JB 82-01

Paper Entitled

"Existence, Uniqueness and Tail Behavior of Solutions to Zakai Equations with Unbounded Coefficients"

From the Proceedings of

Advances in Filtering and Stochastic Control and Information Sciences
Vol. 42, pp. 1-17

New York
1982

EXISTENCE, UNIQUENESS AND TAIL BEHAVIOR OF SOLUTIONS TO ZAKAI EQUATIONS WITH UNBOUNDED COEFFICIENTS*

W.E. Hopkins, Jr., J.S. Baras and G.L. Blankenship
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

ABSTRACT

Conditions are given to guarantee the existence and uniqueness of solutions to the Duncan-Mortensen-Zakai equation for nonlinear filtering of multivariable diffusions with unbounded coefficients. Sharp upper and lower bounds on the tail of conditional densities are also obtained. A methodology is described to treat these problems using classical p.d.e. methods applied to the "robust" version of the DMZ equation. Several examples are included.

* Supported in part by ONR Contract N00014-79-C-0808.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Recently the problem of filtering a diffusion process $x(t)$ from nonlinear observations $y(t)$ in additive Gaussian noise has been studied by analyzing an unnormalized version of the conditional distribution of $x(t)$ given the past of $y(.)$. If this conditional distribution is absolutely continuous with respect to Lebesgue measure, then it has a density which satisfies a linear stochastic partial differential equation known as the Duncan-Mortensen-Zakai (DMZ) equation [1]. Background information on this equation and other aspects of the nonlinear filtering problem may be found in [2]. In the present paper we focus on existence-uniqueness results for the DMZ equation and on tail estimates of the resulting solutions. Our motivation for these problems stems primarily from the following areas: (a) numerical algorithms for the solution of DMZ equation and their subsequent implementation by special purpose array processors; (b) numerical evaluation of the Kallianpur-Striebel path integral representation of the solution; (c) accuracy and convergence analysis in asymptotic expansions of the solution.

In cases where the process $x(t)$ evolves in a bounded domain in $\mathbb{R}^n$, or when the state space is unbounded but the coefficients of the DMZ equation are bounded, a satisfactory existence-uniqueness theory is available [3]-[5]. More recently, existence-uniqueness of solutions has been established for filtering problems having "strongly" unbounded coefficients. In [6] polynomial observations are studied via a related optimal control problem. In [7] classical results for fundamental solutions of parabolic equations with unbounded coefficients are applied to the "robust" version of the DMZ equation [1], [2], for scalar diffusions. In [8], [9], the method of [7] is extended to multidimensional problems. Furthermore in [7]-[9] tight estimates of the tail behavior of solutions are obtained. In the present paper we review and summarize the method and main results of [7]-[9].

To set the problem, consider the pair of Itô stochastic differential equations

$$\begin{aligned} dx(t) &= f(x,t)dt + g(x,t)dw(t), \quad x(0) = x_0 \\ dy(t) &= h(x,t)dt + dv(t), \quad y(0) = 0 \end{aligned} \tag{1}$$

where $x(t), w(t) \in \mathbb{R}^n$, $y(t), v(t) \in \mathbb{R}^m$, $w(\cdot)$ and $v(\cdot)$ are mutually independent Wiener processes independent of x_0, and x_0 has a density $p_0(.) \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. The coefficients f,g,h are assumed to satisfy $f^i \in HC^{1,0}_{\ell oc}(\omega)$, $h^i \in HC^{2,1}_{\ell oc}(\omega)$, where $\omega = \mathbb{R}^n \times (0,T]$ and $HC^{i,j}_{\ell oc}$ denotes the space of functions having locally Hölder continuous derivatives of order i in x and j in t. Furthermore the generator L of the diffusion process $x(\cdot)$ is assumed to be uniformly elliptic; that is, there exist continuous functions $\theta_i(x,t)$ i=1,2, and a constant $\theta_0>0$ such that for all $\xi\in\mathbb{R}^n$ and $(x,t)\in\mathcal{Q}$, $\theta_0|\xi|^2\leq\theta_1(x,t)|\xi|^2\leq\sum_{i,j=1}^{n}\alpha^{ij}(x,t)\xi_i\xi_j\leq\theta_2(x,t)|\xi|^2$, where $(\alpha^{ij})\triangleq\frac{1}{2}gg^T$. The filtering problem for (1) is to estimate statistics of x(t) given the σ-algebra $\mathcal{F}_t^y=\sigma\{y(s)|0\leq s\leq t\}$. Equivalently we can compute the conditional distribution of x(t) given $\mathcal{F}_t^y$.

Formally, the conditional density of x(t) given $\mathcal{F}_t^y$ is given by

$$p(x,t) = U(x,t)/\int_{\mathbb{R}^n} U(x,t)dx$$

where U(x,t) is a solution of the DMZ equation

$$\begin{aligned} dU &= (L^* - \tfrac{1}{2}|h|^2)Udt + \langle h,dy(t)\rangle U, \quad (x,t)\in\mathcal{Q} \\ U(x,0) &= p_0(x) \qquad , \quad x\in\mathbb{R}^n \end{aligned} \tag{2}$$

Here L^* is the adjoint of L and (2) is written using Stratonovich calculus.

It is well known, particularly in recent studies, that if one introduces the transformation

$$V(x,t) = U(x,t)\ \exp\ (-\langle h(x,t),\ y(t)\rangle) \tag{3}$$

then V satisfies a classical, linear, parabolic partial differential equation, parametrized by the observation paths $y(\cdot)$, which is called the "robust" version of the DMZ equation [1], [2]:

$$\begin{aligned} \frac{\partial V}{\partial t}(x,t) = \sum_{i,j=1}^{u} A^{ij}(x,t)V_{x_ix_j}(x,t) + \sum_{i=1}^{n} B^i(x,t)V_{x_i}(x,t) &+ \\ C(x,t)V(x,t),\ (x,t)\in\mathcal{Q}& \end{aligned} \tag{4}$$

$$V(x,0) = p_0(x)\ ,\ x\in\mathbb{R}^n.$$

The functions $B^i(x,t)$ are pointwise linear functions of y(t), while C(x,t) is pointwise a quadratic function of y(t). Since the paths of $y(\cdot)$ are Hölder continuous, (4) is a classical p.d.e. and classical results on existence-uniqueness of fundamental solutions for linear parabolic equations due to Besala [10] can be fruitfully applied to the robust DMZ equation. Direct analysis of the DMZ equation is rather complicated. However, since the transformation (3) is invertible, positivity preserving, and (2), (4) are linear, existence, uniqueness, and tail behaviour of solutions of the DMZ equation may be obtained by analyzing (4) instead.

2. OUTLINE OF THE METHOD

Our method can be applied to a variety of problems with unbounded coefficients. We outline briefly here the main steps. We shall refer to the zeroth order coefficient of a parabolic equation like (4) as the potential (i.e. $C(x,t)$ in (4)). We use a result of Besala [10] which asserts that if we can find a "weight" of the form $\Phi(x,t) = \exp(\psi(x,t))$ such that the function $\tilde{U}=V\Phi$, satisfies a parabolic p.d.e. where the potential term and the potential term of the adjoint are nonpositive for $(x,t)\epsilon\mathcal{D}$, then there exist a classical fundamental solution for (4). Furthermore integral growth estimates for the fundamental solution are given in [10].

To apply this idea to (4) we must consider stopping time partitions of the interval $[0,T]$; there are special cases where stopping times are not needed. These partitions are defined as follows. Given a small positive number $\varepsilon>0$, we define $0 = t_0<t_1<\ldots<t_N = T$ via

$$t_0 = 0$$
$$t_{k+1} = \begin{cases} \inf\limits_{t_k<t<T} \{t:|y(t)-y(t_k)| =\varepsilon\}, & \text{whenever inf exists} \\ T, & \text{otherwise} \end{cases} \tag{5}$$
$$N = \min \{k:t_k-T\}.$$

Here ε will be fixed by other considerations. Then on each set $\mathcal{D}_k\underline{\Delta}\mathbb{R}^n x(t_k,t_{k-1}]$, $0\leq k\leq N-1$ we introduce the transformations

$$u^k(x,t) = U(x,t) \exp(\phi^k(x,t) - <h(x,t), y(t)>). \tag{6}$$

Then u^k satisfies

$$u^k_t = \mathcal{L}^k u^k, \quad (x,t)\epsilon\mathcal{D}_k, \quad 0\leq k\leq N-1$$

$$u^k(x,t_k) = \begin{cases} p_0(x)\exp(\phi^0(x,0)) & , k = 0 \\ u^{k-1}(x,t_k)\exp(\phi^k(x,t_k) - \phi^{k-1}(x,t_k)), & 1\leq k\leq N-1 \end{cases} \tag{7}$$

where

$$\mathcal{L}^k u^k = \sum_{i,j=1}^{n} a^{ij}u_{x_ix_j} + \sum_{i=1}^{n} b^{ik}u_{x_i} + c^k u$$

$$b^{ik} = - f^i + 2\sum_{j=1}^{n} a^{ij}_{x_j} + 2\sum_{j=1}^{n} a^{ij}(<h_{x_j},y> - \phi^k_{x_j}) \tag{8}$$

$$c^k = c^k_{ess} - \sum_{i,j=1}^{n} a^{ij}(\phi^k_{x_ix_j} - <h_{x_ix_j},y>) -$$

$$- 2\sum_{i,j=1}^{n} a^{ij}_{x_i}(\phi^k_{x_j} - \langle h_{x_j}, y\rangle) + \sum_{i,j=1}^{n} a^{ij}_{x_i x_j} - \sum_{i=1}^{n} f^i_{x_i} - \langle h_t, y\rangle$$

$$c^k_{ess} = \frac{1}{2}|g\nabla\phi^k - g(\langle h_{x_i}, y\rangle)^n_{i=1} +$$

$$+ (g^{-1})^T f|^2 - \frac{1}{2}|h|^2 - \frac{1}{2}|(g^{-1})^T f|^2 + \phi^k_t.$$

Thus u^k satisfy linear parabolic equations of the same type (4). We then choose ϕ^k on each subinterval so as to render the potential term c^k of $\mathcal{L}^k$ as well as the potential term of $(\mathcal{L}^k)^*$ nonpositive, and apply Besala's results.

The guidelines for this construction are relatively simple to understand. The actual computations and resulting conditions depend on the case at hand. First one poses appropriate assumptions on f,g,h so that c^k and the adjoint potential are dominated by c^k_{ess} as $|x|\to\infty$. Then one fixes the parameters in ϕ^k (in particular ϕ^k_t) so that c^k_{ess} is ≤ 0. To construct ϕ^k one first constructs the piece of ϕ^k corresponding to the Fokker-Planck equation for $x(\cdot)$. This is natural since the diffusion itself must be well-behaved: no explosions, nice solutions of the Fokker-Planck equation, etc. Indeed if we set h=0 in (4) we get the Fokker-Planck equation. The construction of this first part of ϕ (here we mean that typically $\phi=\phi_1 + \phi_2$) brings us naturally into contact with Khas'minskii's test for nonexplosions and generalizations thereof. To construct ϕ_2, after obtaining ϕ_1, one returns to the full expressions for c_k and the adjoint potential and appropriately balances growth conditions on h with those of f,g, so as to make both potentials ≤ 0. For uniqueness we use classical weak maximum principles. The procedure gives us upper bounds on the tails by means of the uniqueness class identified. To obtain lower bounds on the tails we choose ϕ^k such that the potential c^k of $\mathcal{L}^k$ is positive; i.e. using classical comparison theorems for linear parabolic equations. We refer to the references cited above [7]-[9] for the details of these constructions. In the remainder of the paper we summarize the results in particular cases.

3. THE SCALAR CASE n=1, m=1

In this section we describe our results for the scalar case including polynomial nonlinearities. Our assumptions on the coefficients in (1.6) are stated in terms of the original functions f,g,h. To state these succinctly, we will use the relative order notation:

Definition. Let $F,G,:R\to R$ and

$$L = \limsup_{|x|\to\infty} |F(x)/G(x)| \varepsilon [0,\infty]$$

Then $F = O(G)$ if $L<\infty$ and $F = o(G)$ if $L = 0$.

The coefficients of the diffusion x are assumed to satisfy

(A1) $f\in C^1(R)$, $g\in C^2(R)$, f_x, g_{xx} are locally Hölder continuous;

(A2) $g(x)\geq\lambda>0, \forall x\varepsilon R$ and some λ;

(A3) $-\int_0^x (f/g^2)(\xi)d\xi \geq M, \forall x\varepsilon R$ and some M;

(A4) $(f/g^2)_x = o(f^2/g^4)$, $f_x = o(f^2/g^2)$; and

(A5) the martingale problem for (f,g) is well-posed.

The last condition implies that the stochastic differential equation for x has a unique weak solution for all $t\geq 0$. A sufficient condition for this is the existence of a Lyapunov function for the backwards Kolmogorov equation associated with the process x[11]. If the integral in (A3) diverges to $+\infty$ as $|x|\to\infty$, then it could serve as the Lyapunov function. If the martingale problem is not well posed, then the process x may have "explosions" (escape times which are finite with probability one). In this case the conditional distribution of x(t) given $\mathcal{F}_t^y$ may have singular components which are not computed by the DMZ-equation.

The observation function h is assumed to satisfy:

(B1) $h\in C^2(R)$, h_{xx} is locally Hölder continuous;

(B2) either g^2h_{xx}, $(g^2h_x)_x = o(h^2)$, or

g^2h_{xx}, $(g^2h_x)_x = o(g^2h_x^2)$

(B3) either $gh_x = O(h)$ or $gh_x = o(f/g)$;

(B4) either $(g^2)_{xx} = o(h^2)$ or $(g^2)_{xx} = o(f^2/g^2)$;

(B5) one of the two mutually exclusive cases holds:

(i) either $h = O(f/g)$ or $h = O(gh_x)$; or

(ii) both $f/g = o(h)$ and $gh_x = o(h)$; in addition, gh_x, $g_xh = o(h^2)$

(B6) in case (B5)(i),

$$\lim_{|x|\to\infty} \max\{|h(x)|, -\int_0^x (f/g^2)(\xi)d\xi\} = +\infty;$$

and in case (B5)(ii),

$$\lim_{|x|\to+\infty} \left|\int_0^x (h/g)(\xi)d\xi\right| = +\infty$$

Remarks. (1) the growth conditions are relatively easy to understand in the case when f,g,h are polynomials, especially $f(x) = f_0x^j$, $g(x) =$

$g_0(1+x^2)^k$, $h(x) = h_0 x^\ell$.

(2) The conditions (A1)–(B6) are not necessary; different choices of the weight functions used in the proofs would lead to different growth restrictions. In fact, one could consider optimizing the choice of the weight functions. Here the weights are chosen as

$$\phi^k(x,t) = \psi^k(x) - \gamma^k t \tag{9}$$

where

$$\psi^k(x) = \alpha\phi_1(x) + \beta_1^k\phi_2(x) + \beta_2[1 + \phi_2^2(x)]^{\frac{1}{2}}$$

$$\phi_1(x) = \begin{cases} -\int_0^x (f/g^2)(\xi)d\xi & \text{in case (B5)(i)} \\ 0 & \text{in case (B5)(ii)} \end{cases} \tag{9a}$$

$$\phi_2(x) = \begin{cases} h(x) & \text{in case (B5)(i)} \\ \int_0^x (h/g)(\xi)d\xi & \text{in case (B5)(ii)} \end{cases}$$

The parameters $\alpha, \beta_2, \{\beta_1^k, \gamma^k, t_k\}_{k=0}^\infty$ will be functionals of the path $y(t)$, $t \geq 0$; for their explicit definition see [7].

Assumptions (A3) and (B6) together with the constraints $\alpha>0$, $\beta_2>0$, $\beta_2>|\beta_1^k|$, imply that the weight functions $\psi^k(x)$ diverge to $+\infty$ as $|x|\to\infty$. The remaining growth conditions serve to identify the dominant terms (as $|x|\to\infty$) in the potential $c^k(t,x)$ in (7) and in the potential of the adjoint of (7). Assumption (B3) permits us to select the functions ψ^k and the constants γ^k so that these potentials are nonpositive. This in turn permits the use of a maximum principle.

Under these assumptions we show that the robust equation (4) has a fundamental solution which may be used to construct a unique solution to the DMZ-equation within a certain class of functions. To describe this class, we define the constants

$$\eta_i = \lim_{|x|\to\infty} \sup |g(\phi_i)_x|/[h^2+f^2/g^2]^{\frac{1}{2}} \tag{10}$$

$$\nu_i = \lim_{|x|\to\infty} \inf |g(\phi_i)_x|/[h^2+f^2/g^2]^{\frac{1}{2}}, \quad i = 1,2$$

The assumptions imply $\eta_1, \nu_1 \in [0,1]$ and $\eta_2, \nu_2 \in [0,\infty)$ when (B5)(i) holds, while $\eta_1 = 0 = \nu_1$, $\eta_2 = 1 = \nu_2$ when (B5)(ii) holds. The assumption that either (B5)(i) or (ii) holds implies $(\nu_1 + \nu_2)>0$.

Theorem 1. Suppose (A1) - (A5), (B1) - (B6) hold. Let $p_o(x)$ be continuous, $p_o(x)\geq 0$, and assume that there exist constants $\theta_i>0$, $i=1,2$, such that $0<\theta_1\eta_1 + \theta_2\eta_2<1$, and

$$p_o(x)\exp[\theta_1\phi_1(x) + \theta_2|\phi_2(x)|]\leq M, \ \forall x\varepsilon R \tag{11}$$

and some $M<\infty$. Then for any constants $\tilde{\theta}_i, 0<\tilde{\theta}_i<\theta_i, i=1,2,$ there exists a unique solution to the DMZ-equation (2) within the class of functions satisfying

$$\lim_{|x|\to\infty} \sup U(x,t) \exp[\tilde{\theta}_1\phi_1(x) + \tilde{\theta}_2|\phi_2(x)|] = 0, \ \forall t\geq 0 \tag{12}$$

This solution satisfies $U(t,x) = U^k(t,x)$, $t\varepsilon(t_k,t_{k+1})$

$$U^k(x,t) = e^{h(x)y(t)}\int_{-\infty}^{\infty}\Gamma_k(t,x;z,t_k)U^{k-1}(t_k,z)dz \tag{13}$$

$$U^0(x,0) = p_o(x), \quad k = 1,2,\ldots$$

where Γ_k is the fundamental solution of (7).

Theorem 2. Suppose (A1) - (A5), (B1) - (B6) hold, and assume when case (B5)(i) holds with $\nu_1>0$, $\nu_2>0$, that $-f(x)$ sgn(x) and $h_x(x)$ sgn$(xh(x))$ are non-negative for $|x|$ sufficiently large. Let $p_o(x)$ satisfy the conditions in Theorem 1, and suppose further that there exist $M_0>0, K_0>0$ such that

$$M_0 \exp[-K_0\phi(x)]\leq p_o(x), \forall x\varepsilon R \tag{14}$$

where

$$\phi(x) = \phi_1(x) + |\phi_2(x)| \tag{15}$$

Then for any $T<\infty$, there exist positive constants M_1, M_2, K_1, K_2, which may depend on the path $\{y(t), 0\leq t\leq T\}$ such that the solution of the DMZ-equation given by (13) satisfies

$$M_1 \exp[-K_1\phi(x)]\leq U(x,t)\leq M_2 \exp[-K_2\phi(x)] \tag{16}$$

$$\forall (x,t)\varepsilon R\times[0,T]$$

To illustrate our results and make contact with other recent work on nonlinear filtering (e.g., [6],[12]), we consider a class of systems with polynomial f,h.

So let f,h, be polynomials with f odd and stable, i.e.,

$$f(x) = \sum_{i=0}^{2q-1} f_i x^i, \quad -f_{2q-1}>0 \tag{17}$$

$$h(x) = \sum_{j=0}^{s} h_j x^j, \quad h_s \neq 0$$

where q,s are positive integers. Suppose

$$g(x) = g_o (1+x^2)^{r/2}, \quad g_o > 0 \tag{18}$$

where $r \in [0,\infty)$. Our conditions for existence and uniqueness and estimates of the asymptotic behavior of the density depend on whether or not $g(x)$ is globally Lipschitz and on the degree of $h(x)$ relative to the degree (or stability) of $f(x)$. There are two cases covered by Theorems 1-2.

<u>Case 1</u>: $r \in [0,1]$, $q > r$, $s \geq 1$, $q \geq 1$.

The restrictions (A1)-(A5),(B1)-(B6), applied here require $g(x)$ to satisfy a linear growth constraint $r \in [0,1]$, that f be at least a cubic polynomial, $q \geq 2$, where $g(x)$ is of linear growth, $r=1$, and that $h(x)$ be non-constant.

Then from Theorem 2 for any $0 \leq t_1 \leq t_2$, there exist constants M_i, K_i depending on the observation path such that

$$M_1 \exp[-K_1 |x|^\rho] \leq U(x,t) \leq M_2 \exp[-K_2 |x|^\rho] \tag{19}$$

where

$$\rho = \begin{cases} s-r+1 & , r<1 \text{ and } r+s>2q-1 \\ \max[s, 2(q-r)], & \text{otherwise} \end{cases} \tag{20}$$

We refer to [7] for details. Although it is not covered in the present case, the situation $r=0$, f and its first two derivatives are bounded, and h is asymptotic to a non-constant polynomial, can be easily treated by adapting the arguments in Theorems 1-2. In particular, the inequalities (19) hold with $\rho = s+1$, and this result overlaps [6] [12]. For example, if $f=0$, $g=1$, $h(x) = h_s x^s$, $h_s \neq 0$, then

$$0 < K_2 < |h_s|/(s+1), \quad \rho = s+1. \tag{21}$$

This was obtained by Sussmann for $s=3$ in [12].

<u>Case 2</u>: $r > 1$, $q > r + \frac{1}{2}s$, $s \geq 1$, $q \geq 2$.

Here $g(x)$ is of super linear growth, $f(x)$ is at least a cubic polynomial, and $h(x)$ is dominated, as indicated, by the dynamics of the state process. In this case the asymptotic behavior of the conditional density is the same as that of the <u>a priori</u> density (of $x(t)$).

4. THE SCALAR BILINEAR FILTERING PROBLEM

The example presented here illustrates a different grouping of terms and different growth conditions, necessary for this class of problems. It is presented here as another application of the method described earlier.

Consider the system

$$\begin{aligned} dz(t) &= f(z(t))dt + z(t)dw(t) \\ dy(t) &= h(z(t))dt + dv(t) \\ z(0) &= z_o,\ y(0) = 0,\ 0 \leq t \leq T < \infty \end{aligned} \tag{22}$$

with z_o having density $p_o(z)$, and z_o, w,v mutually independent as before. Since z(t) will eventually be trapped in either the positive or negative half space, we shall arrange that $z(t) \in [0,\infty)$ by taking $f \in C^1(0,\infty)$ satisfying

$$\text{(C1)} \quad \begin{cases} f(z) \leq K(1+z) \text{ for some } K>0 \\ f(0) \geq 0 \end{cases} \tag{23}$$

and by taking $p_o(z)$ defined on $(0,\infty)$ and continuous and integrable there. We also assume that $h \in C^2(0,\infty)$ with h_z, h_{zz} locally Hölder continuous (and so, bounded at zero).

We impose the following growth conditions on f,h.

(C1') $f_z(z)$ is bounded and locally Hölder continuous

(C2) $\lim\limits_{z \downarrow \infty} [f(z)/z] = >0$

(C3) $\lim\limits_{z \to \infty} |h(z)|/\log z = +\infty$

(C4) $\lim\limits_{z \to \infty} [h_{zz}(z)/h_z^2(z)] = 0$

(C5) for some constants K_i, M_i, i=1,2

$$M_1 + K_1|zh_z(z)| \leq |h(z)| \leq M_2 + K_2|zh_z(z)|$$

Note that these conditions are satisfied when f(z) is affine and h(z) is a non-constant polynomial; we do not consider the case h(z) constant. The assumptions that f,h and their derivatives are bounded at the origin are made for convenience only. They can be relaxed by introducing more complex growth restrictions. The other assumptions (C2)-(C5) are essential (to our method).

The DMZ-equation associated with (22) is

$$dU(z,t) = [\tfrac{1}{2}(z^2U)_{zz} - (fU)_z - \tfrac{1}{2}h^2U]dt + h\,U\,dy(t)$$
$$U(z,0) = p_o(z)\ ,\quad (z,t)\in[0,\infty)\times[0,T] \tag{24}$$

Because the generator for the diffusion x in (22) is not uniformly elliptic our theorems are not directly applicable. If we make the logarithmic change of coordinates $x=\log z$ and let $W(x,t) = U(e^x,t)$, $x\in R$, (24) becomes

$$dW(x,t) = \{\tfrac{1}{2} W_{xx} + [3/2 - e^{-x}f(e^x)]W_x + [1-f_z(e^x) - \tfrac{1}{2}h^2(e^x)]W\}dt + h(e^x)Wdy(t) \tag{25}$$
$$W(x,0) = p_o(e^x)$$

The methods of Besala [10] as used in the proofs of Theorems 1 and 2 can be directly applied to the robust version of this equation. The proofs of Theorems 1-2 go through when the weight functions $\psi^k(x)$ are defined by (9) with $x=\log z$ and

$$\phi_1(z) = -\int_1^z f(\xi)/\xi^2 d\xi$$
$$\phi_2(z) = h(z) \tag{26}$$

Similarly as before define

$$\nu_1 = \eta_1 = \lim_{z\downarrow 0}|f/z|/[h^2 + f^2/g^2]^{1/2}$$
$$= 1 \text{ by (C2)} \tag{27}$$

$$\nu_2 = \liminf_{z\to+\infty} z|h_z(z)|/[h^2 + f^2/g^2]^{1/2}\in(0,\infty)$$
$$\eta_2 = \limsup_{z\to+\infty} z|h_z(z)|/[h^2 + f^2/g^2]^{1/2}\in(0,\infty) \tag{28}$$

Then we have obtained [7] the following result.

Theorem 3. Suppose (C1)-(C5) hold. Let $\theta_i>0$, $0<\theta_1 + \eta_2\theta_2<1$, _and suppose_ $p_o(z)$ _satisfies_

$$p_o(z)\le M_1 \exp\,[\theta_1\int_1^z(f(\xi)/\xi^2)d\xi-\theta_2|h(z)|] \tag{29}$$

for all $z\in(0,\infty)$ _and some_ $M_1>0$. _Then for any_ $\hat\theta_i<\theta_i$ _the DMZ-equation (24) has a unique solution in the class of functions satisfying, for all_ $t\ge 0$

$$\limsup U(z,t)\exp\,[-\hat\theta_1\int_1^z(f(\xi)/\xi^2)d\xi + \hat\theta_2|h(z)|] = 0 \tag{30}$$

Moreover, if there exist constants, M_2, $\hat\theta_i$, $0<\hat\theta_i<\theta_i$, _such that for all_

$z \in (0,\infty)$

$$M_2 \exp\left[\hat{\theta}_1 \int_1^z (f(\xi)/\xi^2)d\xi - \hat{\theta}_2 |h(z)|\right] \le p_o(z) \tag{31}$$

<u>then for all</u> $t \geq 0$ <u>the solution</u> $U(z,t)$ <u>is asymptotic to</u>

$$\exp\left[\int_1^z (f(\xi)/\xi^2)d\xi - |h(z)|\right] \tag{32}$$

<u>in the sense of (16) in Theorem 2</u>.

For example, when $f(z) = az + b$, assumption (C2) implies either $b>0$ or $a>0$ and $b = 0$. Then

$$\phi_1(z) = bz^{-1} - b - a \log z$$

and, whenever (29) (30) are satisfied, $U(z,t)$ is asymptotic to

$$\begin{cases} \exp\left[-|h(z)| - b/z\right], & \text{if } b>0 \\ z^a \exp\left[-|h(z)|\right], & \text{if } b=0 \text{ and } a>0, \end{cases} \tag{33}$$

5. THE MULTIVARIABLE CASE

In [8] [9] the previous results have been generalized to multivariable diffusions. We shall briefly describe one of the cases here and refer the reader to [8], [9] for other cases and further details. The assumptions on the functions f,g,h are of two types basically: relative order relations implying the potentials of $\mathcal{L}^k$ and its adjoint are dominated by certain terms in c^k_{ess} as $|x| \to \infty$, and inequalities providing for control of these dominant terms. These assumptions may be stated succintly using the following definitions.

<u>Definition</u>. Let $f,g, \in C(\mathbb{R}^n \times [0,T])$ with $g \geq 0$. Then $f = o_b(g)$ if for every $\varepsilon>0$ there exists a constant $K(\varepsilon)$ such that for all $(x,t) \in \mathscr{D}$

$$|f(x,t)| \le \varepsilon g(x,t) + K(\varepsilon)$$

<u>Definition</u>. A nonnegative function $r \in H^2_{loc}(\mathbb{R}^n)$ is said to be a scale function if

(i) there exist positive constants θ^1, R such that $|\nabla r(x)|^2 \geq \theta^1$ for all $|x| \geq R$

(ii) $\lim_{R\to\infty} \min_{|x|=R} r(x) = +\infty$

We shall use on occas ion the notation

$$A_r(x,t) \triangleq 2 \sum_{i,j=1}^{n} a^{ij}(x,t) r_{x_i}(x) r_{x_j}(x) = |g(x,t)\nabla r(x)|^2 \geq 0.$$

<u>Definition</u>. Let $\mathscr{V}$ denote the collection of pairs of functions $(F(z,t), r(x))$ satisfying

$F \in HC^{2,1}_{loc}(\mathbb{R} \times [0,T])$, $r \varepsilon H^2_{loc}(\mathbb{R}^n)$ and

$$\vartheta(i)\quad F_z(z,t) = o_b(F^2(z,t))$$

$$\int_0^{r(x)} F(z,t)dz, \int_0^{r(x)} F_t(z,t)dz = o_b(F^2 A_r)$$

$$\vartheta(ii)\quad F \sum_{i,j=1}^{n} a^{ij} r_{x_i x_j},\ F \sum_{i,j=1}^{n} a^{ij}_{x_i} r_{x_j} = o_b(F^2 A_r)$$

Definition. Two time-varying vector fields $f_1(x,t)$, $f_2(x,t)$ are said to be compatible if there exists a constant $R>0$ such that for all $|x| \geq R$, $t \varepsilon [0,T]$,

$$\sum_{i,j=1}^{n} a^{ij}(x,t) f_1^i(x,t) f_2^i(x,t) \geq 0.$$

If we now let $(\bar{a}^{ij})$ denote the inverse of (a^{ij}) we can state the assumptions on the coefficients f,g of the diffusion as follows.

Hypothesis F. There exist a scale function $r(x)$, nonnegative functions $\overline{F}$, $\underline{F} \varepsilon HC^{1,1}_{loc}(\mathbb{R} \times [0,T])$ satisfying $0 \leq \overline{F} \leq \underline{F}$, and constants $F_0 \geq 0$, $F_1 > 0$, $R>0$ such that $(\overline{F},r)$, $(\underline{F},r) \varepsilon \vartheta$, and for all $|x| \geq R$, $t \varepsilon [0,T]$,

$$\text{(i)}\quad \overline{F}(r(x),t) \leq (F_0 - \sum_{i=1}^{n} f^i(x,t) r_{x_i}(x))/A_r(x,t)$$

$$\text{(ii)}\quad \underline{F}(t(x),t) \geq (-F_0 - \sum_{i=1}^{n} f^i(x,t) r_{x_i}(x))/A_r(x,t)$$

$$\text{(iii)}\quad A_r(x,t)\overline{F}^2(r(x),t) \geq 2F_1[-F_0 + (\frac{\theta_2}{\theta_1} \sum_{ij=1}^{n} a^{-ij} f^i f^j)(x,t)]$$

$$\text{(iv)}\quad \operatorname{div}(f),\ \sum_{i,j=1}^{n} a^{ij}_{x_i x_j}(x,t) = o_b(\overline{F}^2 A_r)$$

Hypothesis G. There exist constants $0 \leq \mu \leq \nu \leq 2$ and positive constants R, K_0, K_1 such that for all $|x| \geq R$, $t \varepsilon [0,T]$,

$$K_1 |x|^{2-\nu} \leq \theta_1(x,t)$$

$$|a^{ij}(x,t)| \leq K_0 |x|^{2-\mu}$$

$$|a^{ij}_{x_i}(x,t)| \leq K_0 |x|^{1-\nu}$$

$$|a^{ij}_{x_i x_j}(x,t)| \leq K_0$$

If hypothesis G is not satisfied, it will be assumed that

Hypothesis L. $\lim_{|x|\to\infty} \int_0^{r(x)} \overline{F}(x,t)dx = +\infty$, uniformly in $[0,T]$.

One sees easily that F(i) is a generalization of Khas'minskii's test for explosions. In fact we have the following result [8] [9].

Theorem 4. If either F(i), (iii) and G hold, or F(i) and L hold, then the martingale problem for the pair (f,g) is well-posed.

Next the observation nonlinearity is assumed to satisfy the following conditions. Let

$$h^+(x,t) = ([\vec{1} + h^k(x,t)]^{\frac{1}{2}})_{k=1}^m$$

where $\vec{1} = (1,1,\ldots,1)^T \in \mathbb{R}^n$.

Hypothesis H. There exists a scale function s(x), nonnegative functions $\overline{H}$, $\underline{H} \in HC^{2,1}_{loc}(\mathbb{R}\times[0,T])$ satisfying $0 \le \overline{H} \le \underline{H}$, and constants $R>0$, $H_0 \ge 0$ such that $(\overline{H},s)$, $(\underline{H},s) \in \mathcal{A}$ and for all $|x| \ge R$, $t \in [0,T]$,

(i) $\overline{H}^2(s(x),t) \le (|h(x,t)|^2 + H_0)/A_s(x,t)$

(ii) $\underline{H}^2(s(x),t) \ge (|h(x,t)|^2 - H_0)/A_s(x,t)$

(iii) $|h_t| = o_b(|h|^2)$, $|h_t| = o_b(\overline{H}^2 A_s)$

$\sum_{i,j=1}^n a^{ij}|h_{x_i x_j}|$, $\sum_{i,j=1}^n a^{ij}_{x_i}|h_{x_j}|$, $\sum_{i,j=1}^n a^{ij}\langle h^+,\vec{1}\rangle_{x_i x_j}$,

$\sum_{i,j=1}^n a^{ij}_{x_i}\langle h^+,\vec{1}\rangle_{x_j} = o_b(\sum_{i,j=1}^n a^{ij}|h_{x_i}||h_{x_j}|)$.

(iv) $\sum_{i,j=1}^n |a^{ij}||h_{x_i}||h_{x_j}| = o_b(\overline{H}^2 A_s + \overline{F}^2 A_r)$

(v) $|h(x,t)| = o_b(\int_0^{s(x)} \overline{H}(z,t)dz)$

Finally we need the following hypothesis which help control the growth of the potentials.

Hypothesis I.

(i) $x_i|h_{x_i}| = o_b(\overline{H}^2 A_s)$

(ii) $\sum_{i=1}^n f^i|h_{x_i}| = o_b(\overline{H}^2 A_s + \overline{F}^2 A_r)$

(iii) $\overline{F}\sum_{i,j=1}^n a^{ij}r_{x_i}|h_{x_j}|$, $\overline{H}\sum_{i,j=1}^n a^{ij}s_{x_i}|h_{x_j}| = o_b(\overline{H}^2 A_s + \overline{F}^2 A_r)$

$\underline{F}\sum_{i,j=1}^n a^{ij}r_{x_i}|h_{x_j}|$, $\underline{H}\sum_{i,j=1}^n a^{ij}s_{x_i}|h_{x_j}| = o_b(\underline{H}^2 A_s + \underline{F}^2 A_r)$

To compute lower bounds on the density we shall use

Hypothesis K. (i) ∇r and ∇s are compatible

(ii) $\vec{x}$ is compatible with both ∇r and ∇s.

We can now state the following existence result. For a proof see [9].

Theorem 5. (Existence of fundamental solutions). Suppose hypotheses F, H, I (ii) and I (iii) hold. Then for each Hölder continuous path $\{y(t),\ 0\le t\le T\}$ of the observation process, there exists a classical fundamental solution of the robust DMZ equation (4).

To describe our results for uniqueness and tail behavior we need the definition of certain function classes, given below.

Definition. Let $f\in C(\mathbb{R}^n \times [0,t])$ and $\phi\in C(\omega)$. Then $f\in\mathcal{E}(\phi)$ if $\exists$ constant K s.t. $|f|\le K \exp(\phi)$ for all $(x,t)\in\omega$, and

$f\in\mathcal{E}_0(\phi)$ if $\lim_{|x|\to\infty} |f| \exp(-\phi) = 0$, uniformly in $t\in[0,T]$.

Let

$$\bar{\psi}^0(x,t) = \begin{cases} \log|x| \quad , \quad \text{if } \mu=0 \\ \qquad\qquad\qquad\qquad\qquad , \text{ for } |x|\ge R \\ |x|^\mu/\mu, \text{ if } \mu\in(0,2] \\ \text{any } C^\infty\text{-time invariant extension for } |x|\le R \end{cases}$$

$$\bar{\psi}^1(x,t) = \int_0^{r(x)} \bar{F}(z,t)\,dz$$

$$\bar{\psi}^2(x,t) = \int_0^{s(x)} \bar{H}(z,t)\,dz$$

Theorem 6. [9] (Existence and uniqueness of solutions). Suppose F, G, H, I hold and that p_o is continuous, nonnegative and integrable. Then there exist positive constants a_1, $\underline{\alpha}_i$, $\bar{\alpha}_i$, $i = 1,2$ such that whenever

$$p_o(x)\in\mathcal{E}\,(-\alpha_0\bar{\psi}^0(x) - \alpha_1\bar{\psi}^1(x,0) - \alpha_2\bar{\psi}^2(x,0))$$

for some constants $\alpha_0\in(0,\bar{\alpha}_0)$, $\alpha_1\in(\underline{\alpha}_1,\bar{\alpha}_1)$, $\alpha_2\in(0,\bar{\alpha}_2)$, the DMZ equation has a unique solution in

$$\mathcal{E}_0\left(\frac{\bar{\alpha}_0}{1-a_1 t}\bar{\psi}^0(x) - \underline{\alpha}_1\bar{\psi}^1(x,t) + (1-\varepsilon)\underline{\alpha}_2\bar{\psi}^2(x,t)\right)$$

for all $\varepsilon\in(0,1)$. If instead of hypothesis G, hypothesis L holds the result remains valid with $\alpha_0 = \bar{\alpha}_0 = 0$ and $\underline{\alpha}_1$ chosen arbitrarily small.

Theorem 7. [9] (Lower bounds). Suppose the same hypotheses hold as in theorem 6, including the growth assumption on p_o. Suppose in addition that $\mu>0$ and hypothesis K holds.

(a) If $\mathrm{supp}(p_o(x))\supseteq(|x|\le R)$, where R is the maximum of the radii R in F,G,H,K, then there exist constants M, α_i, $i=0,1,2$, depending on the path of the observation process $y(\cdot)$, such that

$$U(x,t) \geq M \exp[-\frac{1}{t}(1 + \sum_{i=0}^{2} \alpha_i \underline{\psi}^i(x,t))].$$

(b) If there exist constants $M^0, \alpha_i^0 > 0$, $i = 0,1,2$, such that for all $x \in \mathbb{R}^n$

$$p_0(x) \geq M^0 \exp[-\sum_{i=0}^{2} \alpha_i^0 \underline{\psi}^i(x,0)]$$

then there exist constants M, α_i, $i=0,1,2$, with M depending on the observation path such that

$$U(x,t) \geq M \exp[-\sum_{i=0}^{2} \alpha_i^0 \underline{\psi}^i(x,t)].$$

There are several other cases where the method described here has been successfully applied. We refer to [9] for the details.

We close this section with a class of relatively simple but interesting examples. Consider the problem of a scalar observation of a two dimensional Wiener process,

$$dx(t) = dw(t)$$

$$dy(t) = h(x_1,x_2)dt + dv(t), \quad y(t) \in \mathbb{R}^1.$$

Then with $\overline{F} = \underline{F} = 0$, the natural upper and lower bounds of Theorems 6, 7 are of the form

$$\exp(-A(x_1^2 + x_2^2) - Bs(x)).$$

Examples of pairs of functions (h,s) for which Theorems 5-7 hold include the following: $h = (x_1^{2p} + x_2^{2q})^{1/2}$, $s = c(\sqrt{1 + x_1^{2p+2}} + \sqrt{1 + x_2^{2q+2}})$; $h = ax_1^p + bx_2^p$, $s = c(h^2 \rho^{1-p} + \rho)$ where $\rho = (x_1^2 + x_2^2)^{1/2}$. Here p,q are positive integers, a,b,c real numbers. Note in particular that the cases $h(x_1,x_2) = x_1^3 \pm x_2^3$ or $h(x_1,x_2) = (x_1^4 + x_2^2)^{1/2}$ are covered.

REFERENCES

[1] M.H.A. Davis, S.I. Marcus, "An introduction to nonlinear filtering", in Stochastic Systems: The Mathematics of Filtering and Identification and Applications, M. Hazewinkel, J.C. Willems, eds., NATO Adv. Study Inst. Ser., Reidel, Dordrecht, 1981, pp. 53-76.

[2] M. Hazewinkel, and J.C. Willems, eds., The Mathematics of Filtering and Identification and Applications, NATO Adv. Study Inst. Ser., Reidel, Dordrecht, 1981.

[3] N.V. Krylov, B.L. Rozovskii, "On the conditional distribution of diffusion processes", Izvestia Akad. Nauk, SSSR, Math. Ser., 42, 1978, pp. 356-378.

[4] E. Pardoux, "Stochastic partial diff. eqs. and filtering of diff.

processes", Stoch., 3, 1979, p. 127.

[5] D. Michel, "Regularite des lois conditionnelles en theorie du filtrage non-lineaire et calcul des var. stoch." J. Funct. Anal., 41, 1981, pp. 8-36.

[6] W.H. Fleming, S.K. Mitter, "Optimal control and pathwise nonlinear filtering of nondeg. diff.", 20th IEEE Conf. Dec. Contr., San Diego, Dec. 1981.

[7] J.S. Baras, G.L. Blankenship and W.E. Hopkins, Jr., "Existence, Uniqueness, and Asymptotic Behavior of Solutions to a Class of Zakai Equations with unbounded Coeff," to appear in IEEE Trans. Aut. Contr.

[8] W.E. Hopkins, Jr., "Nonlinear Filtering of Multivariable Diffusions with Strongly Unbounded Coefficients", to appear in Proc. of 1982 Conference on Information Sciences and Systems, Princeton Univ.

[9] W.E. Hopkins, Jr., Ph.D. Thesis, Electrical Engineering Dept., University of Maryland, in preparation.

[10] P. Besala, "Fundamental Solution and Cauchy Problem for a Parabolic System with Unbounded Coefficients" J. Diff. Eqs., 33, 1979, pp. 26-38.

[11] D.W. Stroock and S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, New York, 1979.

[12] H.J. Sussmann, "Rigorous Results on the Cubic Sensor Problem", in [2], above.