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ON THE POINT PROCESS DISORDER PROBLEM AND ITS APPLICATIONS TO URBAN TRAFFIC ESTIMATION

by

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Abstract

We consider here the point process disorder problem in its generality and we develop the maximum likelihood and MAP estimates of the disorder time T. Applications of the results to the platoon determination problem in urban traffic show that both estimators outperform previous solutions to the problem.

1. Introduction

In a series of papers [1, 2] we have demonstrated the applicability of point process techniques in urban traffic estimation problems. The present paper is a continuation of [2] where a simple stochastic model was proposed for the headway process as observed at a traffic detector and then subsequently utilized to derive estimates of platoon "passage" time and platoon "length". It was recognized in [2] that the appropriate starting point for such problems is a complete statistical characterization of the headway (or interarrival time) process. The simple model proposed there states that the first order density for headways is:

$$p(h) = \psi\, p_0(h) + (1-\psi)\, p_1(h) \tag{1.1}$$

where $p_0(h)$ = following headway probability density function = lognormal density

$$= \begin{cases} \dfrac{1}{\sigma h \sqrt{2\pi}} \exp\left[\dfrac{-(\ell n\, h - \mu)^2}{2\sigma^2}\right] & , h \geq 0 \\ 0 & , h < 0 \end{cases} \tag{1.2}$$

$p_1(h)$ = nonfollowing headway probability density function = displaced negative exponential density

$$= \begin{cases} \lambda \exp(-(h-\tau)\lambda) & , h \geq \tau \\ 0 & , h < \tau \end{cases} \tag{1.3}$$

ψ = a variable denoting the degree of interaction between following and nonfollowing headway processes.

Furthermore, to a good approximation consecutive headways are independent. To analyze the platoon passage time (or platoon size) problem in [2] ψ was modelled as a switching variable assuming the value 1 when a platoon passes over the detector and the value 0 when freely flowing traffic passes over the detector. This lead to the formulation of the problem of estimating the passage time of a platoon over a detector as a point process disorder problem [2]. The general point process disorder problem is: Given a point process with counting process N_t, which is characterized by a rate process λ_t^0 up to a random time T and by a rate process λ_t^1 afterwards, estimate the switching time T, from the observed process N_t. We are interested in the case where T coincides with one of the occurence times (detector activation times in the traffic context) T_i. Furthermore due to the possibility of feedback we do not assume that the events $\{T=T_i\}$ and N_t are independent. This problem has been recently solved in the filtering context (i. e. under the constraint to utilize only the past of N_t for estimation) by Wan and Davis [3]. Letting

$$x_t = I_{\{t \geq T\}} \tag{1.4}$$

it can be shown that the dynamics are given by [2, 3]

$$dx_t = (1-x_t)\, q_t \lambda_t^0\, dt + dv_t \tag{1.5}$$

and the observations by

$$dN_t = ((1-x_t)\lambda_t^0 + x_t \lambda_t^1)\, dt + dw_t \tag{1.6}$$

where

$$q_t = \sum_i q_i\, I_{\{T_{i-1} \leq t < T_i\}} ,$$
$$q_i = \frac{p_i}{\sum_{k \geq i} p_k}, \quad p_i = \Pr\{T=T_i\} \tag{1.7}$$

where p_i, q_i are allowed to be functions of t, $T_1, T_2, \ldots, T_{i-1}$, and w_t, v_t are martingales with respect to $\mathcal{B}_t = \sigma\{x_s, N_s, s \leq t\}$. Noting that

$$\Pr\{T \leq t \mid \mathfrak{F}_t\} = E\{x_t \mid \mathfrak{F}_t\} = \hat{x}_t \tag{1.8}$$

where $\mathfrak{F}_t = \sigma\{N_s, s \leq t\}$, the minimum variance filtered estimate of T (in the sense of computing the conditional distribution (1.8)) was found in [2, 3] to satisfy

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$$d\hat{x}_t = -(\lambda_t^1 - \lambda_t^0)\hat{x}_t(1-\hat{x}_t)\,dt + \frac{(\lambda_t^1-\lambda_t^0)\hat{x}_t(1-\hat{x}_t) + q_t\lambda_t^0(1-\hat{x}_t)}{(\lambda_t^1-\lambda_t^0)\hat{x}_t + \lambda_t^0}\,dN_t \qquad (1.9)$$

$$\hat{x}_0 = E[x_o]$$

which for the traffic problem described above was solved explicitly in [2]. Two scalar estimates of T and of the platoon size were developed and evaluated in [2]. One which performed reasonably well was the <u>maximum jump estimator</u>

$$\hat{T}_{MJ} = T_{j^*} \quad \text{where } j^* = \arg\{\max_{1<j\le N}(\hat{x}_{T_j} - \hat{x}_{T_{j-1}})\} \qquad (1.10)$$

and N is the maximum number of vehicles which might cross the detector within a cycle of the signal.

In this paper, we first solve for the maximum likelihood estimate of the switching time T given an upper bound N on the number of occurences (vehicles which might cross the detector in one cycle) on which to base our estimate. Next we present the solution to the smoothing problem for T, in the sense of computing the conditional distribution

$$\Pr\{T \le t|\mathfrak{F}_N\} = E\{x_t|\mathfrak{F}_N\} = \hat{x}_{t|N} \qquad (1.11)$$

where $\mathfrak{F}_N = \sigma\{N_s, s \le T_N\}$. This result is then used to derive the maximum aposteriori (MAP) estimator for T. All results are recursive on the length of data used. Finally the theory is applied to the platoon determination problem and the resulting estimators are evaluated against our previous solution (i.e. the maximum jump estimator).

2. Maximum Likelihood Estimator

The maximum likelihood (ML) estimator derived here applies to a memory 1 point process. However it should be clear from the derivation that the general result for a finite memory (say ℓ) point process can be derived mutatis-mutandis and its form is going to be very similar to the result reported here.

Since we have a memory 1 point process the interarrival times are independent and distributed with density p_0 when the rate is λ_0 and density p_1 when the rate is λ_1. Therefore the ML estimate is simply given by

$$\hat{T}_{ML}(n) = T_{j^*_{(n)}} \quad \text{where } j^*_{(n)} = \arg\{\max_{1\le j\le n} p_{T_1,T_2,\ldots,T_n}(t_1,t_2,\ldots,t_n|T=T_j)\} \qquad (2.1)$$

where n is the length of the data record used (i.e. the n arrival (activation) times). Then in view of our assumptions

$$j^*_{(n)} = \arg\{\max_{1\le j\le n} \prod_{i=1}^{j} p_0(h_i) \prod_{\ell=j+1}^{n} p_1(h_\ell)\} \qquad (2.2)$$

since $p_{T_1,T_2,\ldots T_n}(t_1,t_2,\ldots,t_n|T=T_j) = p_{H_1,H_2,\ldots,H_n}(h_1,\ldots,h_n|T_j)$ where H_i are the interarrival times $T_{i+1}=T_i+H_{i+1}$ $i=1,2,\ldots,$ $T_1=0$. We now give a recursive computation of the ML estimator as the data record length increases.

<u>Theorem 2.1:</u> Under the above hypotheses (i.e. independent interarrival times and $T=T_i$ for some i), $\hat{T}_{ML}(n)$ is obtained via

$$T_{ML}(n) = T_{j^*(n)} \quad \text{where}$$

$$j^*(n+1) = \begin{cases} j^*(n) & \text{if } \ell n\, p_1(h_{n+1}) > \ell n\, p_0(h_{n+1}) \text{ or} \quad (2.3) \\ & \text{if } j^*(n)<n, \ell n\, p_0(h_{n+1}) \ge \ell n\, p_1(h_{n+1}) \text{ and} \\ & V(j^*(n),n+1) > V(n+1,n+1) \quad (2.4) \\ n+1 & \text{if } \ell n\, p_0(h_{n+1}) \ge \ell n\, p_1(h_{n+1}) \\ & \text{and } j^*(n)=n \text{ or} \quad (2.5) \\ & \text{if } j^*(n)<n, \ell n\, p_0(h_{n+1}) \ge \ell n\, p_1(h_{n+1}) \text{ and} \\ & V(j^*(n),n+1) \le V(n+1,n+1) \quad (2.6) \end{cases}$$

where

$$V(j,n) = \sum_{i=1}^{j} \ell n\, p_0(h_i) + \sum_{i=j+1}^{n} \ell n\, p_1(h_i) \qquad (2.7)$$

<u>Proof:</u> Note that V(j,n) is just the natural logarithm of the conditional density (i.e. the right hand side of (2.2)). For (2.3) observe that

$$V(j^*(n),n) + \ell n\, p_1(h_{n+1}) = \sum_{i=1}^{j^*(n)} \ell n\, p_0(h_i) + \sum_{\ell=j^*(n)+1}^{n+1} \ell n\, p_1(h_\ell)$$

$$\ge V(k,n) + \ell n\, p_1(h_{n+1}), \quad \forall k \le n$$

$$> V(k,n) + \ell n\, p_0(h_{n+1}), \quad \forall k \le n$$

Therefore

$$V(j^*(n),n+1) \ge V(k,n+1), \quad \forall k \le n$$

$$\text{and} \quad > V(n+1,n+1)$$

which proves (2.3). Now suppose $j^*(n)=n$. Then if $\ell n\, p_0(h_{n+1}) \ge \ell n\, p_1(h_{n+1})$ we have

$$V(j^*(n),n) + \ell n p_0(h_{n+1}) = \sum_{i=1}^{j^*(n)+1} \ell n\, p_0(h_i)$$

$$\geq V(k,n) + \ell n p_0(h_{n+1}), \quad \forall\, k \leq n$$

$$\geq V(k,n) + \ell n p_1(h_{n+1}), \quad \forall\, k \leq n$$

from which (2.5) follows.
Next suppose $j^*(n) < n$. Then if the r.h.s. of (2.4) is assumed we have

$$V(j^*(n),n) + \ell n p_1(h_{n+1}) = \sum_{i=1}^{j^*(n)} \ell n p_0(h_i) + \sum_{\ell=j^*(n)+1}^{n+1} \ell n p_1(h_\ell)$$

$$\geq V(k,n) + \ell n\, p_1(h_{n+1}), \quad \forall\, k \leq n$$

$$> V(k,n) + \ell n p_0(h_{n+1}), \quad \forall\, k \leq n$$

and the result follows. Similarly one obtains (2.6).

It is clear that the implementation of the ML estimator presents no problem once the probability densities p_o, p_1 are specified. The initial conditions are $\hat{T}_{ML}(1) = T_1$ with $V(1,1) = \ell n p_0(h_1)$. Then $\hat{T}_{ML}(n)$ is computed from (2.3) - (2.6) while updating $V(j^*(n),n)$ using (2.7) also. Furthermore the running sum $V(n,n)$ needs to be updated. An explicit example relevant to urban traffic is given in the last section of the paper.

If we relax the memory 1 hypothesis and instead assume that the interarrival times $\{H_i\}$ are dependent with memory ℓ when the point process is represented by rate λ_0 and that they are independent when the rate is λ_1. Then

$$V'(j,n) = \ell n p_{0\, H_1,\ldots,H_j}(h_1,\ldots,h_j) + \sum_{i=j+1}^{n} \ell n\, p_1(h_i). \qquad (2.8)$$

A recursion on the ML estimator can be derived again but it is more involved.

3. Maximum Aposteriori Estimator

In this section we derive the maximum aposteriori estimate of the switching time given n observations:

$$\hat{T}_{MAP}(n) = T_{j^*(n)} \quad \text{where}$$

$$j^*(n) = \arg\{\max_{1 \leq j \leq n} \Pr\{T = T_j \mid \mathfrak{F}_{T_n}\}\} \qquad (3.1)$$

To solve this problem we first solve for the smoothed estimate of the switching time T, which has the representation [4, (3.2)] given the model (1.5) - (1.7),

$$\hat{x}_{t|T} = \hat{x}_{t|t} + \int_t^{\tau} \frac{1}{\hat{\lambda}_s} E^{\mathfrak{F}_s}\{(x_t - \hat{x}_{t|t})(\lambda_s - \hat{\lambda}_s)\}\, d\nu_s \qquad (3.2)$$

where

$$\lambda_t = (1-x_t)\lambda_t^0 + x_t\lambda_t^1, \quad \hat{\lambda}_t = E^{\mathfrak{F}_t}\{\lambda_t\}$$

and ν_t is the innovations process

$$\nu_t = N_t - \int_0^t \hat{\lambda}_s\, ds \qquad (3.3)$$

But

$$\lambda_s - \hat{\lambda}_s = (\lambda_s^1 - \lambda_s^0)(x_s - \hat{x}_{s|s})$$

and since $\hat{x}_{t|t}$ is $\mathfrak{F}_t$ measurable, while $\mathfrak{F}_t \subseteq \mathfrak{F}_s$ for $t \leq s$ (3.2) becomes

$$\hat{x}_{t|\tau} = \hat{x}_{t|t} + \int_t^{\tau} \left(\frac{\lambda_s^1 - \lambda_s^0}{\hat{\lambda}_s}\right) E^{\mathfrak{F}_s}\{x_t(x_s - \hat{x}_{s|s})\}\, d\nu_s$$

$$= \hat{x}_{t|t} + \int_t^{\tau} \left(\frac{\lambda_s^1 - \lambda_s^0}{\hat{\lambda}_s}\right)\{E^{\mathfrak{F}_s}(x_t x_s) - \hat{x}_{s|s}\hat{x}_{t|s}\}\, d\nu_s \qquad (3.4)$$

By definition (recall (1.4)) and since $t \leq s$

$$x_t x_s = \begin{cases} 0 & \text{if } t < T \\ 1 & \text{if } T \leq t \end{cases}$$

so that

$$E^{\mathfrak{F}_s}\{x_t x_s\} = \hat{x}_{t|s} \qquad (3.5)$$

and therefore (3.2) becomes

$$\hat{x}_{t|\tau} = \hat{x}_{t|t} + \int_t^{\tau} \left(\frac{\lambda_s^1 - \lambda_s^0}{\hat{\lambda}_s}\right) \hat{x}_{t|s}(1 - \hat{x}_{s|s})\, d\nu_s. \qquad (3.6)$$

The first step in reducing (3.6) is to use the nature of $d\nu_t = dN_t - \hat{\lambda}_s ds$ (from (3.3)), namely that (3.6) has a continuous part and a jump part. Considering t fixed in (3.6) and letting $T_j \leq t < \tau < T_{j+1}$ we need concern ourselves only with the continuous part in (3.6).
Thus if

$$y_\tau = \hat{x}^c_{t|\tau} = \text{continuous part of } \hat{x}_{t|\tau}$$

$$y_\tau = \hat{x}_{t|t} - \int_t^{\tau} (\lambda_s^1 - \lambda_s^0)\, y_s\,(1 - \hat{x}_{s|s})\, ds \qquad (3.7)$$

for $T_j \leq t < \tau < T_{j+1}$, j arbitrary.

or

$$\frac{dy_\tau}{y_\tau} = -(\lambda_\tau^1 - \lambda_\tau^0)(1 - \hat{x}_{\tau|\tau})\, d\tau$$

or

$$y_\tau = y_t \exp\left[-\int_t^{\tau} (\lambda_s^1 - \lambda_s^0)(1 - \hat{x}_{s|s})\, ds\right],$$

and therefore

$$\hat{x}^{c}_{t|\tau} = \hat{x}_{t|t} \exp[-\int_t^\tau (\lambda_s^1 - \lambda_s^0)(1-\hat{x}_{s|s})\, ds] \quad (3.8)$$

for $T_j \le t < \tau < T_{j+1}$, j arbitrary.

Now at $\tau = T_{j+1}$ we have to calculate the contribution from the jump part in (3.6), which leads to (where t_- denotes left limit)

$$\hat{x}_{t|T_{j+1}} = \hat{x}_{t|T_{j+1}^-} + \hat{x}_{t|T_{j+1}^-} \frac{(\lambda^1_{T_{j+1}^-} - \lambda^0_{T_{j+1}^-})(1-\hat{x}_{T_{j+1}^-|T_{j+1}^-})}{(\lambda^1_{T_{j+1}^-} - \lambda^0_{T_{j+1}^-})\hat{x}_{T_{j+1}^-|T_{j+1}^-} + \lambda^0_{T_{j+1}^-}} . \quad (3.9)$$

Then (3.8) and (3.9) give

$$\hat{x}_{t|T_{j+1}} = \hat{x}_{t|T_{j+1}^-} \frac{\lambda^1_{T_{j+1}^-}}{\hat{\lambda}_{T_{j+1}^-}} = \quad (3.10)$$

$$= \frac{\lambda^1_{T_{j+1}^-}}{\hat{\lambda}_{T_{j+1}^-}} \hat{x}_{t|t} \exp\left[-\int_t^{T_{j+1}^-} (\lambda_s^1 - \lambda_s^0)(1-\hat{x}_{s|s}) ds\right]$$

for $T_j \le t < T_{j+1}$, j arbitrary.

To proceed let

$$\gamma(s) = (\lambda_s^1 - \lambda_s^0)(1-\hat{x}_{s|s}) . \quad (3.11)$$

Then by (3.6) and (3.10)

$$\hat{x}_{t|\tau} = \hat{x}_{t|t} - \int_t^{T_{j+1}^-} \gamma(s)\hat{x}_{t|s}\, ds + \hat{x}_{T_j|T_{j+1}^-}\left(\frac{\lambda^1_{T_{j+1}^-}}{\hat{\lambda}_{T_{j+1}^-}} - 1\right) - \int_{T_{j+1}}^{\tau} \gamma(s)\hat{x}_{t|s}\, ds$$

$$= \hat{x}_{t|T_{j+1}} - \int_{T_{j+1}}^{\tau} \gamma(s)\hat{x}_{t|s}\, ds \quad (3.12)$$

for $T_j \le t < T_{j+1}$, $T_{j+1} \le \tau < T_{j+2}$.

Therefore similarly as in (3.7), (3.8),

$$\hat{x}^{c}_{t|\tau} = \hat{x}_{t|T_{j+1}} \exp\left[-\int_{T_j}^{\tau} \gamma(s)\, ds\right] \quad (3.13)$$

for $T_j \le t < T_{j+1}$, $T_{j+1} \le \tau < T_{j+2}$.

So similarly as in (3.9) (3.10),

$$\hat{x}_{t|T_{j+2}^-} = \hat{x}_{t|T_{j+1}} \exp\left[-\int_{T_{j+1}}^{T_{j+2}^-} \gamma(s)\, ds\right]. \quad (3.14)$$

$$\hat{x}_{t|T_{j+2}} = \hat{x}_{t|T_{j+2}^-} \cdot \frac{\lambda^1_{T_{j+2}^-}}{\hat{\lambda}_{T_{j+2}^-}} \quad (3.15)$$

Then (3.10) - (3.15) imply

$$\hat{x}_{t|T_{j+2}} = \hat{x}_{t|t} \prod_{i=j+1}^{j+2} \frac{\lambda^1_{T_i^-}}{\hat{\lambda}_{T_i^-}} \exp\left(-\int_t^{T_{j+1}^-} \gamma(s) ds - \int_{T_{j+1}}^{T_{j+2}^-} \gamma(s) ds\right) \quad (3.16)$$

It is now apparent that the general formula is

$$\hat{x}_{t|T_n} = \hat{x}_{t|t}\left[\prod_{i=j+1}^{n} \frac{\lambda^1_{T_i^-}}{\hat{\lambda}_{T_i^-}}\right] \exp\left[-\int_t^{T_{j+1}^-} \gamma(s) ds - \sum_{i=j+1}^{n-1} \int_{T_i}^{T_{i+1}^-} \gamma(s) ds\right] \quad (3.17)$$

for $T_j \le t < T_{j+1}$ and $j = 1, 2, \ldots n-1$.

Since we assume that T (the switching time) coincides with one of the T_i's we are clearly interested only in the values of the smoothed probabilities given by (3.17) at $t = T_j$, $j = 1, \ldots, n-1$. Furthermore since x_t switches only at occurence times we expect the smoothed probabilities to be constant between occurence times. Indeed considering the continuous part of the filtering equations (1.9) we see that for s between occurence times

$$-\gamma(s)ds = -(\lambda_s^1 - \lambda_s^0)(1-\hat{x}_{s|s})ds = \frac{d\hat{x}_{s|s}}{\hat{x}_{s|s}}$$

and therefore for $T_i \le t < T_{i+1}$

$$\int_t^{T_{i+1}^-} \frac{d\hat{x}_{s|s}}{\hat{x}_{s|s}} = -\int_t^{T_{i+1}^-} \gamma(s)\, ds$$

or

$$\frac{\hat{x}_{T_{i+1}^-|T_{i+1}^-}}{\hat{x}_{t|t}} = \exp\left[-\int_t^{T_{i+1}^-} \gamma(s)\, ds\right]. \quad (3.18)$$

for $T_i \le t < T_{i+1}$, $i = 1, \ldots n-1$.

Consequently (3.17), (3.18) lead to

$$\hat{x}_{t|T_n} = \hat{x}_{T_j|T_j} \prod_{i=j}^{n-1}\left[\frac{\lambda^1_{T_{i+1}^-}}{\hat{\lambda}_{T_{i+1}^-}} \cdot \frac{\hat{x}_{T_{i+1}^-|T_{i+1}^-}}{\hat{x}_{T_i|T_i}}\right] \quad (3.19)$$

for $T_j \le t < T_{j+1}$ and $j = 1, 2, \ldots, n-1$. This demonstrates the constancy of smoothed probabilities between occurence times mentioned above. Given n occurence times the quantities of interest are $\hat{x}_{T_j|T_n}$, $j = 1, \ldots, n-1$. Notice that we also have the recursion

$$\hat{x}_{T_j|T_{n+1}} = \frac{\lambda^1_{T_{n+1}^-}}{\hat{\lambda}_{T_{n+1}^-}} \cdot \frac{\hat{x}_{T_{n+1}^-|T_{n+1}^-}}{\hat{x}_{T_n|T_n}} \hat{x}_{T_j|T_n} . \quad (3.19a)$$

We thus have

Theorem 3.1: The smoothed probabilities of switching, given n occurence times, are given by

$$\Pr\{T=T_j|\mathcal{F}_{T_n}\} = V(j,n)$$

$$= \hat{x}_{T_j|T_n} - \hat{x}_{T_{j-1}|T_n} \;, \; j=1,2,\ldots,n \;,$$

where $\hat{x}_{T_j|T_n}$ are computed in (3.19).

Having obtained explicit expressions for the smoothed probabilities we can now give several estimators. We are primarily interested here in the MAP estimator which from (3.1) and theorem 3.1 is given by

$$\hat{T}_{MAP}(n) = T_{j*(n)} \text{ where } j*(n)=\arg\{\max_{1\le j\le n} V(j,n)\}. \quad (3.20)$$

We then have

Theorem 3.2: Let

$$\beta(j,j-1)=\hat{x}_{T_j|T_j} - \hat{x}_{T_j^-|T_j^-} \frac{\lambda^1_{T_j^-}}{\hat{\lambda}_{T_j^-}} \cdot \frac{\hat{x}_{T_j^-|T_j^-}}{\hat{x}_{T_{j-1}|T_{j-1}}} \quad (3.21)$$

and

$$\tilde{V}(j,n)=\sum_{i=j}^{n-1}\left[\ell n\frac{\lambda^1_{T_{i+1}^-}}{\hat{\lambda}_{T_{i+1}^-}} + \ell n\frac{\hat{x}_{T_{i+1}^-|T_{i+1}^-}}{\hat{x}_{T_i|T_i}}\right] + \ell n\,\beta(j,j-1)$$

for $j=1,2,\ldots n$.

Then the MAP estimator satisfies the recursion

$$j*(n+1)=\begin{cases} j*(n) & \text{if } \tilde{V}(j*(n),n+1) \ge \ell n\,\beta(n+1,n) \\ n+1 & \text{if } \tilde{V}(j*(n),n+1) < \ell n\,\beta(n+1,n)\end{cases}$$

Proof: From (3.19)

$$V(j,n) = \prod_{i=j}^{n-1}\left[\frac{\lambda^1_{T_{i+1}^-}}{\hat{\lambda}_{T_{i+1}^-}} \cdot \frac{\hat{x}_{T_{i+1}^-|T_{i+1}^-}}{\hat{x}_{T_i|T_i}}\right]$$

$$\cdot\left\{\hat{x}_{T_j|T_j} - \hat{x}_{T_{j-1}|T_{j-1}} \frac{\lambda^1_{T_j^-}}{\hat{\lambda}_{T_j^-}} \cdot \frac{\hat{x}_{T_j^-|T_j^-}}{\hat{x}_{T_{j-1}|T_{j-1}}}\right\}$$

and therefore $\tilde{V}(j,n) = \ell n\, V(j,n)$ has the form shown in the theorem statement. Suppose $j*(n)$ is optimal. Then

$$\tilde{V}(j*(n),n) \ge \tilde{V}(k,n) \;, \quad \forall\; k \le n.$$

Since

$$\tilde{V}(j,n+1) = \begin{cases} \tilde{V}(j,n) + \ell n\frac{\lambda^1_{T_{n+1}^-}}{\hat{\lambda}_{T_{n+1}^-}} + \ell n\frac{\hat{x}_{T_{n+1}^-|T_{n+1}^-}}{\hat{x}_{T_n|T_n}} & \text{for } j=1,2,\ldots,n \\ \ell n\,\beta(n+1,n) & \text{for } j=n+1 \end{cases}$$

we have

$$\tilde{V}(j*(n),n+1) \ge \tilde{V}(k,n) + \ell n\frac{\lambda^1_{T_{n+1}^-}}{\hat{\lambda}_{T_{n+1}^-}} + \ell n\frac{\hat{x}_{T_{n+1}^-|T_{n+1}^-}}{\hat{x}_{T_n|T_n}}$$

for $k \le n$, since $j*(n) \le n$.

So

$$\tilde{V}(j*(n),n+1) \ge \tilde{V}(k,n+1), \;\forall\; k \le n.$$

Hence if

$$\tilde{V}(j*(n),n+1) \ge \ell n\,[\beta(n+1),n)]$$

then $j*(n+1) = j*(n)$. Otherwise $j*(n+1) = n+1$.

Several remarks are now in order. First observe that the MAP estimator can be directly implemented using the filter equations (1.9). Next for uniform prior probabilities $\Pr\{T=T_j\}$ the MAP gives identical results to ML which can be actually implemented in a much simpler fashion (recall theorem 2.1). Furthermore one can combine the recursive implementation of the MAP estimator with the filter in order to obtain a measure of the degree of certainty that the switch has occured (via $\hat{x}_{T_n|T_n}$).

4. Algorithms and Applications to Urban Traffic

Our results described in Theorems 2.1 and 3.2 lead to the following explicit algorithms to compute the ML and MAP estimate for the switching time T. Both algorithms are recursive, but only the second (MAP) requires the filter's implementation.

ALGORITHM ML

1. Set j=1, n=1, $\hat{T}_1 = T_1$, and $V_{OP}=V_{RUN}=0$
2. $V_{RUN} \leftarrow V_{RUN} + \ell n\; p_0(h_{n+1})$
3. If $\ell n\; p_1(h_{n+1}) > p_0(h_{n+1})$

   then $\hat{T}_{n+1} = T_j$, $V_{OP} \leftarrow V_{OP} + \ell n\, p_1(h_{n+1})$

   and go to 6.

   Otherwise go to 4.
4. If j=n the j=n+1, $\hat{T}_{n+1}=T_j$, $V_{OP} \leftarrow V_{OP} + \ell n\, p_0(h_{n+1})$.

   Otherwise go to 5
5. If $V_{OP}+\ell n\, p_0(h_{n+1}) > V_{RUN}$ then $\hat{T}_n=T_j$ and

   $V_{OP} \leftarrow V_{OP} + \ell n\, p_1(h_{n+1})$.

Otherwise $j=n+1$, $\hat{T}_{n+1}=T_j$, $V_{OP} \leftarrow V_{RUN}$

Go to 6.

6. Set $n=n+1$ and if $n \leq N_{max}$ to to 2. Otherwise stop.

ALGORITHM MAP

1. Set $j=1$, $n=1$, $\hat{T}_1=T_1$ and $V_{OP}=\hat{x}_{T_1|T_1}$

2. Compute $A = \dfrac{\lambda^1_{T_{n+1}^-}}{\hat{\lambda}_{T_{n+1}^-}} \cdot \dfrac{\hat{x}_{T_{n+1}^-|T_{n+1}^-}}{\hat{x}_{T_n|T_n}}$

$V_{OP} \leftarrow V_{OP} \cdot A$

$V_{RUN} \leftarrow \hat{x}_{T_{n+1}|T_{n+1}} - \hat{x}_{T_n|T_n} \cdot A$

3. If $V_{OP} \geq V_{RUN}$ then $\hat{T}_{n+1} = T_j$

Otherwise $j=n+1$, $V_{OP} \leftarrow V_{RUN}$ and $\hat{T}_{n+1}=T_j$.

4. $n \leftarrow n+1$ and if $n \leq N_{max}$ to to 2. Otherwise stop.

These algorithms have been applied to the platoon determination problem [2], where the two densities p_0, p_1 are given by (1.2), (1.3). As expected in critical runs these estimators perform better than the previously proposed ones [2]. In table 1 we show one trial run.

TABLE 1 $\mu = 1.0$ $\sigma^2=0.1681$, $\lambda=0.1$
$\psi = 0.8$ $\tau = 1.0$

SEQ NUMBER	MJ EST	MAP EST	ML EST	ACTUAL
0	6	4	4	4
1	5	5	5	5
2	6	6	6	5
3	10	10	10	10
4	6	6	6	6
5	4	4	4	3
6	6	6	6	5
7	3	3	3	3
8	13	13	13	13
9	2	2	2	2
10	7	7	7	7
11	10	10	10	8
12	9	9	9	9
13	5	5	5	5
14	1	6	6	6
15	7	7	7	7
16	1	2	2	2
17	11	11	11	9
18	3	3	3	3
19	13	13	13	9

References

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