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GROUP INVARIANCE METHODS IN NONLINEAR FILTERING OF DIFFUSION PROCESSES

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Abstract

Given two "nonlinear filtering problems" described by the processes

$$dx(t)^i = f^i(x^i(t))dt+g^i(x^i(t))dw^i(t)$$

$$dy(t)^i = h^i(x^i(t))dt+dv^i(t), \quad i = 1,2,$$

we define a notion of strong equivalence relating the solutions to the corresponding Mortensen-Zakai equations

$$du_i(t,x) = \mathcal{L}^i u_i(t,x)dt + \mathcal{L}_1^i u_i(t,x)dy_t^i \quad , i=1,2,$$

which allows solution of one problem to be obtained easily from solutions of the other. We give a geometric picture of this equivalence as a group of local transformations acting on manifolds of solutions. We then show that by knowing the full invariance group of the time invariant equations

$$du_i(t,x) = \mathcal{L}^i u_i(t,x)dt \quad , i=1,2,$$

we can analyze strong equivalence for the filtering problems. In particular if the two time invariant parabolic operators are in the same orbit of the invariance group we can show strong equivalence for the filtering problems. As a result filtering problems are separated into equivalent classes which correspond to orbits of invariance groups of parabolic operators. As specific example we treat V. Beneš's case establishing from this point of view the necessity of the Riccati equation.

1. Introduction

Very recently new ideas and techniques have been applied to a long standing problem in stochastic systems theory: "the nonlinear filtering problem". The approach taken in these recent studies is markedly different from previous efforts in that innovative and rather unusual (from the point of view of classical probability theory) mathematical tools are brought to bear on this long standing problem. A large portion of this new work is geometrical in nature. Thus Brockett [1]-[2] and Mitter [3]-[4] have emphasized the significance of a certain Lie-algebra of partial differential operators associated with each nonlinear filtering problem, while Marcus et al [5] and Baras and Blankenship [6] have provided explicit examples where these concepts lead to significant developments in the solution of nonlinear filtering problems. In a different direction but one that influences at a fundamental level the geometric constructions, Davis [7][8] and Clark [9] emphasized pathwise solutions of the crucial stochastic partial differential equation which governs the evolution of the conditional statistics. Finally Pardoux [10], Baras and Blankenship [11] and Baras, Mitter and Ocone [12] have analyzed evolution properties of such stochastic p.d.e's and path integral representations of solutions. A good reference to all these developments is the forthcoming proceedings volume of a recent symposium on these topics [13].

Our objective here is to describe a geometric way of characterizing computationally equivalent nonlinear filtering problems. This work is inspired by similar ideas in the theory of ordinary differential equations which go under the names "Similarity Methods" or "Group Invariance Methods" [14]-[15].

We will only briefly discuss the focal points of our current understanding of the nonlinear filtering problem and we will refer the reader to [16], [17] or the references [1]-[13] for details. Thus the "nonlinear filtering problem for diffusion processes" consists of a model for a "signal process" $x(t)$ via a stochastic differential equation

$$dx(t) = f(x(t))dt + g(x(t))dw(t) \tag{1.1}$$

which is assumed to have unique solutions in an appropriate sense (strong or weak, see [17]). In addition we are given "noisy" observations of the process $x(t)$ described by

$$dy(t) = h(x(t))dt+dv(t). \tag{1.2}$$

Here $w(t)$, $v(t)$ are independent standard Wiener processes and h is such that y is a semimartingale. The problem is to compute conditional statistics of functions of the signal process $\phi(x(t))$ at time t given the data observed up to time t, i.e. the σ-algebra

$$\mathcal{F}_t^y = \sigma\{y(s), 0\le s\le t\} \tag{1.3}$$

Clearly the maximum information about conditional statistics is obtained once we find ways to compute the conditional probability density of $x(t)$ given F_t^y. Let us denote this conditional density by $p(t,x)$. Now one of the main points of the new developments has been to emphasize a different function, so called unnormalized conditional density, $u(t,x)$ which produces p after normalization

$$p(t,x) = \frac{u(t,x)}{\int u(t,z)dz} \tag{1.4}$$

The reason for the emphasis put on u is that it satisfies a linear stochastic p.d.e. driven directly by the observations. This is the so called Mortensen-Zakai stochastic p.d.e., which in Itô's form is

$$du(t,x) = \mathcal{L}u(t,x)dt+h^T(x)u(t,x)dy(t) \tag{1.5}$$

Here $\mathcal{L}$ is the adjoint of the infinitesimal generator of the diffusion process $x(\cdot)$ =

$$[\mathcal{L}\phi](x)=\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i}[\sigma_{ij}(x)\phi]-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[f_i(x)\phi(x)] \tag{1.6}$$

which is also called the Fokker-Planck operator associated with $x(\cdot)$. In (1.6) the matrix σ is given by

$$\sigma(x) = g(x)g(x)^T \quad , \tag{1.7}$$

and we shall assume that σ is possitive definite, i.e. the elliptic operator $\mathcal{L}$ is nondegenerate. When applying geometric ideas to (1.5) it is more convenient to consider the Stratonovich version

$$\frac{\partial u(t,x)}{\partial t} = (\mathcal{L}-\frac{1}{2}h(x)^T h(x))u(t,x)+h^T(x)u(t,x)\frac{dy(t)}{dt}\ . \tag{1.8}$$

We shall primarily work with (1.8) in the present paper. Letting

$$A := \mathcal{L} - \frac{1}{2}h^T h$$
$$B_j := \text{Mult. by } h_j \text{ (jth comp. of h)} \tag{1.9}$$

we can rewrite (1.8) as an infinite dimensional bilinear

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equation

$$\frac{du(t)}{dt} = (A+\sum_{j=1}^{p} B_j \dot{y}_j(t))u(t). \tag{1.10}$$

Throughout the paper we shall assume that every equation of the form (1.8) considered has a complete existence and uniqueness theory established on a space X. Furthermore we shall assume that continuous dependence of solutions on $y(\cdot)$ has been established. For results of this type we refer the reader to [6][10][12].

The estimation Lie algebra introduced by Brockett [2] and analyzed in [1]-[6] is the Lie algebra

$$\Lambda(E) = \text{Lie algebra generated by } A \text{ and } B_j,\ j=1,\dots p. \tag{1.11}$$

Again we shall assume that for problems considered the operators A, B_j have a common, dense invariant set of analytic vectors in X [18] and that the mathematical relationship between $\Lambda(E)$ and the existence-uniqueness theory of (1.8) is well understood. For results of this nature we refer to [6][12].

A central problem in the current developments of nonlinear filtering theory is to develop a methodology for recognizing mathematically "equivalent" problems. Equivalence here carries the following meaning: two nonlinear filtering problems should be equivalent when knowing the solution of one, the solution of the other can be obtained by relatively simple additional computations. This problem is of course the reason for the emphasis on geometric methods in recent efforts. Examples discovered by Beneš [19], created certain excitement for the possibility of a complete classification theory. We shall see how transparent Beneš's examples become from the point of view proposed in this paper.

It will be apparent from the present paper that the fundamental concept in this problem of "equivalence" is that of invariance groups of (1.8). To make things precise consider two nonlinear filtering problems (vector)

$$\begin{aligned} dx^i(t) &= f^i(x^i(t)dt + g^i(x^i(t))dw^i(t) \\ dy^i(t) &= h^i(x^i(t))dt + dv^i(t) \ ;\ i=1,2 \end{aligned} \tag{1.12}$$

and the corresponding Mortensen-Zakai equations in Stratonovich form

$$\frac{\partial u_i(t,x)}{\partial t} = (\mathcal{L}^i - \frac{1}{2}\|h^i(x)\|^2)u_i(t,x) + h^{iT}(x)u_i(t,x)y^i(t) \qquad i=1,2 \tag{1.13}$$

<u>Definition</u>: The two nonlinear filtering problems above are <u>strongly equivalent</u> if u_2 can be computed from u_1, and vice versa, via the following types of operations:

<u>Type 1</u>: $(t,x^2) = \alpha(t,x^1)$, where α is a diffeomorphism.

<u>Type 2</u>: $u_2(t,x) = \psi(t,x)u_1(t,x)$, where $\psi(t,x) \geq 0$ and $\psi^{-1}(t,x) \geq 0$.

<u>Type 3</u>: Solving a set of ordinary (finite dimensional) differential equations (i.e. quadrature).

Brockett [2], has analyzed the effects of diffeomorphisms in x-space and he and Mitter [4] the effects of so called "g ge" transformations (a special case of our type 2 operations) on (1.8). Type 3 operations are introduced here for the first time, and will be seen to be the key in linking this problem with mathematical work on group invariance methods in o.d.e. and p.d.e.'s

Our approach starts from the abstract version of (1.13) (i.e. (1.10)):

$$\frac{\partial u_i}{\partial t} = (A^i + \sum_{j=1}^{p} B_j^i \dot{y}_j(t))u_i \quad , \quad i=1,2 \ , \tag{1.14}$$

where A^i, B_j^i are given by (1.9). We are thus dealing with two parabolic equations. We will first examine whether the evolutions of the time invariant parts can be computed from one another. This is a classical problem and the methods of section 3,4 apply. In section 5 we shall give an extension to the full equation (1.14) under certain conditions on B_j^i. We shall then apply this result to the examples studied by Beneš and recover the Riccati equations as a consequence of strong equivalence. Further results and details can be found in [20] which will appear elsewhere.

2. A Motivating Example from Parabolic Equations

The most common starting point in descriptions of group invariance in partial differential equations is the discussion of invariance properties of the heat equation:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}. \tag{2.1}$$

It is well known [14],[21] that (2.1) is invariant under the variable transformation

$$\begin{aligned} x &\mapsto e^s x \\ t &\mapsto e^{2s} t \ . \end{aligned} \tag{2.2}$$

That is to say if $u(t,x)$ is a solution of (2.1), so is $u(e^{2s}t, e^s x)$. Clearly the initial data should be changed appropriately. So if ϕ is the initial data for u, the initial data for the transformed (under 2.2)) solution are $\phi(e^s x)$. This elementary invariance can be written symbolically as

$$e^{tD^2} e^{sxD}\phi = e^{sxD} e^{e^{2s}tD^2}\phi \ . \tag{2.3}$$

Here

$$\begin{aligned} D: &= \frac{\partial}{\partial x} \\ D^2: &= \frac{\partial^2}{\partial x^2} \ . \end{aligned} \tag{2.4}$$

Often in this paper we shall give double meaning to exponentials of partial differential operators. Thus while $\exp(tD^2)$ in (2.3) denotes the semigroups generated by D^2 [22], $\exp(sxD)$ is viewed as an element of the Lie group of transformations generated by xD. It is easy to verify that

$$\phi(e^s x) = [\exp(sxD)\phi](x) \ , \tag{2.5}$$

where we view $\exp(sxD)$ as such a transformation, with parameter s. Now the association

$$(t,s) \mapsto e^{tD^2} e^{sxD} \tag{2.6}$$

defines a two parameter semigroup with product rule

$$(t,s)\cdot(t_1,s_1): = (t_1\exp(-2s) + t, s+s_1) \ , \tag{2.7}$$

because of the invariance (2.3). A one parameter subgroup is

$$\begin{aligned} t &= a(\exp(2cr)-1) \\ s &= -cr \end{aligned} \tag{2.8}$$

where a,c are positive constants and $r>0$ is the group parameter. To this subgroup (2.6) associates the one parameter semigroup of operators

$$\begin{aligned} H(r): &= \exp^{a(\exp(2cr)-1)D^2 - crxD} \\ &= e^{-crxD} e^{a(1-\exp(-2cr))D^2}. \end{aligned} \tag{2.9}$$

It is straightforward to compute the infinitesimal generator of H

$$M\phi: = \lim_{r\to 0} \frac{H(r)\phi - \phi}{r} = 2acD^2\phi - cxD\phi. \tag{2.10}$$

But in view of (2.9) and (2.10) we have the operator identity

$$e^{Mt} = e^{-crtD} e^{a(1-\exp(-2ct))D^2} \ . \tag{2.11}$$

To understand the meaning of (2.12) recall that for appropriate functions ϕ, $\exp(Mt)\phi$ is the solution to the initial value problem

$$\left.\begin{aligned}\frac{\partial w(t,x)}{\partial t} &= [Mw](x) = 2ac\frac{\partial^2}{\partial x^2}w(t,x) - cx\frac{\partial}{\partial x}w(t,x)\\ w(0,x) &= \phi(x)\end{aligned}\right\}\quad (2.12)$$

Then (2.11) suggests the following indirect procedure for solving (2.12):

Step 1: Solve the simpler initial value problem

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2}{\partial x^2}u(t,x)\\ u(0,x) &= \phi(x)\end{aligned}\quad (2.13)$$

Step 2: Change independent variables in u to obtain w via

$$w(t,x)=u(a(1-\exp(-2ct)),\exp(-ct)x). \quad (2.14)$$

Here we have interpreted the exponential in (2.11) as a transformation of variables.

This simple example illustrates the main point of the present paper: knowing that a certain partial differential equation (such as (2.1)) is invariant under a group of local transformations (such as (2.8)) can be used to solve a more difficult equation (such as (2.12)) by first solving the simpler equation (such as (2.1)) and then changing variables.

This idea has been developed by S.I. Rosencrans in [15] [23]. It is appropriate to emphasize at this point that this use of a group of invariance of a certain p.d.e. is not quite traditional. The more traditional use of group invariance is discussed at length in [14] [21], and is to reduce the number of independent variables involved in the p.d.e. Thus the traditional use of group invariance, is just a manifestation and mathematical development of the classical similarity methods in o.d.e.

The point of the simple example above is to illustrate a different use of group invariance which goes roughly as follows: given a parabolic p.d.e.

$$u_t = Lu \quad (2.15)$$

and a group of local transformations that leave the solution set of (2.15) invariant, use this group to solve a "perturbed" parabolic p.d.e.

$$w_t = (L + P)w \quad (2.16)$$

by a process of variable changes and the possible solution of an ordinary (not partial) differential equation. The operator P will be referred to as the "perturbation".

Our contribution in this paper can be viewed as an extension of the results of Rosencrans to stochastic partial differential equations of the type (1.5), that play a fundamental role in nonlinear filtering theory.

3. The Invariance Group of a Linear Parabolic Operator.

Consider the general, linear, nondegenerate elliptic partial differential operator

$$L: = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)\text{id}. \quad (3.1)$$

and assume that the coefficients a_{ij}, b_i, c are smooth enough, so that L generates an analytic semigroup [22], denoted by exp(tL), for at least small $t\geq 0$, on some locally convex space of X of initial functions ϕ and appropriate domain Dom(L).

Let V be the set of solutions to

$$\begin{aligned}\frac{\partial u}{\partial t} &= Lu\\ u(0,x) &= \phi(x)\end{aligned}\quad (3.2)$$

in X, as we vary ϕ. The aim is to find a local Lie transformation group G which transforms every element of V into another element of V. Such a group will be called an invariance group of (3.2) or of L.

This of course is a classical topic of mathematical research initiated by Sophus Lie [24]. Lie considered a system of p.d.e.'s:

$$\mathcal{D}(u) = 0 , \quad (3.3)$$

where the independent variables are $x_1, x_2, \ldots, x_n$ while the dependent variables are $u_1, \ldots, u_p$. A solution of (3.3) is an n-dimensional manifold $u = A(x)$ in $M\times\mathbb{R}^p$, where M is typically $\mathbb{R}^n$ but in general is a manifold. That is a solution is a hypersurface in $M\times\mathbb{R}^p$. A local group of transformations G of (3.3) consists of transformations

$$(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p) \underset{g}{\mapsto} (x_1', x_2', \ldots, x_n', u_1', \ldots, u_p') \quad (3.4)$$

where each of the primed variables can depend on all the unprimed variables. The qualifier local means that the group properties (i.e. multiplication and inversion) hold only for a neighborhood of the identity element.

An invariance group acts on $M\times\mathbb{R}^p$ and permutes the solution manifolds

$$u = A(x) \underset{g}{\mapsto} u' = A'(x') \quad (3.5)$$

Note that G induces a group $\tilde{G}$ acting on the space of functions on M with values in $\mathbb{R}^p$, denoted by $\mathcal{F}(M;\mathbb{R}^p)$. The element $\tilde{g}$ corresponding to a g specified by (3.4) will map the function A into A', i.e.

$$A' = \tilde{g}(A) \quad . \quad (3.6)$$

It is easy to show [23] that G and $\tilde{G}$ are isomorphic as groups. We are interested in groups $\tilde{G}$ acting linearly. For that we need:

Definition: $\tilde{G}$ is linear if there exists a Lie group of transformations $\Sigma: M\to M$ such that for each $\tilde{g}\in\tilde{G}$, there exists a $\sigma\in\Sigma$, a pxp matrix "multiplier" $\nu=\nu(x,\tilde{g})$ and a solution ψ of (3.3) such that

$$\tilde{g}(A)(x) = \nu(x,\tilde{g})\, A(\sigma(x)) + \psi(x). \quad (3.7)$$

The meaning of (3.7) is rather obvious. The way $\tilde{G}$ acts on functions is basically via the "coordinate change" group Σ of M. The main result of Rosencrans [23], concerns the case of a single parabolic equation (3.2), i.e. p = 1 in (3.3).

Theorem 3.1 [23]: Every transformation $\tilde{g}$ in the invariance group $\tilde{G}$ of a linear parabolic equation is of the form

$$u(t,x) \mapsto \nu(p(t,x))u(p(t,x)) + \psi(x) \quad (3.8)$$

where p is a transformation acting on the variables (t,x), ψ a fixed solution of the parabolic equation.

Clearly for linear parabolic equations $\tilde{G}$ is always infinite dimensional since it always includes the infinite dimensional subgroup $\tilde{\mathcal{K}}$ consisting of transformations of the form

$$A \mapsto cA + \phi, \tag{3.9}$$

where $A\in\mathcal{F}(M;\mathbb{R})$, c a scalar $\neq 0$, ϕ a fixed solution of (3.2). Because of (3.8) one says that $\tilde{G}$ acts as a multiplier representation of Σ upon the space of solutions of (3.2).

We consider now one-parameter subgroups of the invariance group G of a given partial differential equation, i.e. p = 1 in (3.3). That is we consider subgroups of G of the form $\{X_s\}$ where s "parametrizes" the elements. According to standard Lie theory the infinitesimal generators of these one-parameter subgroups form the Lie algebra $\Lambda(G)$ of the local Lie group G [14]. We shall, using standard Lie theory notation, denote X_s by exp(sX) where X is the infinitesimal generator of the one parameter group $\{X_s\}$. Thus $X\in\Lambda(G)$. Clearly the elements of $\Lambda[G]$ can be considered as first order partial differential operators on $\mathbb{R}^{n+1}$

$$X = \gamma(x,u)\frac{\partial}{\partial u} - \sum_{i=1}^{n} \beta_i(x,u)\frac{\partial}{\partial x_i} \quad . \tag{3.10}$$

Indeed this follows from an expansion of exp(sX)(x,u) for small s. Now $\{X_s\}$ induces a one-parameter subgroup $\{\tilde{X}_s\}$ in $\tilde{G}$, acting on functions. Let $\tilde{X}$ be the infinitesimal generator of $\{\tilde{X}_s\}$. Given a function $A\in\mathcal{F}(\mathbb{R}^n;\mathbb{R})$ let

$$A(s,x) := \tilde{X}_s(A)(x) \quad . \tag{3.11}$$

If x_i,u are transformed to x_i',u' by a specific one-parameter subgroup exp(sX) of G we can expand

$$\begin{aligned} u' &= A(x) + s\gamma(x,A(x)) + 0(s^2) \\ x_i' &= x_i - s\beta_i(x,A(x)) + 0(s^2) \quad . \end{aligned} \tag{3.12}$$

Thus

$$A(x') = A(x) - s\sum_{i=1}^{n} \beta_i(x,A(x))\frac{\partial A(x)}{\partial x_i} + 0(s^2)$$

or

$$\begin{aligned} \tilde{X}(A)(x) &= \lim_{s\to 0}\frac{A(s,x) - A(0,x)}{s} \\ &= \lim_{s\to 0}\frac{A(s,x) - A(x)}{s} \\ &= \lim_{s\to 0}\frac{A(s,x') - A(x')}{s} \\ &= \lim_{s\to 0}\frac{u' - A(x')}{s} \\ &= \gamma(x,A(x)) + \sum_{i=1}^{n}\beta_i(x,A(x)\frac{\partial A(x)}{\partial x_i} \quad . \end{aligned} \tag{3.13}$$

In view of (3.7) the condition for $\tilde{G}$ to be linear is that [23]

$$\beta_{i,u} = \gamma_{uu} = 0 \tag{3.14}$$

The best way to characterize G (or $\tilde{G}$) is by computing its Lie algebra $\Lambda(G)$ (or $\Lambda(\tilde{G})$). A direct way of doing this is the following. By definition $\tilde{X}\in\Lambda(\tilde{G})$ iff

$$\mathcal{D}(A) = 0 \Rightarrow \mathcal{D}(e^{s\tilde{X}}A) = 0 \text{ for small s} \tag{3.15}$$

When $\mathcal{D}$ is linear this reduces to

$$\mathcal{D}(A) = 0 \Rightarrow \mathcal{D}(\tilde{X}(A)) = 0 \quad , \tag{3.16}$$

since

$$\frac{d}{ds}\mathcal{D}(e^{s\tilde{X}}A) = \mathcal{D}(e^{s\tilde{X}}(A))$$

implies (3.16) if we set s = 0. It is not difficult to show that (3.16) leads to a system of partial differential equations for γ and β_i.

The above method of determining G is different from the Lie-Ovsjannikov method [24] [25]. The latter proceeds along the following lines. For an o.d.e.

$$D(u) = \frac{du}{dx} - f(x,u) = 0 \tag{3.17}$$

we wish to find transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$ that permute solution curves. Suppose $h:\mathbb{R}^2\to\mathbb{R}^2$ is a diffeomorphism

$$(x,u) \mapsto (\phi(x,u),\psi(x,u)) . \tag{3.18}$$

Observe that we can extend h to derivatives along the following lines. If a curve passes through (x,u) with slope du/dx, its image passes through (x,u) with slope du'/dx' where $x' = \phi(x,u)$, $u' = \psi(x,u)$ and

$$\frac{du'}{dx'} = \frac{\psi_x + \psi_u\, du/dx}{\phi_x + \phi_u\, du/dx} \quad .$$

The map

$$\begin{matrix} x \\ u \\ \frac{du}{dx} \end{matrix} \;\overset{h^*}{\mapsto}\; \begin{matrix} x' \\ u' \\ du'/dx' \end{matrix} \tag{3.19}$$

is an extension of h from $\mathbb{R}^2\to\mathbb{R}^2$ to $\mathbb{R}^3\to\mathbb{R}^3$. In $\mathbb{R}^3$ the given o.d.e. determines a surface. The fact is that h permutes solution curves iff h* leaves this surface invariant. The Lie-Ovsjannikov method is then to find all h* which have s = f(x,u) as an invariant manifold. Its great popularity is due to its simplicity. In the case of p.d.e.'s one proceeds exactly the same way by computing derivatives in the transformed variables. The important (albeit simple) fact is that the transformed derivative of order k involves only old derivatives up to order k. A more geometric approach can be taken by introducing jet bundles. One views a smooth function on M as a cross-section of the vector bundle M×$\mathbb{R}$. The k-jet bundle has fiber (over $p\in M$) consisting of local cross sections which agree up to order k at p. The extension h* in (3.19) can be considered as a transformation of cross sections of the k-jet bundle. We shall not consider the Lie-Ovsjannikov method any further in the present paper; we refer the interested reader to [26] for many interesting applications.

Returning back to the determination of $\Lambda(\tilde{G})$ via (3.16) we shall consider only the case when $\tilde{G}$ is linear, since it is the only case of importance to our interests. Then in view of (3.14)

$$\left.\begin{aligned} \beta_i(x,u) &= \beta_i(x) \\ \gamma(x,u) &= u\delta(x) + \phi(x) \end{aligned}\right\} \tag{3.20}$$

for some β_i, δ, ϕ. Let us denote by β the vector $[\beta_1,\beta_2,\ldots,\beta_n]^T$. Then if A is a solution of (3.3), another solution is

$$A(s,x) = \exp(s\tilde{X})\; A,$$

which satisfies

$$\left.\begin{aligned}\frac{\partial A(s,x)}{\partial s} &= \delta(x)A(x)+\sum_{i=1}^{n}\beta_i(x)\frac{\partial A(x)}{\partial x_i}+\phi(x)\\ A(0,x) &= A(x)\end{aligned}\right\} \quad (3.21)$$

in view of (3.13) and due to the linearity assumption (3.20). The crucial point is that (3.21) is a first order hyperbolic p.d.e. and thus it can be solved by the method of characteristics. The latter, very briefly, entails the following. Let $\varepsilon(t)$ be the flow of the vector field $\sum_{i=1}^{n}\beta_i\frac{\partial}{\partial x_i}$, i.e. the solution of the o.d.e.

$$\begin{aligned}\frac{d}{dt}\varepsilon(t,x) &= \beta(\varepsilon(t,x))\\ \varepsilon(0,x) &= x\ .\end{aligned} \quad (3.22)$$

Then from (3.21)

$$\frac{d}{dt}A(s-t,\varepsilon(t,x)) = -\delta(\varepsilon(t,x))A(s-t,\varepsilon(t,x)) + \phi(\varepsilon(t,x))$$

and therefore

$$A(s,x) = \exp\left(\int^{s}\delta(\varepsilon(r,x))dr\right)A(\varepsilon(s,x)) + \int_0^{s}\Phi(t,x)dt \quad (3.23)$$

where

$$\Phi(t,x) = \exp\left(\int_0^{t}\delta(\varepsilon(r,x))dr\right)\phi(\varepsilon(t,x))\, . \quad (3.24)$$

By comparison with (3.7) one can view $\exp\left(\int_0^{s}\delta(\varepsilon(r,x))dr\right)$ as the "multiplier" ν. (3.23) clearly displays the linearity of $\tilde{G}$ near the identity.

The most widely known example, for which $\Lambda(\tilde{G})$ has been computed explicitly is the heat equation (2.1). The infinitesimal generators in this case are six, as below

$$\left.\begin{aligned}&\frac{\partial}{\partial t},\ 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\ \frac{\partial}{\partial x}\\ &1,\ 2t\frac{\partial}{\partial x}+x,\ 4t^2\frac{\partial}{\partial t}+4tx\frac{\partial}{\partial x}+x^2\ .\end{aligned}\right\} \quad (3.25)$$

Let us apply these general results to a linear parabolic equation, like (3.2). From Theorem 3.1, then $\tilde{G}$ is linear. The infinitesimal generators of $\tilde{G}$ are given in view of (3.13) (3.20) (note that $x_1 = t$ here) by

$$Z = \alpha(t,x)\frac{\partial}{\partial t}+\sum_{i=1}^{n}\beta_i(t,x)\frac{\partial}{\partial x_i}+\gamma(t,x)\text{id}. \quad (3.26)$$

for some functions α,β_i,γ of t and x. If u solves (3.2) so does

$$v(s) = \exp(sZ)u\quad,\ \text{for small } s\ . \quad (3.27)$$

However v is also the solution of

$$\left.\begin{aligned}\frac{\partial v}{\partial s} &= \alpha\frac{\partial v}{\partial t}+\sum_{i=1}^{n}\beta_i\frac{\partial v}{\partial x_i}+\gamma v\\ v(0) &= u\ ,\end{aligned}\right\} \quad (3.28)$$

a first order hyperbolic p.d.e. (solvable by the method of characteristics). Clearly since $\frac{\partial}{\partial t} - L$ is linear (3.16) applies and therefore

$$Zu\in V \text{ if } u\in V\ . \quad (3.29)$$

The converse is also true: if (3.29) holds for some first order partial differential operator, Z is a generator of $\tilde{G}$.

Now (3.29) indicates how to compute α,β,γ. Namely

$$\left(\frac{\partial}{\partial t} - L\right)u = 0 \quad (3.30)$$

implies

$$\left(\frac{\partial}{\partial t} - L\right)\left(\alpha u_t + \sum_{i=1}^{n}\beta_i\frac{\partial u}{\partial x_i}+\gamma u\right) = 0\ . \quad (3.31)$$

For $u\in V$ the second reads

$$\alpha_t u_t + \sum_{i=1}^{n}\beta_{i,t}u_{x_i} + \gamma_t u + \alpha u_{tt} + \sum_{i=1}^{n}\beta_i u_{x_i,t} + \gamma u_t = LZu\ ,$$

or

$$\frac{d}{dt}Zu = (LZ-ZL)u\ ,$$

or

$$\frac{d}{dt}Z = [L,Z] \text{ on } V\ . \quad (3.32)$$

In (3.32) [,] denotes commutator and $\frac{d}{dt}Z$ is symbolic of $\alpha_t\frac{\partial}{\partial t}+\sum_{i=1}^{n}\beta_{i,t}\frac{\partial}{dx_i}+\gamma_t$ id. Thus the elements of $\Lambda(\tilde{G})$ in this case satisfy a Lax equation. It is immediate from (3.32) that Z form a Lie algebra. Furthermore it can be shown [23] that α is independent of x, i.e. $\alpha(t,x) = \alpha(t)$ and that every Z satisfies an o.d.e.

$$d_\ell\frac{d^\ell Z}{dt^\ell} + d_{\ell-1}\frac{d^{\ell-1}Z}{dt^{\ell-1}} + \dots + d_0 Z = 0 \quad (3.33)$$

where $\ell \leq \dim \tilde{G}$.

4. Using the Invariance Group of a Parabolic P.D.E. in Solving New P.D.E.'s.

In this section we use the results of the previous section, to generalize the ideas presented via the example of section 2. We follow Rosencrans [15] - [23].

Thus we consider a linear parabolic equation like (3.2) and we assume we know the infinitesimal generators Z of the nontrivial part of $\tilde{G}$. Thus if u solves (3.2), so does $v(s) = \exp(sZ)u$ but with some new initial data, say $R(s)\phi$. That is

$$e^{sZ}e^{tL} = e^{tL}R(s) \text{ on } X. \quad (4.1)$$

Now $R(\cdot)$ has the following properties. First

$$\lim_{s\to 0} R(s)\phi = \phi\ . \quad (4.2)$$

Furthermore from (4.1)

$$\begin{aligned}e^{tL}R(r)R(s)\phi &= e^{rZ}e^{tL}R(s)\phi = e^{rZ}e^{sZ}e^{tL}\phi\\ &= e^{(r+s)Z}e^{tL}\phi = e^{tL}R(r+s)\phi\ .\end{aligned} \quad (4.3)$$

Or

$$R(r)R(s) = R(r+s) \quad \text{for } r,s\geq 0 \quad (4.4)$$

rom (4.3), (4.4), $R(\cdot)$ is a semigroup. Let M be its generator:

$$M\phi = \lim_{s\to 0} \frac{R(s)\phi-\phi}{s}, \ \phi\epsilon \text{Dom}(M) . \qquad (4.5)$$

It is straightforward to compute M, given Z as in (3.26). Thus

$$M\phi = \alpha(0)L\phi + \sum_{i=1}^{n} \beta_i(0,x) \frac{\partial\phi}{\partial x_i} + \gamma(0,x)\phi . \qquad (4.6)$$

Note that M is uniquely determined by the Z used in (4.1). The most important observation of Rosencrans [ ] was that the limit as t→0 of the transformed solution v(s) = exp(sZ)u, call it w, solves the new initial value problem

$$\left.\begin{aligned} \frac{\partial w}{\partial s} &= Mw \\ w(0) &= \phi \end{aligned}\right\} \qquad (4.7)$$

That is

$$e^{sZ}e^{tL} = e^{tL}e^{sM} \text{ on } X$$

or $\qquad (4.8)$

$$Ze^{tL} = e^{tL}M \text{ on Dom}(L) .$$

This leads immediately to the following generalization of discussions in section 2:

To solve the initial value problem

$$\left.\begin{aligned} \frac{\partial w}{\partial s} &= Mw \\ w(0) &= \phi \end{aligned}\right\} \qquad (4.9)$$

where

$$M = \alpha(0)L + \sum_{i=1}^{n} \beta_i(0,x)\frac{\partial}{\partial x_i} + \gamma(0,x) \text{ id} \qquad (4.10)$$

follow the steps given below.

Step 1: Solve $u_t = Lu$, $u(0) = \phi$.

Step 2: Find generator Z of $\tilde{G}$ corresponding to M and solve

$$\left.\begin{aligned} \frac{\partial v}{\partial s} &= \alpha(t) \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \beta_i(t,x) \frac{\partial v}{\partial x_i} + \gamma(t,x)v \\ v(0) &= u \end{aligned}\right\} \qquad (4.11)$$

via the method of characteristics. Note this step requires the solution of ordinary differential equations only.

Step 3: Set t=0 to v(s,t,x).

This procedure allows easy computation of the solution to the "perturbed" problem (4.10) if we know the solution to the "unperturbed" problem (3.2). The "perturbation" which is of degree ≤ 1st, is given by the part of M:

$$P = \sum_{i=1}^{n} \beta_i(0,x) \frac{\partial}{\partial x_i} + \gamma(0,x). \text{ id}. \qquad (4.12)$$

We shall denote by $\Lambda(P)$ the set of all perturbations like (4.12), that permit solutions of $u_t = (L+P)u$ to be computed from solutions of $u_t = Lu$, by integrating only an additional ordinary differential equation. We would like to show that $\Lambda(P)$ is a Lie algebra strongly related to the Lie algebra $\Lambda(\tilde{G})$ of the invariance group of L.

Definition: The Lie algebra $\Lambda(P)$ will be called the perturbation algebra of the elliptic operator L.

To see the relation between $\Lambda(\tilde{G})$ and $\Lambda(P)$, observe first that each generator Z in $\Lambda(\tilde{G})$ uniquely specifies an M, via (3.26), (4.6). Conversely suppose M is given. From the Lax equation (3.32) we find that

$$\left.\frac{dZ}{dt}\right|_{t=0} = [L,Z]\Big|_{t=0} = [L,M]$$

$$= \alpha_t(0)L + \sum_{i=1}^{n} \beta_{i,t}(0,x)\frac{\partial}{\partial x_i} + \gamma_t(0,x)\text{id} . \qquad (4.13)$$

Note that the right hand side of (4.13) is another perturbed operator M'. Thus given an M, by repeated bracketting with L all initial derivatives of Z can be obtained. Since from (3.33) Z satisfies a linear ordinary differential equation, Z can be determined from M. So there exists a 1-1 correspondence between $\Lambda(\tilde{G})$ and the set of perturbed operators M, which we denote by $\Lambda(M)$. It is easy to see that $\Lambda(M)$ is a Lie algebra isomorphic to $\Lambda(\tilde{G})$. Indeed let Z_i correspond to M_i, i=1,2. Then from (4.8) we have

$$e^{tL}[M_1,M_2]\phi = e^{tL}M_1M_2\phi - e^{tL}M_2M_1\phi$$

$$= Z_1e^{tL}M_2\phi - Z_2e^{tL}M_1\phi = Z_1Z_2e^{tL}\phi$$

$$- Z_2Z_1e^{tL}\phi = [Z_1,Z_2]\, e^{tL}\phi . \qquad (4.14)$$

This establishes the claim. Since each perturbation P is obtained from an M by omitting the component of M that involves the unperturbed operator L, it is clear that $\Lambda(P)$ is a Lie subalgebra of $\Lambda(M)$. Moreover the dimension of $\Lambda(P)$ is one less than that of $\Lambda(M)$. In view of the isomorphism of $\Lambda(M)$ and $\Lambda(\tilde{G})$ we have established [15]:

Theorem 4.1: The perturbation algebra $\Lambda(P)$ of an elliptic operator L, is isomorphic to a Lie subalgebra of $\Lambda(\tilde{G})$ (i.e. of the Lie algebra of the invariance group of L). Moreover $\dim(\Lambda(P)) = \dim(\Lambda(\tilde{G}))-1$.

One significant question is: can we find the perturbation algebra $\Lambda(P)$ without first computing $\Lambda(\tilde{G})$, the invariance Lie algebra? The answer is affirmative and is given by the following result [15].

Theorem 4.2: Assume L has analytic coefficients. An operator P_0 of order one or less (i.e. of the form (4.12)) is in the perturbation algebra $\Lambda(P)$ of L iff there exist a sequence of scalars $\lambda_1, \lambda_2, \ldots$ and a sequence of operators $P_1, P_2, \ldots$ of order less than or equal to one such that

$$[L,P_n] = \lambda_n L + P_{n+1} \quad , n\geq 0$$

and $\Sigma\lambda_k t^k/k!$, $\Sigma P_k t^k/k!$ converge at least for small t.

It is an easy application of this result to compute the perturbation algebra of the heat equation in one dimension or equivalently of $L = \frac{\partial^2}{\partial x^2}$. It turns out that $\Lambda(P)$ is 5-dimensional and spanned by

$$\Lambda(P) = \text{Span}(1,x,x^2, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}). \qquad (4.15)$$

So the general perturbation for the heat equation looks like

$$P = (ax+b) \frac{\partial}{\partial x} + (cx^2 + dx + e)\text{id} \qquad (4.16)$$

where a,b,c,d,e are arbitrary constants. Note that the invariance group of the heat equation is 6-dimensional (3.25). It is straightforward to rework the example of section 2, along the lines suggested here.

The implications of these results are rather significant. Indeed consider the class of linear parabolic equations $u_t = Lu$, where L is of the form (3.1). We can define an equivalence relationship on this class by : "L_1 is equivalent to L_2 if $L_2=L_1+P$ where P is an element of the perturbation algebra $\Lambda^1(P)$ of L_1". Thus elliptic operators of the form (3.1), or equivalently linear parabolic equations are divided into equivalent classes (orbits); within each class (orbit) $\{L(k)\}$ (k indexes elements in the class) solutions to the initial value problem $u(k)_t=L(k)u(k)$ with fixed data ϕ (independent of k) can be obtained by quadrature (i.e. an o.d.e. integration) from any one solution $u(k_0)$.

We close this section by a list of perturbation algebras for certain L, from [15].

Elliptic operator L	Generators of perturbation algebra $\Lambda(P)$
D^2	$1,x,x^2,D,xD$
xD^2	$1,x,xD$
x^2D^2	$x\log xD, xD, \log x, (\log x)^2, 1$
x^3D^2	$1,x^{-1},xD$
e^xD^2	$1,e^{-x},D$

Table 4.1. Examples of perturbation algebras.

5. Sufficient Conditions for Strong Equivalence and Applications.

We return now to the problem posed in section 1. Namely to discover conditions that imply strong equivalence of two nonlinear filtering problems. Our main result is:

Theorem 5.1: Given two nonlinear filtering problems (see (1.12)), such that the corresponding Mortensen-Zakai equations (see (1.13)) have unique solutions, continuously dependent on $y(\cdot)$. Assume that using operations of type 1 and 2 (see definition in section 1) these stochastic p.d.e. can be transformed in bilinear form

$$\frac{\partial u_i}{\partial t} = (A^i + \sum_{j=1}^{p} B_j^i \xi_j^i(t))u_i \quad i=1,2$$

such that:

(i) A^i, i=1,2, are nondegenerate elliptic, belonging to the same equivalence class (see end of section 4)

(ii) B_j^i, j=1,...p, i=1,2 belong to the perturbation algebra $\Lambda(P)$ of (i).

Then the two filtering problems are strongly equivalent.

Proof: Only a sketch will be given here. The proof is rather lengthy and can be found in [20]. One first establishes that is enough to show computability of solutions for piecewise constant ξ, from one another, by the additional computation of solutions of an o.d.e. For piecewise constant ξ the solution to any one of the p.d.e.'s in bilinear form is given by

$$u_i=e^{(A^i+B^i_{j_m}\xi^i_{j_m})(t_m-t_{m-1})}.e^{(A^i+B^i_{j_{m-1}}\xi^i_{j_{m-1}})(t_{m-1}-t_{m-2})} \ldots e^{(A^i+B^i_{j_1})t_1}\phi \text{ ; } i=1,2 \quad . \tag{5.1}$$

Since A^1, A^2 belong to the same equivalence class there exist $Z^{12}\epsilon\Lambda(G)$, (where $\Lambda(G)$ is the Lie algebra of the invariance group for the class) and $P^{12}\epsilon\Lambda(P)$ (where $\Lambda(P)$ is the perturbation algebra of the class) such that (see (4.8)):

$$\begin{aligned} A^2 &= A^1 + P^{12} \\ e^{sZ^{12}}e^{tA^1} &= e^{tA^1}e^{sA^2} \quad , t,s\geq 0 \ . \end{aligned} \tag{5.2}$$

That is consider A^2 as a "perturbation" of A^1. We know by now what (5.2) means: to compute the semigroup generated by A^2, we first compute the semigroup generated by A^1, we then solve the o.d.e. associated with the characteristics of the hyperbolic p.d.e.

$$\frac{\partial v}{\partial s} = Z^{12}v \tag{5.3}$$

and we have:

$$e^{sA_2} = [e^{sZ^{12}}e^{tA^1}]\Big|_{t=0} \quad . \tag{5.4}$$

More generally since $A^1 + B^1_j$, $A^2 + B^2_k$ belong to the same class there exist $Z^{12}_{jk}\epsilon\Lambda(G)$, $P^{12}_{jk}\epsilon\Lambda(P)$ such that

$$\begin{aligned} A^2 + B^2_k &= A^1 + B^1_j + P^{12}_{jk} \\ e^{sZ^{12}_{jk}}e^{t(A^1+B^1_j)} &= e^{t(A^1+B^1_j)}e^{s(A^2+B^2_k)} \ . \end{aligned} \tag{5.5}$$

It is now apparent that if we know (5.1) explicitly for i=1, we obtain u_2 from (5.5) with the only additional computations being the integration of the o.d.e.'s associated with the characteristics of the hyperbolic p.d.e.'s

$$\frac{\partial v}{\partial s} = Z^{12}_{jk}v \ , \ k,j = 1,\ldots p \quad . \tag{5.6}$$

This completes the proof.

Let us apply this result to the Beneš case. We consider the linear filtering problem (scalar x,y)

$$\left.\begin{aligned} dx(t) &= dw(t) \\ dy(t) &= x(t)dt+dv(t) \end{aligned}\right\} \tag{5.7}$$

and the nonlinear filtering problem (scalar x,y)

$$\left.\begin{aligned} dx(t) &= f(x(t))dt + dw(t) \\ dy(t) &= x(t)dt + dv(t) \ . \end{aligned}\right\} \tag{5.8}$$

The corresponding Mortensen-Zakai equations in Stratonovich form are: for the linear

$$\frac{\partial u_1(t,x)}{\partial t} = \frac{1}{2}(\frac{\partial^2}{\partial x^2}-x^2)u_1(t,x)+x\dot{y}(t)u_1(t,x) \ ; \tag{5.9}$$

for the nonlinear

$$\frac{\partial u_2}{\partial t} = \frac{1}{2}(\frac{\partial^2}{\partial x^2}-x^2)u_2(t,x)-\frac{\partial}{\partial x}(fu_2) + + x\dot{y}(t)u_2(t,x) \ . \tag{5.10}$$

We wish to show that (5.7)(5.8) are strongly equivalent only if f (the drift) is a global solution of the Riccati equation

$$f_x + f^2 = ax^2 + bx + c \quad . \tag{5.11}$$

First let us apply to (5.8)(5.10) an operation of type 2. That is let (defines v_2)

$$u_2(t,x) = v_2(t,x)\exp(\int_0^x f(u)du) \quad . \tag{5.12}$$

Then the new function v_2 satisfies

$$\frac{\partial v_2(t,x)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2 - V(x)\right)v_2(t,x) + x\dot{y}(t)v_2(t,x) \quad , \qquad (5.13)$$

where

$$V(x) = f_x + f^2 \quad . \qquad (5.14)$$

Existence, uniqueness and continuous dependence on $y(\cdot)$ for (5.9)(5.10) have been established in [12] using classical p.d.e. results. We apply the theorem to (5.9)(5.13). So

$$A^1 = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)$$
$$A^2 = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2 - V\right) \qquad (5.15)$$

while

$$B^1 = B^2 = \text{Mult. by } x \quad . \qquad (5.16)$$

From the results of section 4, the only possible equivalence class is that of the heat equation. Clearly from (4.16) or the table 4.1 $A^1, B^1, B^2 \epsilon \Lambda(P)$ for this class. For $A^2 \epsilon \Lambda(P)$ it is necessary that V be quadratic, which is the same as f satisfying the Riccati equation (5.11), in view of (5.14).

Recall that the solution of (5.9) is

$$u_1(t,x) = \exp\left[-\frac{(x-\mu(t))^2}{2\sigma(t)}\right] \qquad (5.17)$$

where

$$d\mu(t) = \sigma(t)(dy(t) - \mu(t)dt); \mu(0) = \xi$$
$$d\sigma(t) = 1 - \sigma^2(t) \;;\; \sigma(0) = 0 \qquad (5.18)$$

Beneš [19], using a path integral computation showed that the solution of (5.13), when (5.11) is satisfied is given by

$$v_2(t,x) = \exp\left[-\frac{(x-\mu(t))^2}{2\sigma(t)}\right] \qquad (5.19)$$

where

$$d\mu(t) = -(a+1)\sigma(t)\mu(t)dt - \frac{1}{2}\sigma(t)b\,dt + \sigma(t)dy(t)$$
$$d\sigma(t) = 1 - (a+1)\sigma^2(t) \quad . \qquad (5.20)$$

This result has since been established by more straightforward (and short) methods in [12]. As Beneš [19] remarks there were strong hints that his class was essentially equivalent to a linear problem; compare (5.17)(5.19). Indeed as Mitter and Ocone [27] observed the estimation Lie algebras for the two problems are isomorphic; a fact that enabled Ocone [28] to show using a Wei-Norman type construction that the ways of solving (5.9)(5.10) are identical. What we have shown here is a converse, from the point of view that strong equivalance of (5.7)(5.8) implies the Riccati equation. We also maintain that knowledge of group invariance theory makes the result immediate at the level of comparing (5.9) with (5.13).

Other examples can be found in [20], where a different use of group invariance methods is also developed.

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