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STOCHASTIC CONTROL OF TWO COMPETING QUEUES

JOHN S. BARAS AND STAVROS A. BELBAS
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

ABSTRACT

We consider optimal server time allocation to two parallel queues. The server has available complete past observations of the queue sizes for his decisions. The infinite time discounted version of the problem is analyzed here. It is shown that the optimal strategy is stationary. The optimal value function is shown to be the unique solution of the Bellman equation. Finally, analysis of degenerate Bellman equations, of the type appearing in this problem is presented. Numerical methods of solution can be derived from the results presented here.

1. INTRODUCTION

We consider the problem of selecting which of two parallel queues to serve with a single server. The system is depicted in Figure 1. Customers arrive into stations 1 and 2 according to two independent Poisson streams with constant rates λ_1, λ_2 respectively. The two queues compete for the services of an exponential server with constant service rate μ. Let $x_{i,t}$ be the number of customers in queue i at time t, the customer in service included. The control to be selected is clearly of a switching type. When $u_t=1$ and the server completes a service the next customer to be served comes from queue 1, while if $u_t=0$ the next customer comes from queue 2. The queue that is being served forms an M/M/1 system with the server. This is a simple sequencing problem, where the sequencing variable is u_t. Let $x_t=(x_{1,t},x_{2,t})$ denote the state at time t. u_t is to be selected knowning x_t.

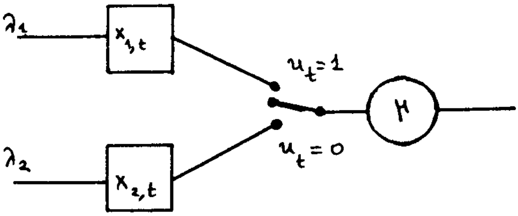

Fig. 1 The server time allocation problem

The server allocation time is to be selected to minimize delays, weighted according to c_1, c_2, two positive constants. Thus the cost per unit time with queues $x_{1,t}$, $x_{2,t}$ is $c_1x_{1,t}+c_2x_{2,t}=c^Tx_t$. It is shown that the policies which minimize the infinite time discounted average cost are characterized by a switching curve $S:x_2=S(x_1)$. That is $u_t=1$ if $x_{2,t}<S(x_{1,t})$, while $u_t=0$ qhwn $x_{2,t}>S(x_{1,t})$. Thus the optimal strategy is stationary.

The present paper is an outgrowth of earlier work [1] by one of the authors.

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There are two important differences between the work presented in [1] and the problems treated here. First the queues are observed here while only partial observations were assumed in [1]. Second finite time problems were treated in [1] while infinite time discounted problems are considered here.

The problem described can be thought of as a dynamic priority (non-preemptive) problem in a single queue with two classes of customers [2] [3]. The optimal static priority assignment (i.e. open loop control) which minimizes average cost per unit time, under steady state conditions was obtained by Cox and Smith [2]. The result, in the terminology used here, is that queue 1 has priority over queue 2 iff $c_1>c_2$ (the opposite holds when $c_2<c_1$). Actually their result is easily extended to include general service time distributions, as well as different distributions for different classes and to general m queues (classes) problems. Dynamic priorities minimizing average cost per unit time, under steady state conditions were derived by Rykov and Lembert [3]. The analysis presented in [3] is rather formal, however the results appear to be correct. The result, in the terminology used here, is that the optimal dynamic (feedback) priority coincides with the optimal static priority of Cox and Smith [2]. The result is valid under general service time distributions. In both cases the results were obtained by classical queueing methods and are in agreement with intuition, given the steady state assumptions and the average cost per unit time criterion used.

The problem considered here is quite different from those analyzed in [2] [3], since infinite time discounted average delays are considered. We use two methods to analyze the problem. The first uses an embedded Markov chain and is inspired by the analysis of a simple tandem queueing system studied by Rosberg, Varaiya and Walrand [6]. The second centers on direct analysis of the associated Bellman equation. Both methods suggest numerical schemes for computation of the optimal strategy, and are useful for more general queueing systems problems as well.

Other examples of dynamic queueing control problems can be found in [4] [5].

2. A METHOD BASED ON THE ASSOCIATED EMBEDDED MARKOV CHAIN PROBLEM

We consider the discounted continuous time problem for the two competing queues system described in figure 1. Let $\alpha\geq 0$ be the discount rate and

$$V_t^\alpha(x) := \min_{\gamma\in\Gamma} E\{\int_0^t c^T x_\sigma \, e^{-\alpha\sigma} d\sigma\} \tag{2.1}$$

be the minimum total expected discounted delay, when the time horizon is $t\geq 0$ and the starting state at time 0 is $x_0=x$. In (2.1) Γ is the set of all admissible policies

$$\Gamma := \{\gamma=\{u_t(.), t\geq 0\} \text{ such that } u_t : Z\text{x}Z\rightarrow\{1,0\}\} \tag{2.2}$$

where $Z = \{0,1,2,3...\}$.

The main objective is to characterize the optimal cost for the infinite time problem V_∞^α and the corresponding optimal strategy γ_∞^α. Considering the sizes of both queues x_t, as the state of the system, we have a Markov chain with countable state space ZxZ. The transitions of x_t are easy to describe. They correspond to arrivals at queue 1 or queue 2, and to service completions. The transitions induced by arrivals occur at arrival times in queues 1 and 2 respectively denoted by

$$\mathcal{A}^1 = \{t_0^1,\ t_1^1,\ t_2^1,\ \ldots\} \qquad (2.3)$$
$$\mathcal{A}^2 = \{t_0^2 m\ t_1^2,\ t_2^2,\ \ldots\}$$

The control strategy does not affect arrivals. Therefore if we let A_1, A_2 be functions describing the new state after a transition induced by arrivals we see that for u = 1 or 0, the transition probabilities are

$$\left.\begin{array}{ll} A_1(x_1,x_2) = (x_1+1,x_2) & \text{with Prob. } \lambda_1 dt + o(dt) \\ A_2(x_1,x_2) = (x_1,x_2+1) & \text{with Prob. } \lambda_2 dt + o(dt \end{array}\right\} \qquad (2.4)$$

On the other hand transitions induced by service completions occur at service completion times

$$\mathcal{S} = \{t_0^s,\ t_1^s,\ t_2^s, \ldots\} \qquad (2.5)$$

and depend on the value of the control. Thus when u=1, D_1 is possible where

$$D_1(x_1,x_2) = ((x_1-1)^+,x_2) \text{ with Prob. } \mu dt + o(dt), \qquad (2.6)$$

while when u = 0, D_2 is possible where

$$D_2(x_1,x_2) = (x_1,(x_2-1)^+) \text{ with Prob. } \mu dt + o(dt) \qquad (2.7)$$

Here $(x)^+ = \max\{x,0\}$. The transition epochs [2], [3], [6], for the state process x_t are

$$\mathcal{T} = \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{S} \qquad (2.8)$$

The embedded Markov chain method analyzes the behavior of the state process x_t at transition epochs from $\mathcal{T}$ [2], [3], [6]. It is a consequence of our formulation that the sets $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{S}$ are disjoint (i.e. no simultaneous occurrence is possible).

To treat the infinite time discounted problem we need some results of Lippman [8], particularly since we do not have bounded costs for all x. The action space in the problem treated here is finite: {0,1}. The set of states accessible from a given state (x_1,x_2) in one transition is $\{(x_1+1,x_2), (x_1,x_2+1), ((x_1-1)^+,x_2), (x_1,(x_2-1)^+), (x_1,x_2)\}$, independent of the transition epoch. Furthermore the cost is linear in the state. As a consequence Assumptions 1 and 2 in Lippman [8, pp. 719] hold.

The strategies in Γ are Markovian [9]. Let us also consider the more general class of randomized nonanticipative strategies:

$$\tilde{\Gamma} := \{\gamma = (\pi_t(0), \pi_t(1)),\ t \geq 0\} \text{ such that } \pi_t \text{ is a function of the past history } h^t = (x_s,\ u_s,\ 0 \leq s \leq t). \qquad (2.9)$$

Here $\pi_t(i)$, i = 0,1 is the probability of choosing u=0, or 1 at time t. A strategy in Γ is stationary whenever $u_t(.)$ is independent of t. We now have the following result:

Theorem 1:

The infinite time discounted problem (α>0), for the two competing queues, has an optimal strategy over $\tilde{\Gamma}$, which is Markovian and stationary. Furthermore the optimal value function V^α_∞ is the unique solution of the Bellman equation

$$V^{\alpha}_{\infty}(i_1,i_2) = \frac{c_1}{\mu+\alpha}\, i_1 + \frac{c_2}{\mu+\alpha} + \frac{\mu}{\mu+\alpha} \min_{v\varepsilon\{0,1\}} \{\sum_{j_1,j_2} P^1_{i_1,j_1}(v) P^2_{i_2,j_2}(v) V^{\alpha}_{\infty}(j_1,j_2)\} \tag{2.10}$$

$$i_1 i_2 \geq 0,$$

where $P^1_{i_1,j_1}(v)$, $P^2_{i_2,j_2}(v)$ are given in (2.21) below.

Proof: Since Assumptions 1, 2 of Lippman [8] are satisfied and the action set is finite, the first result is an immediate consequence of Theorem 1 in Lippman [8, p. 719]. Since the optimal policy is stationary we can restrict consideration to stationary Markovian strategies which change values only at service completion times, $t^s_k \varepsilon \mathscr{S}$. We denote this restricted class by Γ_0. As a result we need find the equivalent discrete time stochastic control problem for the embedded Markov chain with transition epochs $\mathscr{S}$ and not $\mathscr{T}$. For ease of notation we denote x_{t_k} by x_k and u_{t_k} by u_k. It is plain that the intertransition intervals $t^s_{k+1} - t^s_k$ are independent, identically distributed random variables with an exponential distribution

$$\text{Prob } \{t^s_{k+1} - t^s_k > t\} = \exp(-t\mu). \tag{2.11}$$

Consider now a policy in Γ_0 and let us compute the corresponding cost over the random interval $[0,t^s_n]$ and initial state x:

$$E_x \int_0^{t^s_n} e^{-\alpha t} c^T x_t \, dt = E_x \sum_{k=0}^{n-1} \int_{t^s_k}^{t^s_{k+1}} e^{-\alpha t} c^T x_k \, dt$$

$$= E_x \sum_{k=0}^{n-1} \{c^T x_k \, E \int_{t^s_k}^{t^s_{k+1}} e^{-\alpha t} dt\} =$$

$$= \frac{1}{\alpha} E_x \sum_{k=0}^{n-1} \{c^T x_k E[e^{-\alpha t^s_k} - e^{-\alpha t^s_{k+1}}]\}$$

$$= \frac{1}{\alpha} E_x \sum_{k=0}^{n-1} c^T x_k (\beta^k - \beta^{k+1}) = \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} \beta^k c^T x_k \;, \tag{2.12}$$

provided $\beta<1$, while when $\beta=1$ it equals $E_x \sum_{k=0}^{n-1} c^T x_k$. In the sequel we shall ignore the constant $\frac{1-\beta}{\alpha}$ in (2.12). Here

$$\beta = \frac{\mu}{\alpha+\mu}, \tag{2.13}$$

so $\beta=1$, corresponds to the discount rate $\alpha=0$. So the infinite time cost for the embedded Markov chain is

$$W^{\beta}_{\infty}(\gamma,x) = E_x\{\sum_{k=0}^{\infty} \beta^k c^T x_k\} \tag{2.14}$$

when policy $\gamma\varepsilon\Gamma_0$ is used. Obviously for $\alpha>0$,

$$V^{\alpha}_{\infty} = \frac{1-\beta}{\alpha} W^{\beta}_{\infty} \tag{2.15}$$

where W^{β}_{∞} is the optimal over $\gamma\varepsilon\Gamma_0$ of (2.14). We next compute the transition probabilities for the embedded Markov chain x_k. Let $\xi_{1,k}$, $\xi_{2,k}$ be the total arrivals for queues 1, 2 during the interval $[t^s_k, t^s_{k+1}]$. Note that given $t^s_{k+1} - t^s_k = \delta$, $\xi_{1,k}, \xi_{2,k}$ are independent random variables, with Poisson distributions, with means $\lambda_1\delta$, $\lambda_2\delta$ respectively. Let $N_{1,t}$, $N_{2,t}$ be the counting processes for queues 1,2.

Then

$$\Pr\{N_{1,k+1}-N_{1,k}=\xi_{1,k}\} = \int_0^{\infty} \frac{e^{-\lambda_1\delta}(\lambda_1\delta)^{\xi_{1,k}}}{\xi_{1,k}!}\,\mu e^{-\mu\delta}\,d\delta$$

$$= \left(\frac{\lambda_1}{\lambda_1+\mu}\right)^{\xi_{1,k}} \frac{\mu}{\lambda_1+\mu} \tag{2.16}$$

Similarly

$$\Pr\{N_{2,k+1}-N_{2,k}=\xi_{2,k}\} = \left(\frac{\lambda_2}{\lambda_2+\mu}\right)^{\xi_{2,k}} \frac{\mu}{\lambda_2+k}\ . \tag{2.17}$$

Now if $u_k = 1$

$$\left.\begin{aligned} x_{1,k+1} &= x_{1,k} - 1 + \xi_{1,k} \quad, && \text{if } x_{1,k}\neq 0 \\ x_{1,k+1} &= \xi_{1,k} \quad, && \text{if } x_{1,k} = 0 \\ x_{2,k+1} &= x_{2,k} + \xi_{2,k} && . \end{aligned}\right\} \tag{2.18}$$

Similarly if $u_k = 0$

$$\left.\begin{aligned} x_{1,k+1} &= x_{1,k} + \xi_{1,k} \\ x_{2,k+1} &= x_{2,k} - 1 + \xi_{2,k} \quad, && \text{if } x_{2,k}\neq 0 \\ x_{2,k+1} &= \xi_{2,k} \quad, && \text{if } x_{2,k} = 0 \end{aligned}\right\} \tag{2.19}$$

So the (reduced) embedded Markov chain has state space ZxZ and transition probabilities

$$P_{i_1,i_2,j_1,j_2}(u) := \Pr\{x_{k+1}=(j_1,j_2)\,|\,x_k=(i_1,i_2),u_k = v\}$$

$$= \Pr\{x_{1,k+1}=j_1\,|\,x_{1,k} = i_j,\ u_k = v\}.\Pr\{x_{2,k+1}=j_2\,|\,x_{2,k}=i_2,u_k=v\} \tag{2.20}$$

$$= P^1_{i_1,j_1}(v)\ P^2_{i_2,j_2}(v)$$

From (2.16) - (2.19) there result

$$P^1_{i_1,j_1}(1) = \begin{cases} (\lambda_1/\lambda_1+\mu)^{j_1}(\mu/\lambda_1+\mu) \quad, & \text{if } i_1 = 0,\ j_1\geq 0 \\ (\lambda_1/\lambda_1+\mu)^{j_1+1-i_1}(\mu/\lambda_1+\mu) \quad, & \text{if } i_1\geq 1,\ j_1\geq i_1-1 \\ 0 \quad, & \text{otherwise} \end{cases} \tag{2.21a}$$

$$P^1_{i_1,j_1}(0) = \begin{cases} (\lambda_1/\lambda_1+\mu)^{j_1-i_1}\ (\mu/\lambda_1+\mu) \quad, & \text{if } j_1\geq i_1 \\ 0 \quad, & \text{otherwise} \end{cases} \tag{2.21b}$$

$$P^2_{i_2,j_2}(1) = \begin{cases} (\lambda_2/\lambda_2+\mu)^{j_2-i_2}(\mu/\lambda_2+\mu) \quad , & \text{if } j_2\geq i_2 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{2.21c}$$

$$P^2_{i_2,j_2}(0) = \begin{cases} (\lambda_2/\lambda_2+\mu)^{j_2}(\mu/\lambda_2+\mu) \quad , & \text{if } i_2=0,\ j_2\geq 0 \\ (\lambda_2/\lambda_2+\mu)^{j_2+1-i_2}(\mu/\lambda_2+\mu) \quad , & \text{if } i_2\geq 1,\ j_2\geq i_2-1 \\ 0 \quad , & \text{otherwise} \quad . \end{cases} \tag{2.21d}$$

Then W^β_∞ is the unique solution of

$$W^\beta_\infty(i_1,i_2) = c_1 i_1+c_2 i_2+\beta \min_{v\varepsilon\{0,1\}} (\sum_{j_1,j_2} P_{i_1,i_2,j_1,j_2} W^\beta_\infty(j_1,j_2)) \tag{2.22}$$

Multiplying (2.22) by $\frac{1-\beta}{\alpha}$ we obtain (2.10), and this completes the proof of the theorem.

It is now clear how to obtain the switching curve in ZxZ, once V^α_∞ is known:

$$S = \{(i_1,i_2)\varepsilon ZxZ:$$

$$\sum_{j_1,j_2} P^1_{i_1,j_1}(1)P^2_{i_1,j_2}(1)V^\alpha_\infty(j_1,j_2) = \sum_{j_1,j_2} P^1_{i_1,j_1}(0)P^2_{i_2,j_2}(0)V^\alpha_\infty(j_1,j_2)\} \tag{2.23}$$

From (2.21) easily follows that $(0,0)\varepsilon S$. The optimal stationary policy $\gamma^\alpha_\infty=\{f^\alpha,f^\alpha,\ldots\}$ is determined by the function $f^\alpha:ZxZ\to\{0,1\}$ as follows. If (i_1,i_2) is such that

$$\sum_{j_1,j_2} P^1_{i_1,j_1}(1)P^2_{i_2,j_2}(1)V^\alpha_\infty(j_1,j_2)<\sum_{j_1,j_2} P^1_{i_1,j_1}(0)P^2_{i_2,j_2}(0)V^\alpha_\infty(j_1,j_2) \tag{2.24a}$$

then

$$f^\alpha(i_1,i_2) = 1 \tag{2.24b}$$

and

$$f^\alpha(i_1,i_2) = 0 \tag{2.24c}$$

when the inequality in (2.24) is reversed. From (2.21), (2.24) easily follows that $f^\alpha(0,i_2) = 0$ and $f^\alpha(i_1,0) = 1$. These results agree with intuition.

The result of Theorem 1, suggests the following scheme for the numerical computation of V^α_∞ and f^α.

Let

$$V^\alpha_1(i_1,i_2) = \frac{c_1}{\mu+\alpha}i_1+\frac{c_2}{\mu+\alpha}i_2 \quad , \quad i_1,i_2\geq 0 \tag{2.25}$$

and define V^α_n recursively via

$$V_n^\alpha(i_1,i_2) = \frac{c_1}{\mu+\alpha} i_1 + \frac{c_2}{\mu+\alpha} i_2 + \frac{\mu}{\mu+\alpha} \min_{v\varepsilon\{0,1\}} \{\sum_{j_1,j_2} P^1_{i_1,j_1}(v) P^2_{i_2,j_2}(v) V^\alpha_{n-1}(j_1,j_2)\} \tag{2.26}$$

Let

$$f_n^\alpha : Z\times Z \to \{0,1\}$$

be the function defined by the minimization in (2.26). Then we can show that

$$\left.\begin{aligned} &\lim_{n\to\infty} V_n^\alpha = V_\infty^\alpha \\ &\lim_{n\to\infty} f_n^\alpha = f^\alpha . \end{aligned}\right\} \tag{2.27}$$

3. DIRECT ANALYSIS OF THE BELLMAN EQUATION

First recall from [10] the following result on Bellman equations over an arbitrary Hilbert lattice V:

$$\max_{a\varepsilon\tilde{A}} (L^a u - f^a) = 0 \quad , \quad u\varepsilon V,\ f^a \varepsilon V' \tag{3.1}$$

where L^a is coercive, and $\tilde{A}$ is the action set.

Let

$$K = \{v\varepsilon V : L^a v \leq f^a \quad , \quad \text{all } a\varepsilon\tilde{A}\} . \tag{3.2}$$

Next for arbitrary $a\varepsilon\tilde{A}$, let φ be the solution of the variational inequality

$$\langle L^a\varphi - f^a, v-\varphi\rangle \geq 0 \quad \text{for all } v\varepsilon K, \varphi\varepsilon K. \tag{3.3}$$

Then (see [10]) φ is the maximal element of the set of all subsolutions of (3.1) (which implies that φ is independent of the choice of a from $\tilde{A}$ in (3.2)). Furthermore, under appropriate technical assumptions [10], φ is also the strong solution of (3.1). In view of the result of Theorem 1 in section 2, it is only necessary to consider stationary problems.

For the case of interest here $\tilde{A} = \{1,0\}$, i.e. we have two operators L_1, L_0. Furthermore we have two state variables x_1, x_2 which are integer valued. Let us define

$$f(i_1,i_2) = \frac{c_1}{\mu+\alpha} i_1 + \frac{c_2}{\mu+\alpha} i_2 \ , \ i_1,i_2 \geq 0 . \tag{3.4}$$

Furthermore for a function $v(i_1,i_2)$ let

$$\begin{aligned} [L^0 v](i_1,i_2) &= v(i_1,i_2) - \frac{\mu}{\mu+\alpha} \sum_{j_1,j_2}^{\infty} P^1_{i_1,j_1}(0) P^2_{i_2,j_2}(0) v(j_1,j_2) \\ [L^1 v](i_1,i_2) &= v(i_1,i_2) - \frac{\mu}{\mu+\alpha} \sum_{j_1,j_2}^{\infty} P^1_{i_1,j_1}(1) P^2_{i_2,j_2}(1) v(j_1,j_2) . \end{aligned} \tag{3.5}$$

Then (2.10) can be written

$$\max\,(L^0 v, L^1 v) = f \tag{3.6}$$

The main difficulty arising here is due to the fact that L_0, L_1 are not coercive, but degenerate. Thus, it is necessary to modify the method of [10] to apply to degenerate problems; this is done in [11]; here we present the method for the particular case of two operators.

It is easier to explain the main features of this method by introducing some functional spaces. Because our arguments do not depend on the maximum principle, they are applicable to both, the continuous and the discrete problems.

Let $\{1,2,\ldots,n\} = A \cup B$, where n is the dimension of the spatial variable and $A \cap B \neq \phi$. Let $\Omega = \Omega_A \times \Omega_B \subseteq R^n$, where $\Omega_A \subseteq R^A$ and $\Omega_B \subseteq R^B$ are bounded open sets in the respective spaces. (The case of an unbounded set Ω can be handled similarly, by introducing weighted functional spaces).

Let

$$L_1 \equiv L_{1,1} + L_{1,2},\ L_0 \equiv L_{0,1} + L_{0,2}$$

where $L_{1,1}$ is a first-order hyperbolic (differential or finite difference) operator in the A-variables and $L_{1,2}$ is coercive in the B-variables, while the situation is reversed for L_0.

Now let

$$W_A = H_0^1(\Omega_A),\ W_B = H_0^1(\Omega_B)$$

$$\tilde{D}_A = \{v \varepsilon L^2(\Omega):\ v|<L_{1,1},n>|^{\frac{1}{2}}\ \varepsilon L^2(\partial\Omega_A),\ v = 0 \text{ on } \Gamma_A^-\}$$

where

$$\Gamma_A^- = \{x \varepsilon \partial\Omega_A:\ <L_{1,1},n><0\}.$$

$\tilde{D}_B$ is defined similarly.

(Notation: if $L = \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i} + b_0$, then $<L,n> = \Sigma b_i n_i$; $n(x)$ is the outer normal at point $x \varepsilon \partial\Omega_A$).

Finite differences must be substituted instead of derivatives for the discrete problem. Further we denote by H_A, H_B the spaces $L^2(\Omega_A)$, $L^2(\Omega_B)$ (or $\ell^2(\Omega_A)$, $\ell^2(\Omega_B)$, for finite difference equations). We define the spaces

$$V_1 = \tilde{D}_A \otimes W_B,\quad V_0 = W_A \otimes \tilde{D}_B;\ U_1 = H_A \otimes W_B,\ U_0 = W_A \otimes H_B.$$

(Notation: $\otimes$ signifies tensor product.)

Let the convex set K be defined by

$$K = \{v \varepsilon V_1 \cap V_0:\ L_1 v \leq f_1,\ L_0 v \leq f_0\}.$$

We observe that K is sup-stable, i.e. $u, w \varepsilon K$ implies $u \cup w \varepsilon K$.

Let $\overline{K}_1$ be the closure of K in U_1, and $K_1 = \overline{K}_1 \cap V_1$.

Let u be the solution of the variational inequality

$$<L_{1,1}v + L_{1,2}u - f_1, v - u> - \frac{1}{2}\int_{\Omega} [\nabla . L_{1,1}]\,(v-u)^2 dx - \frac{1}{2}\int_{\Gamma_A^- \times\Omega_B} <L_{1,1}, n_A> v^2 d\sigma \geq 0$$

for all $v \varepsilon K_1$; $u \varepsilon \overline{K}_1$ (3.3)

Then we have the following:

Theorem 2:

There exists a solution u of (3.3) and is equal to the maximal element of $\overline{K}_1$.

Existence follows by a penalization argument as in [12]. The claim that u is the maximal element of $\overline{K}_1$ follows by methods similar to those of [10].

When $f^a \geq 0$, we have the following iterative scheme for computing the solution of (2).

Let u_0^a, a = 1,0 be subsolutions of (1), i.e. $u_0^a \varepsilon V$, $L_a u_0^a \leq f^a$. Suppose that $u_0^a \varepsilon \ell^\infty(\Omega)$. Let u_n^a be defined inductively as the solutions of the variational inequalities

$$<L_a u_n^a - f^a, v - u_n^a> \geq 0 \text{ for all } v \varepsilon \overline{K}_n^a;\ u_n^a \varepsilon K_n^{a'} \qquad (3.4)$$

where $K_n^a = \{v\ V_a : v \geq u_{n-1}^{a+1}\}$,

$\overline{K}_n^a = \{\text{closure of } K_n^a \text{ in } U_a\}$,

$K_n^{a'} = \overline{K}_n^a \cap V_a$

a = 1,0.

The variational inequalities (3.4) have strong solutions if we make the regularity assumption:

(A) $u_n^a \varepsilon V_{a+1}$

Conditions under which the regularity assumption (A) is true are given in [13].

Then, we have the following:

Theorem 3:

$u_n \to u$, solution of (3.1), in weak U_1, and $u \varepsilon L^\infty(\Omega)$ $(u \varepsilon \ell^\infty(\Omega)$, for the finite difference problem).

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