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# STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN OPTICAL COMMUNICATION PROBLEMS

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## ABSTRACT

In this paper we present examples of stochastic partial differential equations of the "multiplicative noise type" arising from laser communication problems. The basic origins and generic structure of these equations are described and their relation to general quantum estimation and filtering theory explained. The major part of the paper is devoted to explicitly analyze certain examples arising from practical applications and to provide exact or approximate solutions with the help of the underlying physical theories. Directions for future theoretical work are discussed.

## 1. INTRODUCTION

In this relatively tutorial paper I want to discuss a variety of examples, most arising from practical optical communication problems, which lead to stochastic partial differential equations (s.p.d.e.) with stochastic inputs multiplying a partial differential operator applied on the state of the system. That is the resulting p.d.e's are parametrically stochastic. Systems of this type are the natural extensions of linear finite dimensional systems with multiplicative stochastic inputs. What is more exciting is that the examples I am going to describe come from quantum optics and involve a considerable amount of physics in their understanding. This provides a link between stochastic variable structure systems in infinite dimensions with a large and lately very active body of theoretical physics research. I believe both fields can profit from this crossfertilization. On one hand the precise theory of stochastic variable structure systems can help in better understanding some of the questions of quantum optics and on the other hand a large body of techniques from quantum optics, which aims at approximate solutions of these extremely difficult problems, can provide inspiration for the development of similar techniques for variable structure stochastic systems.

Since this connection is my focal point, I will be deliberately informal and often speculative, and I will rather emphasize the physical concepts than the math-

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ematical rigor. The references, especially in quantum optics, are by no means complete and they basically reflect my own bias and limited knowledge of the subject.

Although it is not my main topic, I want to mention briefly optical waveguide problems[1] as a source of stochastic p.d.e. problems of the type mentioned above. Popularly known as optical fibers, they provide us with the capabilities to generate, guide, modulate and detect light. They are thin films of dielectric material with a thickness comparable to the wavelength. They have been studied heavily, understood well and are currently used in a great variety of applications in medical instrumentation, communication systems, integrated optics. The simplest example is a planar optical waveguide (figure 1)

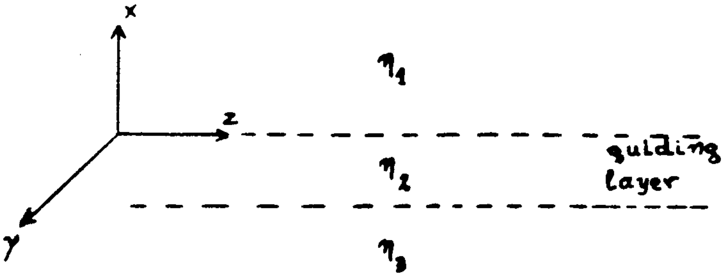


Figure 1. Planar Optical Waveguide

where $n_1$, $n_2$, $n_3$ are the indices of refraction of the three regions. The middle "guiding" layer can support modes when $n_2 > n_1, n_3$. Approximately one has to solve the wave equations

$$\frac{n^2(r)}{c} \quad \frac{\partial^2 E(r)}{\partial t^2} = \Delta E(r) \tag{1}$$

with appropriate boundary conditions, where $r = (x, y, z)$ and $\Delta$ the 3-dimensional Laplace operator. Due to impurities or deliberate variation the index of refraction varies with r. Realistically it is a stochastic process with r as the parameter variable. In corrugated wave guides [1], the boundary is a stochastic surface (figure 2).

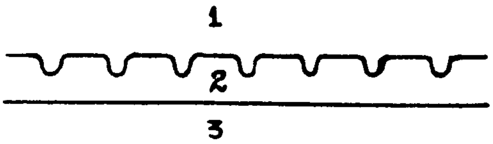


Figure 2. Boundaries of a Corrugated Waveguide

So we have two examples of stochastic p.d.e.'s with multiplicative inputs. Problems in this area include: a) properties of corrugated waveguides, b) distributed feedback lasers (where bulk properties of the medium are perturbed periodically or the boundary is perturbed periodically), c) electrooptic mode coupling e.t.c. There are still interesting mathematical problems here but working devices have been developed!

The non classical characteristics of laser communication systems are due to the non-classical (quantum mechanical) character of noise at light frequencies. At the low frequency part of the electromagnetic spectrum we have large quantum densities of nonergetic quanta while at high frequencies we have very small quantum densities of highly energetic quanta [2]. Accordingly we have to go to frequencies corresponding to wavelengths of 0.1 mm before we get quantum energies which are comparable even with rather low thermal energies [2]. Thus quantum phenomena are masked by thermal noise and are not visible in the low frequency end of spectrum. At radio frequencies (low frequency end of spectrum used for telecommunication) we have thermal noise with power per transverse radiation mode and per bandwidth $\Delta f$

$$P = kT\Delta f \tag{2}$$

where k is the Boltzmann constant [3] and T the absolute temperature in $^{o}K$. At very high frequencies approaching the infrared region of the spectrum equation (2) is no longer true. The thermal noise power drops off very rapidly. "Naive" thinking suggests optical communication systems for better performance due to low noise. However, if you experiment, you will find a new type of noise appearing as soon as the thermal noise begins to decrease. This is the so called "quantum noise". The most common manifestations of quantum noise are: a) spontaneous emission by a laser amplifier and b) detection of light by a photodector. Noise in a) and b) is ultimately determined by the measuring process of quantised radiation and is, as such, a quantum phenomenon. Plank's theory (quantum) gives the power for one mode of the radiation field in thermal equilibrium with material bodies at temperature T (i.e. the thermal noise power) as

$$P = hf\Delta f \left\{ \frac{1}{e^{hf/kT} - 1} + \frac{1}{2} \right\}, \tag{3}$$

where h is Plank's constant [3]. The term $\frac{1}{2}hf$ is commonly known as the "zero-point energy" (or vacuum fluctuations) of the mode [3]. The term

$$\bar{n} = \frac{1}{e^{hf/kT} - 1} \tag{4}$$

is the Bose-Einstein factor and gives the average number of photons in one radiation mode at temperature T [3]. In the low frequency end of the spectrum $hf << kT$ and omitting the zero-point power we get back equation (2) from (3). For $T=300^{\circ}K$ and $\Delta f = 1$ cps the two parts of (3) are shown in figure 3 [3].

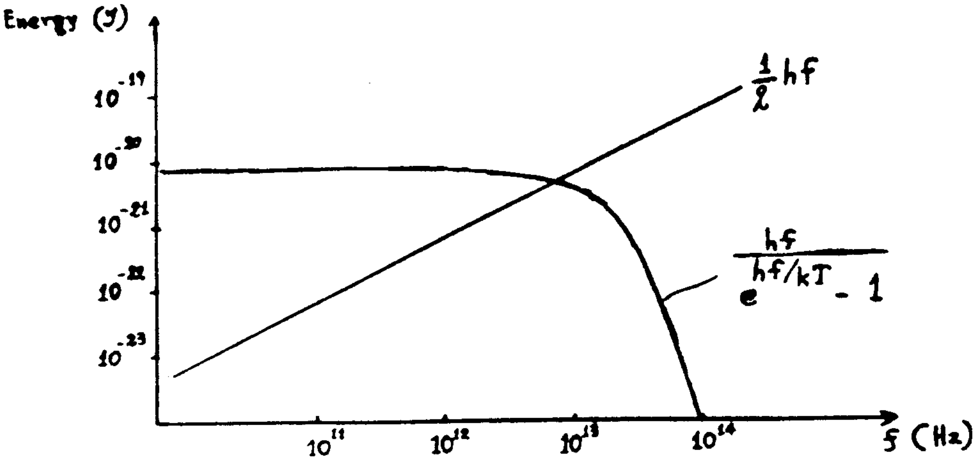


Figure 3. "Thermal" versus "quantum" noise power as function of frequency from [3].

This is meant only as a brief review of the fundamental difference between noise in communication systems at low and optical frequencies. For details I refer the reader to [3].

Optical communication in addition to a description of noise sources, requires the existence of devices that can produce waves with very controllable characteristics. One qualitative distinction between low frequency and high frequency radiation is the degree of control we have over it. For a very nice discussion of this and related issues see [2]. The point is, briefly stated, that while at low frequencies we have almost complete spatial and temporal control of the waveforms we generate, at high (optical) frequencies we loose this possibility rather quickly (for example we can have spatial control of the field but the amplitudes tend to fluctuate uncontrollably). The invention of the laser provided a source of very controllable light fields, since it is an extremely intense source of essentially identical photons. This changed the situation for optics and lead to new experiments that required more than classical physics for their explanation [4], [5]. The use of quantum electrodynamics allowed a fairly complete treatment of the laser which has been verified by experiments [4], [5], [6]. A typical gas laser (figure 4) works as follows: a gas discharge excites the electrons of the atoms to higher energy levels resulting in an inverted population between some atomic

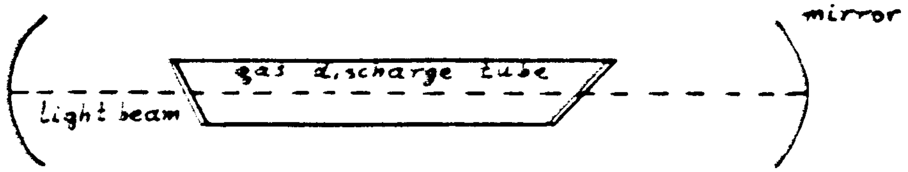


Figure 4. A typical gas laser

levels, which amplifies electromagnetic radiation at a frequency corresponding to the atomic energy level difference. Atoms of the gain medium can absorb photons or emit photons. The emission can be spontaneous, or stimulated due to activation by the discharge (pump mechanism). The spontaneously emitted power appears in all radiation modes regardless of whether they are already occupied by photons or not. On the other hand stimulated emission produces additional power for modes already occupied by photons. This increase of power by stimulated emission is a coherent process since it simply increases the existing excitation of the radiation mode at a rate that is proportional to the energy already stored in that mode. Spontaneous emission, on the other hand, is in no way related to the existing excitation of a mode and occurs at a rate independent of the energy stored in that mode. Spontaneous emission is therefore incoherent and is example a) of quantum noise noted earlier. In the laser via the pump mechanism (and the resulting inverted population) stimulated emission is enhanced while photon absorption is far less probable. The result is a low noise amplifier. For details about lasers I refer the reader to [1],[3]. In view of this discussion it should not be surprising that quantum mechanical concepts and formalisms play a fundamental role in the analysis of laser communication systems.

## 2. QUANTIZATION OF THE ELECTROMAGNETIC FIELD, COHERENT STATES, FOCK SPACE

The need for the quantum mechanical treatment in optics, and therefore in optical communication systems, arises primarily from the way light is detected, i.e. the photoelectric effect. Photon counters are the inevitable device in a very large class of optical communication receivers. Classical optics theory is equivalent to one-photon quantum mechanical optics theory. That is you make the approximation that there is one photon in the EM field and calculate the probability that it will be somewhere in the screen (on which you observe the diffraction pattern). This is equivalent to classical intensity calculation. The Hanbury-Brown-Twiss experiment [2] is the first one to detect correlations between pairs of photons. It is this kind of statistics that became important since the invention of laser. This point is exemplified in [2] where I refer for an enlightening discussion. As Glauber [2] states "The laser is intrinsically a non-linear device; it only works when the field intensities become so high that the photons know a great deal about each other's presence". So classical (or one-photon type) arguments are clearly at a disadvantage.

I would like now to give a brief description of the quantization of the EM field in an effort to make this paper as self-sustained as possible. I refer the interested reader to [4] [5] [7] for further details.

Let E(r, t), H(r, t), A(r, t) be the electric field, magnetic field, and magnetic potential for the classical Maxwell theory, where r = (x, y, z) indicates the spatial variables. Solving the wave equation, that A satisfies, with boundary conditions obtained from a box-normalization we obtain

$$A(r,t) = \sum_n a_n(t)\, u_n(r) \tag{5}$$

where

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2\, q_n(t) = 0 . \tag{6}$$

Then we usually think of $q_n$ as the "position" variables and introducing the "momentum" variables via

$$p_n(t) = \frac{d q_n(t)}{dt} \tag{7}$$

we obtain the Hamiltonian for the electromagnetic field

$$H = \sum_n \tfrac{1}{2}\, [\, p_n^2(t) + \omega_n^2\, q_n(t)\, ] . \tag{8}$$

That is the EM field is equivalent to a set of harmonic oscillators. To quantize the EM field [8] we replace all physical variables with operators. Thus we now have position and momentum operators $q_n$, $p_n$ which satisfy the commutation relation

$$[q_n, p_m] = i\hbar\,\delta_{nm} . \tag{9}$$

Furthermore one introduces creation $a_n^*$ and annihilation $a_n$ operators for each mode

$$a_n^* = \frac{1}{\sqrt{2\hbar\omega_n}}\,(\omega_n q_n - i p_n) \tag{10}$$

$$a_n = \frac{1}{\sqrt{2\hbar\omega_n}}\,(\omega_n q_n + i p_n)$$

to obtain for the magnetic potential operator the expression

$$A(r,t) = c \sum_n \left( \frac{\hbar}{2\omega_n} \right)^{\frac{1}{2}} [a_k u_k(r) e^{-i\omega_k t} + a_k^* \bar{u}_k(r) e^{i\omega_k t}] . \tag{11}$$

The electric and magnetic field operators are

$$E(r,t) = - \frac{\partial A(r,t)}{\partial t} , \quad H(r,t) = \frac{1}{\mu} \nabla \times A(r,t), \tag{12}$$

and the hamiltonian operator becomes

$$H = \sum_n \hbar \omega_n (a_n^* a_n + \tfrac{1}{2}) \tag{13}$$

where the $\frac{1}{2} \hbar \omega_n$ term is usually omitted since it represents the zero point fluctuations. The operators $a_n, a_n^*$ are adjoints of each other [9] and satisfy the canonical commutation relation (C.C.R.) for a Boson field:

$$[a_n, a_m^*] = \delta_{nm}, \; [a_n, a_m] = 0, \; [a_n^*, a_m^*] = 0. \tag{14}$$

In standard quantum mechanics one works with a complex Hilbert space $\mathcal{H}$ which represents the states of the system [9], and utilizes selfadjoint operators V (called observables) on $\mathcal{H}$ to represent measurable variables. If $< \cdot , \cdot >$ is the inner product on $\mathcal{H}$ and v the outcome (a classical random variable) of the measurement represented by V, the expected value of the outcome when the system is in state y is

$$E\{v\} = < y, Vy > . \tag{15}$$

This formalism is inadequate however, whenever we do not have exact knowledge of the state. So one introduces probabilities $p_i$ that the state is $y_i$. Then the expected value formula becomes

$$E\{v\} = \sum_i < y_i, Vy_i > p_i = \mathrm{Tr}[\sum_i p_i < y_i, \cdot > y_i] V = \mathrm{Tr}[\rho V] \tag{16}$$

where $\rho$ is the selfadjoint, trace class ($\mathrm{Tr}\,\rho = 1$) [9] operator on $\mathcal{H}$ defined via

$$\rho x = \sum_i p_i < y_i, x > y_i \; . \tag{17}$$

This operator $\rho$ is called the <u>density operator</u> or the <u>state</u> of the quantum system.

In a standard notation [7]-[9] we denote by the dyad $|n_k><n_k|$ the pure state of the EM field, where the kth mode has $n_k$ photons. The ground state of a mode has no photons, energy equal to the zero point fluctuations and is denoted by $|0_k><0_k|$. The name of the operators $a_n, a_n^*$ comes from their action on such states

$$a_k|n_k> = n_k^{\frac{1}{2}}|n_k-1>$$
$$a_k^*|n_k> = (n_k+1)^{\frac{1}{2}}|n_k+1> \ . \tag{18}$$

The vacuum (ground state) has the property

$$a_k|0_k> = 0 \ . \tag{19}$$

These states (so called energy eigenstates) are useful in the analysis of problems involving few photons. However they are not convenient in situations with many photons (e.g. lasers). A different set of states are very appropriate for the latter. These are the so called coherent states introduced originally by Schrödinger and emphasized by Glauber [10]. They are the eigenvectors of the annihilation operator a (for simplicity we discuss a single mode here), and are usually denoted by $|\alpha>$:

$$a|\alpha> = \alpha|\alpha> \ . \tag{20}$$

Note that (20) can be interpreted as: substraction of a photon from the state leaves it unaltered! So indeed we do not have a specified number of photons in a coherent state. Rather, the probability that a coherent state $|\alpha>$ has n photons is given by the Poisson probability

$$\frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!} \ . \tag{21}$$

These states form an overcomplete set in the sense that their inner product gives

$$<\beta|\alpha> = \exp\{\bar{\beta}\alpha - \tfrac{1}{2}|\beta|^2 - \tfrac{1}{2}|\alpha|^2\} \tag{22}$$

while

$$\frac{1}{\pi}\int|\alpha><\alpha|d^2\alpha = I \ . \tag{23}$$

The extension of this model to a multimode field gives rise to the so called Fock space of quantum mechanics [11]. If you like to see concrete objects, the Fock space for a single mode is isomorphic to the following function analytic model due to Bargmann[12]. The Hilbert space consists of all entire functions of a complex variable with inner product

$$<f,g> = \int\bar{f}(z)g(z)\frac{1}{\pi}\exp(-\bar{z}z)\,dz. \tag{24}$$

The operators a, $a^*$ look like $\frac{\partial}{\partial z}$ and "multiplication by z" respectively. This space of entire functions has very nice properties [12]. Strong convergence implies pointwise convergence. The evaluation functionals

$$f \longmapsto f(a), \ a \text{ complex} \qquad (25)$$

are bounded and give thus rise to <u>principal vectors</u> $e_a$ such that

$$< e_a, f> = f(a) \qquad (26)$$

and

$$e_a(z) = e^{\bar{a}z}. \qquad (27)$$

The coherent state $|\alpha>$ corresponds to

$$\exp(-\frac{\bar{\alpha}\,\alpha}{2})\ e_{\bar{\alpha}} \ . \qquad (28)$$

It is a reproducing kernel Hilbert space with kernel

$$K(w, z) = e^{w\bar{z}} \ . \qquad (29)$$

Every bounded operator on the Fock space is represented with an intergal operator with kernel

$$T_L(w, z) = < e_w, L e_z > . \qquad (30)$$

The Weyl operator [11]

$$W(\lambda) = \exp(\lambda a^* - \bar{\lambda} a) \qquad (31)$$

gives rise to the Weyl characteristic function for the density operator $\rho$ via

$$X(\lambda) = \mathrm{Tr}\{\rho \exp(\lambda a^* - \bar{\lambda} a)\}. \qquad (32)$$

This is really the analog to the classical characteristic function, save for the fact that Fourier transform is non-commutative here. The inverse Fourier transform of $X(\cdot)$ is the Wigner-distribution function [11]. We clearly have other choices for the Fourier transform since $a$, $a^*$ do not commute. The normal characteristic function is given by

$$X_N(\lambda) = \mathrm{Tr}\{\rho \exp(\lambda a^*) \exp(-\bar{\lambda} a)\}. \qquad (33)$$

Now if $\rho$ has a P-representation [8], [10]

$$\rho = \int P(\alpha)\, |\alpha> < \alpha|\, d^2\alpha, \qquad (34)$$

then

$$X_N(\lambda) = \int P(\alpha) \exp(\lambda \bar{a} - \bar{\lambda} a)\, d^2\alpha \qquad (35)$$

and going to real variables we see that $P(\alpha)$ is the Fourier transform of $X_N(\lambda)$, when ever it exists. The importance of the P-representation stems from the following fact: for $M_N(a^*, a)$ a normally ordered operator (i. e. a power series in $a^*$, a, where $a^*$ appears to the left of a) expected values for the corresponding measurement outcome can be easily computed by [8],

$$\mathrm{Tr}\,[\,\rho M_N(a^*, a)] = \int P(\alpha)\, M(\bar{\alpha},\alpha) d^2\alpha \tag{36}$$

where $M(\bar{\alpha}, \alpha)$ is a scalar function of the complex variables $\bar{\alpha}, \alpha$.

To summarize, the importance of coherent states is due primarily to:

1) they are very good models for lasers operating above threshold,
2) they are related to some very important experimental results in optics about higher order coherence of laser beams (for details see [2], [5] ).

## 3. BILINEAR STOCHASTIC P.D.E. FROM QUANTUM OPTICS

The state of the quantum system is an operator $\rho$, trace class, selfadjoint on some complex Hilbert space $\mathcal{H}$, with $\mathrm{Tr}[\rho] = 1$. Let $\mathcal{T}_s(\mathcal{H})$ denote all trace class selfadjoint operators on $\mathcal{H}$. If H is the Hamiltonian operator of a closed system then it is a consequence of the Schrödinger equation, that $\rho$ evolves according to the equation [9], [11]

$$\frac{\partial \rho_t}{\partial t} = -i\,[H, \rho_t] \tag{37}$$

or in integrated form for time independent H,

$$\rho_t = T_t(\rho) = e^{-iHt} \rho\, e^{iHt},$$

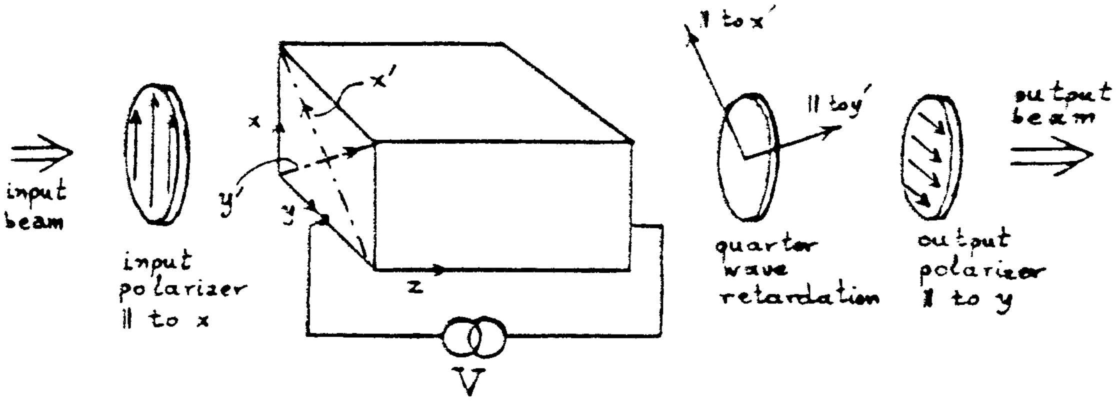
where we have adopted the normalization $\hbar = 1$. Depending on the particular representation of $\mathcal{H}$, the operator H is a typical unbounded selfadjoint operator. The infinitesimal generator of the strongly continuous group of isometries $T_t$ is the operator

$$Z(\rho) = -i\,[H, \rho] = -i\,\mathrm{ad}_H\, \rho$$

with domain $\mathcal{D}(Z) = \{\rho \in \mathcal{T}_s(\mathcal{H}) \mid \rho \mathcal{D}(H) \subseteq \mathcal{D}(H)$, and $H\rho - \rho H$ is norm bounded on $\mathcal{D}(H)$ with an extension to a trace class operator on $\mathcal{H}\}$ [11].

For the time independent case (37) has been studied heavily and for an interesting account I refer to [11]. Quite often however, the situation arises where the Hamiltonian depends upon a random process $x_t$. Then (37) becomes clearly a stochastic operator differential equation with multiplicative excitation. In any representation for $\mathcal{H}$ that is useful for computations, the state operator $\rho$ will appear as an integral operator and then (37) will take the form of a stochastic p.d.e. for the kernel of $\rho$, with multiplicative excitation. This is in short, the way that s.p.d.e. arise from quantum optics. What we have here also is a random field and a way to describe it. Unfortunately even for simple cases the rigorous treatment of (37) with stochastic Hamiltonians is very complicated and very little is known in general. Physicists however have developed an impressive stock pile of formal or approximation techniques to handle such equations depending on the values of some relevant but important parameters. I believe that these methods can provide the inspiration for the development of similar methods for stochastic bilinear systems. I now proceed to give some concrete examples and their solution.

Since I am primarily interested in laser communication problems the first two examples describe two common ways that signals are carried by laser beams. These are amplitude and phase modulation of laser beams using the electrooptic effect. The electrooptic amplitude modulation is described in figure 5 below.



<u>Figure 5.</u> Electrooptic amplitude modulation for laser beam [1].

By applying an electric field along the z direction in certain crystals it is possible to effect a change in the index of refraction that is proportional to the field. Certain crystals have an "ordinary" and "extradordinary" ray with different indices of refraction. The linear electrooptic effect is the change in the indices of the ordinary and extraordinary rays that is caused by and is proportional to an applied electric field. In the figure above x', y' are the electrically induced principal axes of the crystal. When the electric field is on, the polarization of the incident beam is rotated as it goes through the crystal and passes through unattenuated. On the other hand with the field off the input beam is completely attenuated. For small magnitude of the applied voltage V(t) we have $I_0/I_i$ proportional to V(t). Then the interaction Hamiltonian for the quantum mechanical description of the amplitude modulation is modeled by

$$H_{AM} = -i\,g\,f(t)[a - a^*] \tag{38}$$

where g is the coupling constant and f(t) may contain stochastic variations.

Next we consider phase modulation. The same phenomenon can be used as shown below.

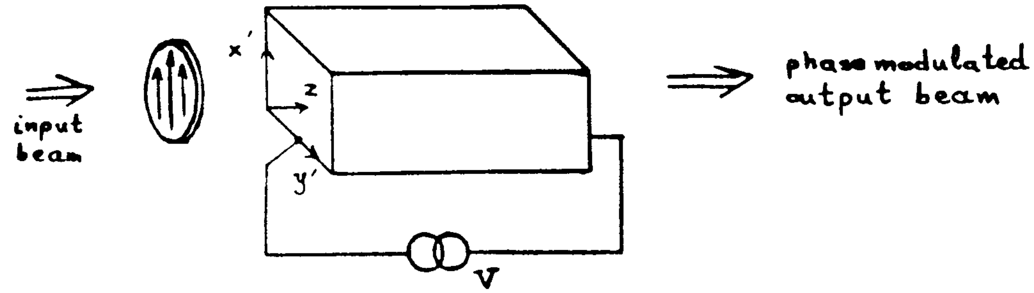


Figure 6. Electrooptic phase modulating of a laser beam [1].

Here the incident field is polarized parallel to x', the induced birefringent axis. In this case the application of the electric field along the z direction does not change the state of polarization but merely the output phase, by an amount proportional to V. In this case the interaction Hamiltonian is modeled by

$$H_{FM} = g\,f(t)\,a^* a \tag{39}$$

where f(t) can have stochastic fluctuations.

The final example is that of optical parametric oscillation and amplification [1]. The classical analog is: consider two slightly damped harmonic oscillators, call them "signal" and "idler" with frequencies $\omega_1$ and $\omega_2$ and damping constants $k_1$ and $k_2$. Let these two oscillators be coupled by some "parameter", and modulate this parameter with a harmonically varying F of frequency $\omega_3$. A simple calculation shows that provided the matching condition

$$|\omega_3 - \omega_1 - \omega_2| \leq k_1, k_2$$

is satisfied, the effective damping in both oscillators is reduced, and above a certain threshold value of F self-sustained oscillations take place. The $\omega_3$ oscillator is called the "pump". Below threshold such a device is called a parametric amplifier, above threshold we have the oscillator region. If the two oscillators coincide, it is called degenerate parametric oscillation or subharmonic generation. In classical physics, one could imagine a situation in which there are no fluctuations. However, the situation changes completely if signal and idler are quantized. Zero-point fluctuations of these oscillators will always be present and these are sufficient for the build up of self-sustained oscillation. So we should study parametirc oscillation in a set up, where quantum phenomena can be observed and therefore the effect of noise can be understood. This is done as follows in optics: signal and idler oscillations are provided by two modes of the electromagnetic field. The coupling parameter is the susceptibility of the medium (P = polarization, E = electric field, $P = \varepsilon_0 \chi_\varepsilon E$) in which the electromagnetic modes propagate (a nonlinear crystal that is). If we choose a medium whose susceptibility depends linearly on the electric field we can modulate this coupling parameter by shining in a powerful beam of laser light of frequency $\omega_3$.

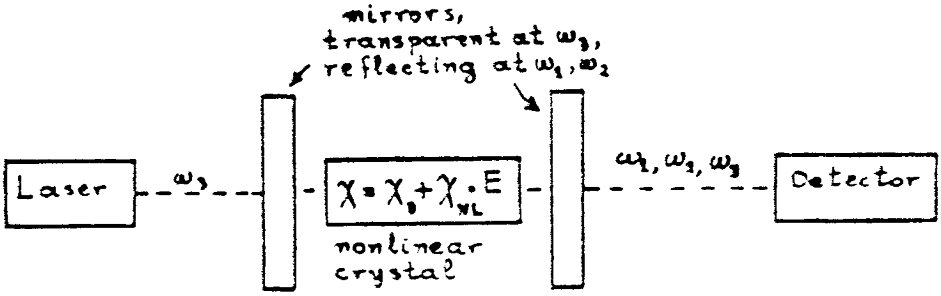


Figure 7. Laser parametric amplifier[1].

Generation of optical signal and idler modes has been achieved experimentally but requires phase and momentum matching. Now the importance of noise is paramount here. Moreover this device is extremely useful because it is "tunable" (contrasted with lasers which are not). The interaction Hamiltonian in this case can be effectively modelled by

$$\begin{aligned} H_{PA} = {} & i\chi\,(a_1^* a_2^* a_3 - a_3^* a_1 a_2) \\ & - i\,(F^*(t)\,a_3 - F(t)\,a_3^*) \end{aligned} \tag{40}$$

Heat baths must be included here to model damping. Again F(t) can contain noise which has serious effects here.

Notice that the examples given here are relatively simple in the sense that the Hamiltonian operator is a memoryless function and linear in the stochastic excitation. Nonlinear memoryless functions as well as functions with memory are possible. All examples I described thus far lead to stochastic evolution equations of the type

$$\frac{d}{dt} f(t) = (Z_0 + A(x_t))\, f(t) \tag{41}$$

where $x_t$ is a stochastic process and f(t) lies in a Banach space $\mathfrak{B}$. When the process $x_t$ is stationary or Gaussian the work of Hida [13] [14] appears to be useful in analyzing this type of stochastic evolution equations. In relation to this see also the work of Balakrishnan [15]. Asymptotic properties based on different time scales for f and $x_t$ have been studied by Papanicolaou and Varadhan [16]. The exact theory of (41) is going to be at best very technical and complicated. Here physicists employ several useful techniques to analyze equations like (41).

To study dissipation and losses in quantum mechanical systems, and in order to simplify dynamical equations, physicists go from closed systems to open systems. So instead of studying the full evolution (37), they try to deduce the dynamical equations for a simple but important (for the particular problem) subsystem. The system becomes open in the following sense: it is represented by a Hilbert space $\mathcal{K}$ and state $\rho$ and is coupled to a reservoir represented by a Hilbert space $\mathfrak{Z}$. This is also motivated by the irreversibility of many real systems. The total system is represented thus on the space $\mathcal{K}\otimes\mathfrak{Z}$ with the Hamiltonian

$$H = H_{\mathcal{K}} \otimes I + I \otimes \mathcal{K}_{\mathfrak{Z}} + \lambda H_I \tag{42}$$

where $\lambda$ is a coupling constant. Initially the two systems are uncoupled and the initial state is $\rho\otimes\rho_\beta$. At time t the state is

$$\rho = e^{-iHt}(\rho\otimes\rho_\beta)e^{iHt}. \tag{43}$$

To find the evolution of the relevant part of the state we trace over the reservoir variables. That is if $\mathcal{B} = \mathcal{T}_s(\mathcal{K}\otimes\mathfrak{Z})$ and $\mathcal{B}_0 = \mathcal{T}_s(\mathcal{K})$ the partial trace is a bounded linear map

$$P_0 : \mathcal{B} \to \mathcal{B}_0 .$$

characterized by

$$\mathrm{Tr}[(P_0X)A] = \mathrm{Tr}[X(A\otimes I)] \tag{44}$$

for all $X\in\mathcal{B}$ and $A\in\mathcal{L}_s(\mathcal{K})$. Identifying $\rho\in\mathcal{B}_0$ with $\rho\otimes\rho_\beta\in\mathcal{B}$, makes $P_0$ a projection of $\mathcal{B}$ onto $\mathcal{B}_0$. Let

$$\begin{aligned} Z(\rho) &= -i\,[H_{\mathcal{K}}\otimes I + I\otimes\mathcal{K}_{\mathfrak{Z}}, \rho] \\ A(\rho) &= -i\,[H_I, \rho] \end{aligned} \tag{45}$$

and by projecting (43) (note $ZP_0 = P_0Z$) we get the dynamics of the system's state

$$\rho_t = P_0[e^{(Z+\lambda A)t}\rho], \ \forall \rho\in\mathcal{B}_0 . \tag{46}$$

This is a special instance of a general procedure which leads to <u>Master equations</u> [6], [11]. Namely let $P_0$ be a projection on a Banach space $\mathcal{B}$ and set $\mathcal{B}_0 = P_0\mathcal{B}$,

$P_1 = I - P_0$, $\mathcal{B}_1 = P_1\mathcal{B}$. Let Z be the generator of a strongly continuous group of isometries $U_t$ on $\mathcal{B}$, such that $U_t P_0 = P_0 U_t$. Suppose A is a bounded perturbation of Z. If $\rho \in B_0$ is the initial state, the state at time t is given by (46), where $\lambda$ is a parameter. Let

$$A_{ij} = P_i A P_j \ , \quad Z_i = Z P_i = P_i Z \ , \quad i = 0, 1. \tag{47}$$

and define

$$U_t^{\lambda} = e^{\{z + \lambda(A_{00} + A_{11})\} t} . \tag{48}$$

It is shown in (11) that $\rho_t$ is a solution of

$$\frac{d}{dt} \rho_t = (Z_0 + \lambda A_{00}) \rho_t + \lambda^2 \int_{\sigma=0}^{t} A_{01} U_{t-\sigma}^{\lambda} A_{10} \rho_\sigma d\sigma \tag{49}$$

which is an exact equation on $\mathcal{B}_0$ (the state space of the subsystem of interest, and not on $\mathcal{B}$). This is the general form of Master equations. At this point approximations are introduced such as neglecting memory effects, which are usually satisfied by the relative length of various relaxation times. This leads from (49) to

$$\begin{aligned} \frac{d}{dt} \rho_t &= (Z_0 + \lambda A_{00} + \lambda^2 K) \rho_t \\ K &= \int_0^{\infty} A_{01} e^{Z_1 \sigma} A_{10} e^{-Z_0 \sigma} d\sigma . \end{aligned} \tag{50}$$

This method relies on the existence of a <u>small parameter</u> and it is usually very successful if coupled with our physical knowledge about the system, and in particular the relative range of coupling parameters. For stochastic evolutions the situation is more complicated. However, results exist utilizing this approach, where the role of $P_0$ is played by the expectation operator over the statistics of the fluctuating input. This way we can generate approximate moment equations. I believe thatthis method can also prove to be very effective in variable structure stochastic systems and large scale systems analysis.

Another approximate method directly related to my main topic is the use of classical or quasi-classical states in quantum optics problems. This utilises the P-representation discussed in the previous section. For a rigorous discussion I refer again to [11]. The idea is then to translate (37) into a p.d.e. with two spatial variables. I restrict the discussion here to the single mode case for simplicity. This program is easily executed once the Hamiltonian is given, since in all cases of interest it is a polynomial in the operators a and $a^*$. For such computations we typically need to calculate the P-representation of operators of the form

$a\rho$, $\rho a$, $a^*\rho$, $\rho a^*$, $\rho a^* a$, $a^* a\rho$, $\rho a a^*$, $a\rho a^*$, $a^*\rho a$. That is each of these operators can be written in the form

$$\int k(\alpha)\, P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha \tag{51}$$

where $k(\alpha)$ is $\alpha$, $\alpha - \frac{\partial}{\partial\bar{\alpha}}$, $\bar{\alpha} - \frac{\partial}{\partial\alpha}$, $\bar{\alpha}$, $\bar{\alpha}\,\alpha - \frac{\partial}{\partial\bar{\alpha}}\,\bar{\alpha}$, $\bar{\alpha}\,\alpha - \frac{\partial}{\partial\alpha}\,\alpha$, $1 + \alpha\bar{\alpha} - \frac{\partial}{\partial\bar{\alpha}}$, $\alpha\bar{\alpha}$, $1+\alpha\bar{\alpha} - \frac{\partial}{\partial\alpha}\,\alpha - \frac{\partial}{\partial\bar{\alpha}}\,\bar{\alpha} + \frac{\partial^2}{\partial\alpha\,\partial\bar{\alpha}}$ for the operators above, in the same order. The end result of such a computation is a p.d.e. of the Fokker-Planck type. We illustrate with the examples above.

First we consider phase modulation. From (13) and (39) the total Hamiltonian is

$$H = \omega a^* a + g\, f(t)\, a^* a. \tag{52}$$

Suppose we are interested in states that have P-representation. Then P satisfies

$$\frac{\partial P(\alpha,t)}{\partial t} = -\, i(\omega - g\, f(t))\, \left(\frac{\partial}{\partial\bar{\alpha}}\,\bar{\alpha} - \frac{\partial}{\partial\alpha}\,\alpha\right) P(\alpha, t) \tag{53}$$

and going to real variables, $\alpha = x + iy$

$$\frac{\partial P(x,y,t)}{\partial t} = -\, (\omega - g\, f(t))\, \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) P(x,y,t) \tag{54}$$

which is clearly a degenerate Fokker-Planck equation. If $f(t)$ is stochastic this is a s.p.d.e. Since $f(t)$ is a signal, it is not natural to assume that it is white noise, rather it is generated by a stochastic differential equation driven by a Wiener process. Then essentially we can solve (54) sample path by sample path. For more details about equations of this type I refer to [17]. In polar coordinates (54) becomes

$$\frac{\partial P(r,\theta,t)}{\partial t} = (\omega - g\, f(t))\, \frac{\partial}{\partial\theta}\, P(r,\theta,t). \tag{55}$$

Now if the initial state is a coherent state then $P_0(\alpha) = \delta^2(\alpha - \alpha_0)$. One way to solve for $P(x,y,t)$ is to add a "diffusion term" in (54)

$$b\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P(x,y,t)$$

which will give a Gaussian solution for a $\delta$-function initial condition and then take the limit as $b \to 0$. This will give us back a $\delta$-function. All this can be made very precise since this new Fokker-Planck equation corresponds to the stochastic d.e.

$$\begin{bmatrix} dx_t \\ dy_t \end{bmatrix} = [\omega - g\, f(t)] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} dt + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dw_t \tag{56}$$

for which we can compute the densities. So if we start with a coherent state we will have a coherent state at all t. Another way to see this is to observe that the solution to (55) is simply

$$P(r, \theta + \omega t - \int_0^t g f(s)\, ds,\ 0).$$

The coherent state is thus the solution of

$$\begin{aligned} \frac{d}{dt}\alpha(t) &= -i(\omega - g f(t))\alpha(t) \\ \alpha(0) &= \alpha_0 . \end{aligned} \tag{57}$$

Now this is an exact result for stochastic inputs with appropriately continuous sample paths. Both (57) and (56) with $b = 0$ are stochastic bilinear equations on the circle, so results on filtering problems for such state equations [18] can be applied here. I want to make two remarks now. First observe that in reality the laser beam will be coupled with a reservoir that produces decay and losses and therefore we will get a "diffusion term" in the Fokker-Planck equation. Second the coherent state approximation is a very good one for lasers operating above threshold (see previous section).

The situation is very similar for amplitude modulation. The total Hamiltonian from (13) and (38) is

$$H = \omega a^* a - i g f(t)\,(a - a^*). \tag{58}$$

In a similar fashion if we use coherent states approximation we obtain the degenerate stochastic Fokker-Planck equation

$$\frac{\partial P(\alpha,t)}{\partial t} = -i\omega\left(\frac{\partial}{\partial\bar{\alpha}}\bar{\alpha} - \frac{\partial}{\partial\alpha}\alpha\right)P(\alpha,t) - g f(t)\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\bar{\alpha}}\right)P(\alpha,t) \tag{59}$$

or

$$\frac{\partial P(x,y,t)}{\partial t} = -\omega\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)P(x,y,t) - g f(t)\frac{\partial}{\partial x}P(x,y,t). \tag{60}$$

Again if we start at a coherent state $\alpha_0 = x_0 + i y_0$ we have a coherent state at all t where

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + g f(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{61}$$

or

$$\alpha(t) = e^{-i\omega t}\alpha_0 + \int_0^t e^{-i\omega(t-s)} g f(s)\, ds. \tag{62}$$

The final example refers to coupling during propagation and resulting losses. This is very important for communication problems and refers to the transmission of the laser through the channel medium. Here one combines master equations and coherent states approximation. The total Hamiltonian is now

$$H = \underbrace{\omega a^* a}_{\text{EM mode}} + \underbrace{\sum_k \omega_k a_k^* a_k}_{\text{reservoir}} + \underbrace{\sum_k g_k (a_k^* a + a^* a_k)}_{\text{interaction}} \tag{63}$$

where $g_k$ are the coupling constants with the heat bath that is usually modelled as an infinite set of harmonic oscillators. The usual assumptions about the reservoir are: i) it is in thermal equilibrium before the interaction and ii) it is a very large system weakly coupled to the field so its thermal equilibrium is never disturbed. Utilizing first master equations [6] one obtain the d.e. for the density of the field

$$\frac{\partial \rho(t)}{\partial t} = - i(\omega + \Delta)[a^* a, \rho(t)] + k\{[a, \rho(t) a^*] + [a\rho(t), a^*]\}$$
$$+ 2k\bar{n}\,[a, [\rho(t), a^*]]\,. \tag{64}$$

The parameters $\Delta$, $k$, $\bar{n}$, characterize the influence of the heat bath and represent a frequency shift, a damping constant and the average number of photons at the frequency of the mode (4). By incorporating the frequency shift in $\omega$ and using P-representation we obtain the Fokker-Planck equation

$$\frac{\partial P(\alpha, t)}{\partial t} = - i\omega\left(\frac{\partial}{\partial \bar{\alpha}}\bar{\alpha} - \frac{\partial}{\partial \alpha}\alpha\right) P(\partial, t) + k\left(\frac{\partial}{\partial \bar{\alpha}}\bar{\alpha} + \frac{\partial}{\partial \alpha}\alpha\right) P(\alpha, t)$$
$$+ 2k\bar{n}\,\frac{\partial^2}{\partial \alpha\, \partial \bar{\alpha}} P(\alpha, t) \tag{65}$$

or

$$\frac{\partial P(x, y, t}{\partial t} = - \omega\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) P(x, y, t) +$$
$$+ k\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\right) P(x, y, t) + \frac{k}{2}\bar{n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P(x, y, t)\,. \tag{66}$$

This is equivalent to the linear stochastic d.e.

$$\begin{bmatrix} dx_t \\ dy_t \end{bmatrix} = \begin{bmatrix} -k & \omega \\ -\omega & -k \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} dt + \begin{bmatrix} dw_{1t} \\ dw_{2t} \end{bmatrix} \tag{67}$$

where $w_{1t}$, $w_{2t}$ are independent $(0, k\bar{n})$ Wiener processes. The solution of (66) with a $\delta$-function initial condition is well known. It is a Gaussian density with mean and variance calculated from (67). In complex form (67) becomes

$$d\alpha_t = (-k - i\omega)\,\alpha_t dt + dw_t, \tag{68}$$

with $w_t$ complex Brownian motion. So the solution to (65) with initial condition $\delta(\alpha - \alpha_0)$ is

$$P(\alpha, t) = \frac{1}{\pi \bar{n}(1-e^{-2kt})} \exp\left\{ - \frac{\left|\alpha - e^{(-k-i\omega)t}\alpha_0\right|^2}{\bar{n}(1-e^{-2kt})} \right\} \tag{69}$$

Thus we see that the effect of such a coupling is to turn a coherent state to a state with a Gaussian $P(\cdot)$ in its P-representation (34).

We can now deduce the state transformations occuring when we first modulate and then transmit a laser operating above threshold. The modulation will shift the initial coherent state and the propagation will turn it into a quasi-classical state with Gaussian P-representation.

## 4. FILTERING PROBLEMS

In communication problems after modulating the carrier by the signal, and propagating through the channel we receive it. At the receiver we must make some sort of measurement and then process the measurement outcomes to obtain estimates of the signal with respect to a fidelity criterion. For us this criterion will be the minimum variance of the error between estimate and signal.

As it was explained in section 1 and 2 quantum mechanical formulation of the measurement process is required, when analysing laser communication systems. However the traditional quantum mechanical formalism (section 2) is not adequate. The reasons is that when transmitting vector valued signals, traditional quantum mehcanics permit simultaneous measurement of compatible observables [7] [9]. Detection and estimation problems in this area demonstrate however that it may be advantageous to attempt "approximate" measurements of noncompatible observables [7] [19] [20]. I refer to these references for further details on the general measurement formulation. Davies in [11] has a very enlightening discussion of this issue and provides a canonical example of "approximate" simultaneous measurement of position and momentum. The resulting model is a positive operator valued measure, that is a mapping M from the Borel $\sigma$-algebra of $\mathbb{R}^n$ (the value space of the measurement process) $\mathcal{B}^n$ into the space $\mathcal{L}_s(\mathcal{H})$ of self-adjoint bounded operators on $\mathcal{H}$ with the properties

$$\left.\begin{aligned} &\sum_i M(B_i) = M(\cup B_i) \\ &M(B) \geq 0 \\ &M(\mathbb{R}^n) = I \end{aligned}\right\} \tag{70}$$

for $B_i$, $B \in \mathcal{B}^n$ and the operator equalities are interpreted in the weak operator sense. The outcome of the measurement is a classical random variable $v \in R^n$ with distribution function

$$F_v(A) = \mathrm{Tr}[\rho M(A)] , \quad A \in \mathcal{B}^n \tag{71}$$

when the quantum system is in state $\rho$. In view of Naimark's theorem [7] [19] this measurement is actually realized by adjoining to the inital system $(\mathcal{H}, \rho)$ an auxiliarysystem $(\mathcal{H}_e, \rho_e)$ and then performing on the augmented system $(\mathcal{H} \otimes \mathcal{H}_e, \rho \otimes \rho_e)$ simultaneous measurements of compatible observables, characterized by the spectral measure $E_M$. That is

$$\mathrm{Tr}[(\rho \otimes \rho_e) E_M(A)] = \mathrm{Tr}[\rho M(A)], \quad A \in \mathcal{B}^n. \tag{72}$$

Finally I would like to bring together all these concepts and methodologies in analyzing a filtering problem in a laser system. We imagine a laser beam operating above threshold which carries a two dimensional real signal as its in phase and quadrature amplitudes, $x_1(t)$, $x_2(t)$. Suppose the two signal processes are inputed on the laser by amplitude and phase modulation techniques. We would like to utilize quantum mechanical measurements on the beam to obtain the minimum square error estimate of $x_1(t)$ and $x_2(t)$. We assume that the laser has one mode i.e. it is monochromatic and that has been contaminated during transmission (i.e. heat bath). Now if we formulate the problem starting from Hamiltonians and all that the problem will become to look unsolvable. Let us utilize what we have developed. Since we are above threshold we can model the laser initially by a coherent state. Then we know that modulation will change the laser beam only by changing the coherent state (see equations (57) and (62)). Finally by amplification we can neglect damping, so that we can take k=0 in (69). So that the received field will have a gaussian $P(\alpha)$.

$$P(\alpha, t) = \frac{1}{\pi \bar{n}} \exp\left\{- \frac{|\alpha - (x_1(t) + i x_2(t)|^2}{\bar{n}}\right\} \tag{73}$$

where by abuse of notation we set the signals $x_1$, $x_2$ as the shifting arguments of the coherent state. Actually $x_1(t)$, $x_2(t)$ are linear functions of the true signals, but we can retrace the operations backwards.

Suppose now we start making measurments on the beam. We represent these measurements by p.o.m.'s. Since we need two measurement outcomes we consider p.o.m.'s on $R^2$. Ideally you would like to make one measurement at discrete instances of time i.e. by beam splitting e.t.c. The problem becomes quickly intractable again because of another fundamental theorem in quantum mechanics [9] [11]: "The state after measurement of a discrete observable <u>changes</u>. If $P_i$ is the projection to the i^th^ eigenspace of the observable the state after the measurement is $\rho_i = \frac{P_i \rho P_i}{Tr(\rho P_i)}$". Instead we restrict ourselves to use one measuring procedure for all times and process its outcomes in an optimal fashion. This at first seems restrictive but we defend it by the following arguments: i) it is certainly very realistic from the engineering point of view ii) it has been shown in similar problems [20] that it is a very good approximation after all! Before we proceed, we know that we can safely assume that if the original modulating signals were derived from a linear stochastic differential equaation then the same is true for $x_1(t)$, $x_2(t)$. So we can, tracing back the calculations, write

$$d\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A(t)x(t)dt + B(t)\,dw_t \tag{74}$$

We further think about the "quantum mechanical meaning" of (73). Recall that $a + a^*$ corresponds to q (or to H) and $a - a^*$ corresponds to p (or to E) from our discussion of the electromagnetic field. Now $x_1$ is clearly related to $Re\,\alpha$, $x_2$ to $Im\,\alpha$, and therefore to $a + a^*$ and $a - a^*$ respectively. So what we have at hand is a problem of simultaneous estimation of p, q. So it makes sense to look for "approximate" simultaneous measurements of p, q. We know that such things exist [11]. It can be shown through our earlier work [20] that the optimum p.o.m for the estimation of $x_1(t)$, $x_2(t)$ is described by

$$M(A) = \frac{1}{\pi} \int_A |\alpha\rangle\langle\alpha| \, d^2\alpha \,. \tag{75}$$

Then we can compute the joint density for the measurement outcomes $y_1$, $y_2$.

$$\begin{aligned} p_{y(t)|x(t)}^{(\xi,\zeta;x_1(t),x_2(t))} &= Tr[\,\rho(x(t))\,|\alpha\rangle\langle\alpha|\,\tfrac{1}{\pi}\,] \\ &= \frac{1}{\pi(\bar{n}+1)} \exp\left(-\frac{(\xi - x_1(t))^2}{\bar{n}+1}\right) \cdot \\ &\quad \cdot \exp\left(-\frac{(\zeta - x_2(t))^2}{\bar{n}+1}\right) \end{aligned} \tag{76}$$

This is the same as saying that the measurement outcomes can be represented as

$$y(t) = x(t) + v_t \tag{77}$$

where $v_t$ is a white noise process with mean zero and variance $\frac{\bar{n}+1}{2} I$.

But then the minimum variance filter for the problem is well known since (74) and (77) constitute a classical problem. It is Kalman filtering. Next question you want to answer is how to implement it. You need the realization He, $\rho_e$, $E_M$. You can show that if you take a local oscillator at its ground state $\rho_e = |0_e\rangle\langle 0_e|$, then M is equivalent to the simultaneous measurement of

$$\frac{a+a}{2} \otimes I - I \otimes \frac{a_e + a_e^*}{2} \qquad \text{on } \mathcal{H} \otimes \mathcal{H}_e. \tag{78}$$

and

$$\frac{a-a^*}{2i} \otimes I + I \otimes \frac{a_e - a_e^*}{2i}$$

and this is called optical heterodyning.

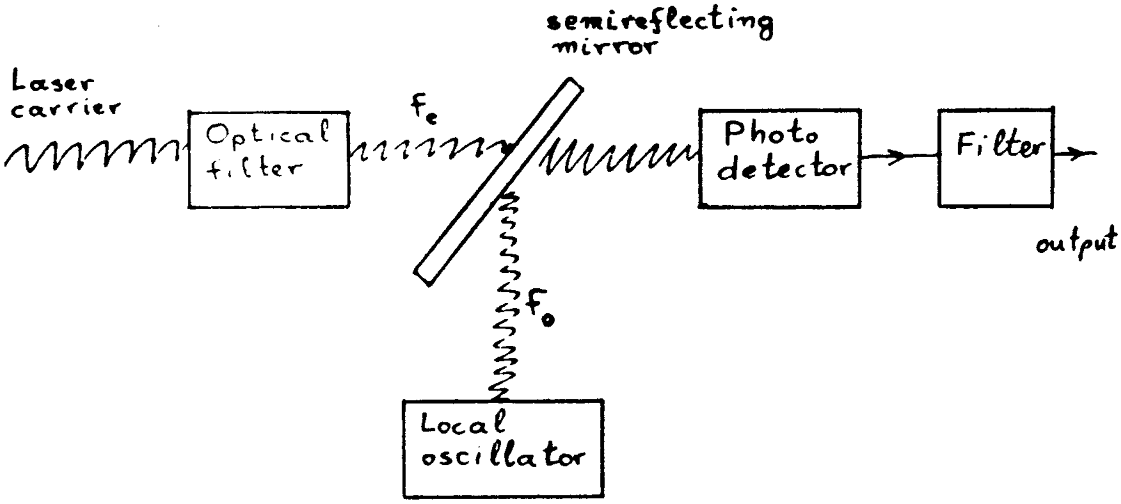


Fig. 8. Optical heterodyning.

Further work is necessary to establish rigorously the solution to similar filtering problems.

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