \underset{p\times p}{D_\ell^{-1}}\,\underset{p\times m}{N_\ell}$$

$N_\ell, D_\ell\ \ N_r, D_r \in \underset{\substack{p\times m\\ m\times m}}{H^\infty}(\Pi^-)$, D_r, D_ℓ inner

Carleson A, B, $\exists$ F, G s.t

$$AF + BG = I$$

iff $\exists\ \delta$ s.t. $\|Fx\| + \|Gx\| \geq \delta > 0$

Euclidean division

Given $\alpha \in H^\infty(\Pi^-)$, $\beta \in H^\infty(\Pi^-)$

$\exists\ k, \rho \in H^\infty(\Pi^-)$ s.t. $\dfrac{\rho}{\beta} \in A^\infty_{M,0}$ $\alpha = k\beta + \rho$

Given $A \in H^\infty_{m\times p}(\Pi^-)$, $D \in H^\infty_{pp}(\Pi^-)$ + inner

$\exists\ Q$ in $m\times p$ + $R \in H^\infty_{m\times p}(\Pi^-)$ s.t

$RD^{-1} \in A^\infty_{M,0}$ and

$$A = QD + R$$

Suppose $f(s) = \begin{bmatrix} f_1(s) \\ \vdots \\ f_m(s) \end{bmatrix}$ $\quad f_i \in H^\infty(\Pi^-)$

other idea
$\deg a < \deg b$ when b has more "zeros"

Again consistent

$\deg f_i = f_i^{inn}$

Why $\quad \deg a < \deg b \quad$ iff

a^{inn} / b^{inn} (we say b^{inn} is a "bigger" than a^{inn})

$a^{inn} H^\infty \quad$ if $\quad b^{inn} = q\, a^{inn}$.

$b^{inn} H^\infty$

then $\quad b^{inn} H^\infty = a^{inn} q H^\infty \subseteq a^{inn} H^\infty$

So $\quad \deg$

$\deg a < \deg b \iff a^{inn} / b^{inn} \iff b H^\infty \subseteq a H^\infty$

(Poly" a, b, $\deg a < \deg b$ iff state space of $\frac{n}{a}$ < state space of $\frac{n}{b}$

or if $(a H^2)^\perp < (b H^2)^\perp$ or $b H^2 < a H^2 \iff a^{inn} / b^{inn}$)

consistent

first guess
$\deg f(s) = \{$ l.c. inner multiple of f_i^{inn} $\}$ ← no good because it contains all zeros of components that is too much
$i = 1, 2, \ldots m)$.

second guess:

$\deg f(s) = \{$ maximal inner function f_i^{inn}, $i = 1, \ldots m\}$ ← no good it [illegible] exist

So now let D be inner matrix $m \times m$

<u>let</u> $k_i(s) =$ l.c. inner multiple of the inner factors of d_{li}, $l = 1, 2 \ldots m$.

Form $K(s)$

London meeting
of Canadian Math. Soc.

$A > 0,\ B > 0$

T. Ando $\quad A:B = (A^{-1} + B^{-1})^{-1}$

or $A:B = \max\{X \geq 0 \mid \begin{bmatrix} X & X \\ X & X \end{bmatrix} \leq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\}$

Kubo – Ando

$\sigma \to \hat{\sigma}$

$$A:B = \int_{(0,1)} (t^{-1}A : (1-t)^{-1}B)\, d\hat{\sigma}(t)$$

2. geometric mean of two operators. $G(A,B) = A^{1/2} V B^{1/2}$, partial isometry

$= \max\{X ;\ \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\}$

or unique solution of $(A+X):(B+X) = X$

or " " " $A:X + B:X = X$, $X \geq A:B$

arithmetic mean, harmonic mean

$$H(A,B) \leq G(A,B) \leq A(A,B)$$

(a) Let $X_0 = A+B$

$X_{n+1} = (A+X_n):(B+X_n)$ then $X_n \downarrow G(A,B)$

$X_0 = A+B$

$X_{n+1} = A:X_n + B:X_n$

$X_n \downarrow G(A,B)$

(b) $X_0 = A(A,B)$; $X_{n+1} = A(X_n, Y_n)$ $\quad X_n \downarrow$

$Y_0 = H(A,B)$; $Y_{n+1} = H(X_n, Y_n)$ $\quad Y_n \uparrow$ $\quad G(A,B)$

3. Geometric mean for more than three operators.

$A_1, \dots, A_N > 0$

$A(\vec{A}) = \frac{1}{N} \sum_{j=1}^{N} A_j$, $H(\vec{A}) = N:(A_1 : A_2 : \cdots A_N) = N \prod_{j=1}^{N}{:}\, A_j$

Kosaki $\quad G(\vec{A}) = \dfrac{1}{\Gamma(1/N)^{N-1}} \displaystyle\int_{\substack{t_1+\cdots+t_N=1 \\ t_j \geq 0}} \left(\prod_{j=1}^{N}{:}\, t_j^{-1} A_j\right) \cdot \left(\prod_{j=1}^{N} t_j\right)^{1/N - 1}$

4. Decomposition (idea consider $A:B$ as a l. bd of A, B)

$X \geq 0$ is A-abs cont. if $\exists\ X_n \geq 0$ s.t. $X_1 \leq X_2 \leq \cdots \to X$ s.t. $X_n \leq a_n A$

$X \geq 0$ is A-singular if $0 \leq Y \leq X$ and $Y \leq A$ imply $Y = 0$. (i.e. $X:A = 0$)

5. Relation to time invariant systems

$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ $\quad \Theta_T(z) = Cz(zI-A)^{-1}B + D$; $\|T\| \leq 1$ iff $\|\Theta_T(z)\| \leq 1$

conversely also

42 $\frac{4}{8}$ Bo x1. $V=$

8,4 23,6

5 | 4
10 | 1,2

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Structural operators and the adjoint equation for time varying retarded F.D.E

extensions of earlier work <u>with</u> <u>Delfour</u>