FA7-11:45

DYNAMIC OBSERVERS AS ASYMPTOTIC LIMITS OF RECURSIVE FILTERS

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ABSTRACT

In this paper we discuss the ansatz: nonlinear observer = limit of nonlinear filter. The results of Vent-tsel' and Freidlin on large deviations are useful in interpreting observer trajectories as 'most-likely' paths.

SUMMARY

We consider the problem of constructing dynamic state observers for certain classes of dynamical systems, including nonlinear ones. To illustrate the basic ideas and the requisite technical steps let us begin with a different solution to the well known linear Gaussian case.

In the linear case we are given the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \quad (1)$$
$$x(0) = x_0$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$ and $y(t)\in\mathbb{R}^p$. We wish to construct a new linear dynamical system called a state observer

$$\dot{z}(t) = Fz(t) + Gu(t) + Hy(t)$$
$$\hat{x}(t) = Kz(t) + Lu(t) + My(t) \quad (2)$$
$$z(0) = z_0$$

such that the error $x(t) - \hat{x}(t)$ decays exponentially fast to zero at a rate controlled by the designer. The problem of course is that the true initial condition x_0 is not known. Therefore the exponential decay should hold for every choice of x_0, z_0. This is a well known problem and well known solutions exist [1] [2].

We propose a new one. First we associate with (1) a filtering problem

$$dx^{\varepsilon,\delta}(t) = Ax^{\varepsilon,\delta}(t)dt + Bu(t)dt + \sqrt{\delta}Ndw(t)$$
$$d\xi^{\varepsilon,\delta}(t) = C\,x^{\varepsilon,\delta}(t) + \sqrt{\varepsilon}Rdv(t) \quad (3)$$

where $w(t)$, $v(t)$ are standard n-dimensional and p-dimensional Brownian motions. The initial values for (3) are: $x^{\varepsilon,\delta}(0)$ is Gaussian independent of $w(\cdot)$, $v(\cdot)$ with

$$E\{x^{\varepsilon,\delta}(0)\} = z_0^{\varepsilon,\delta}$$
$$E\{(x^{\varepsilon,\delta}(0) - z_0^{\varepsilon,\delta})(x^{\varepsilon,\delta}(0)-z_0^{\varepsilon,\delta})^T\} = P_0^{\varepsilon,\delta} \quad (4)$$

The reasons for associating the particular stochastic system (3) to (1) will become apparent only after the construction of the observer. The idea is to get an estimate $\hat{x}^{\varepsilon,\delta}(t)$ for $x^{\varepsilon,\delta}(t)$, using (3), which is computed recursively and then obtain an observer by taking the limit as $\varepsilon,\delta\to 0^+$ in a certain way. Thus we can construct observers as asymptotic limits of certain associated filtering problems. Since (3) is a linear problem the minimum variance (and maximum likelihood) estimate of $x(t)$ is given by the corresponding Kalman filter [3] [4]:

$$d\hat{x}^{\varepsilon,\delta}(t) = A\hat{x}^{\varepsilon,\delta}(t)dt + Bu(t)dt + K^{\varepsilon,\delta}(t)(d\xi^{\varepsilon,\delta}(t) - C\hat{x}^{\varepsilon,\delta}(t)dt) \quad (6)$$
$$\hat{x}^{\varepsilon,\delta}(0) = z_0^{\varepsilon,\delta},$$

where

$$K^{\varepsilon,\delta}(t) = \frac{P^{\varepsilon,\delta}(t)}{\varepsilon} C^T(RR^T)^{-1} \quad (7)$$

and

$$\frac{dP^{\varepsilon,\delta}(t)}{dt} = AP^{\varepsilon,\delta}(t) + P^{\varepsilon,\delta}(t)A^T - \frac{P^{\varepsilon,\delta}(t)}{\varepsilon}C^T(RR^T)^{-1}CP^{\varepsilon,\delta}(t) + \delta NN^T$$
$$P^{\varepsilon,\delta}(0) = P_0^{\varepsilon,\delta} \quad . \quad (8)$$

We have emphasized the dependence of certain matrix valued functions on ε,δ by explicit notation.

The plan is to compute the asymptotic limit as $\varepsilon,\delta\to 0^+$ of (6) (8) and show that under appropriate conditions and interpretations an observer results. The dynamical equation for the Kalman filter error equation is

$$e^{\varepsilon,\delta}(t) = x(t) - \hat{x}^{\varepsilon,\delta}(t)$$
$$de^{\varepsilon,\delta}(t) = Ae^{\varepsilon,\delta}(t)dt - K^{\varepsilon,\delta}(t)Ce^{\varepsilon,\delta}dt + \sqrt{\delta}\,Ndw(t) - \sqrt{\varepsilon}\,K^{\varepsilon,\delta}(t)Rdv(t). \quad (9)$$

If we choose the matrices N and R so that

$[A,N]$ is stabilizable

and

$[A,(RR^T)^{-1/2}C]$ is detectable

we know that for all $\varepsilon,\delta>0$, $P^{\varepsilon,\delta}(t)$ will converge as $t\to\infty$ to the unique positive definite solution of

CH1788-9/82/0000-1126$00.75

$$AP_\infty^{\varepsilon,\delta} + P_\infty^{\varepsilon,\delta} A^T - \frac{P_\infty^{\varepsilon,\delta}}{\varepsilon} C^T(RR^T)^{-1}CP_\infty^{\varepsilon,\delta} + \delta NN^T = 0 \qquad (10)$$

and

$$A - \frac{P_\infty^{\varepsilon,\delta}}{\varepsilon} C^T(RR^T)^{-1}C$$

has all eigenvalues in the open left half plane. These well known facts suggest the following construction. Let

$$Q^\varepsilon(t) = \frac{P^{\varepsilon,\varepsilon}_{(t)}}{\varepsilon} . \qquad (11)$$

Then $Q^\varepsilon(\cdot)$ satisfies

$$\frac{dQ^\varepsilon(t)}{dt} = AQ^\varepsilon(t) + Q^\varepsilon(t)A^T - Q^\varepsilon(t)C^T(RR^T)^{-1}CQ^\varepsilon(t) + NN^T$$

$$Q^\varepsilon(0) = \frac{P^{\varepsilon,\varepsilon}_{(0)}}{\varepsilon} . \qquad (12)$$

Let us choose

$$P^{\varepsilon,\varepsilon}(0) = P_0 \qquad (13)$$

an (independent of ε) positive semidefinite matrix. Then it follows that for every $\varepsilon > 0$

$$\lim_{t\to\infty} Q^\varepsilon(t) = \overline{Q} \qquad (14)$$

where $\overline{Q}$ is the unique positive definite solution of

$$A\,\overline{Q} + \overline{Q}\,A^T - \overline{Q}\,C^T(RR^T)^{-1}\,C\overline{Q} + NN^T = 0 \qquad (15a)$$

$$A - \overline{Q}\,C^T(RR^T)^{-1}\,C \text{ is asymptotically stable} \qquad (15b)$$

It is important to note that $\overline{Q}$ is independent of ε. Using $\overline{Q}$ we define as a candidate observer, the system

$$\frac{dz}{dt} = Az(t) + Bu(t) + \overline{Q}C^T(RR^T)^{-1}(y(t)-Cz(t))$$

$$z(0) = z_0. \qquad (16)$$

Then the error equation for the observer (16) is

$$e_0 = x(t) - z(t)$$

$$\frac{de_0(t)}{dt} = (A - \overline{Q}\,C^T(RR^T)^{-1}C)e_0(t) \qquad (17)$$

The underlying theory that suggests this construction, is the large deviation theory of Vent-sel' and Freidlin [5]. Briefly one compares the observer error equation (17) with the Kalman filter error equation (9) when $\varepsilon=\delta$. The result is that as $\varepsilon=\delta\to0^+$ the support of the measure of the process $e^{\varepsilon,\varepsilon}(\cdot)$, converges to the deterministic path defined by (17). We can now explain the choice of observations in (3). Basically it allows the observer (16) to be driven by the output y(t) and not by the differentiated output dy(t).

Details of the proofs can be found in [6].

We note that the speed of convergence of the observer error equation is controlled by the design matrices R,N via (15a) which determines $\overline{Q}$. The study of this dependence is an interesting algebraic problem. One can associate to (1) more sophisticated parametric filters like (3), in the sense that in general the noise amplification matrices could be chosen as $N(\delta_1, \delta_1,\ldots,\delta_k)$ and $R(\varepsilon_1,\ldots,\varepsilon_k)$; i.e. a multiparameter asymptotic limit can be considered. This latter generalization will permit better control of the spectral properties of the closed loop "A" matrix in the observer (16). Further note that the only assumption needed on the system parameters was that $[A,C]$ is detectable (since then $[A,(RR^T)^{1/2}C]$ is also detectable), which is the same as for Luenberger's construction [1].

Our main goal is to develop constructive methods in designing observers for nonlinear system. The point of view explained above for linear systems extends to certain classes of nonlinear systems as well. A gain the work of Vent-tsel' and Freidlin for non-Gaussian processes [7] is fundamental. Large deviation properties for nonlinear filtering problems have been studied by Hijab in [8]. In cases where finite dimensional filters can be constructed for the associated nonlinear filtering problems, explicit finite dimensional nonlinear observers result. However, in the general case the nonlinear observer is infinite dimensional. The characterization of the observer path in the general case involves the limiting p.d.e. resulting from the Zakai equation for the parameterized nonlinear filtering problem. Explicit results have been obtained in the case of Benes and for linear analytic systems. We refer again for details to [6].

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* Work supported by DOE contract DEAC01-80RAS042.