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# CONTINUOUS QUANTUM FILTERING

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## ABSTRACT

We consider here the general filtering problem for signals carried by quantum mechanical fields. The continuous time, continuous quantum measurement problem is formulated. Simple examples from quantum optics illustrate the approach.

## 1.  INTRODUCTION

Communication theory at optical frequencies requires quantum mechanical description of the various relevant devices and fields in order to analyze properly the phenomenon of "quantum noise". This is the type of noise that appears when the carrier frequency is at the optical or higher range of the spectrum, where thermal noise becomes negligible. For a very nice description of "quantum noise" we refer to [5]. Motivated by detection, estimation problems in quantum optics one is lead to the analysis of similar problems for signals carried by quantum fields in general. For a very comprehensive treatment of such problems we refer to [4].

Recently in [1] - [3] the study of the problem of filtering a signal carried by a quantum field was initiated. Due to the enormous complexity and difficulty in the formulation of this problem several simplifying assumptions were made to render the problem feasible. It is the purpose of the present paper to introduce some initial efforts towards a general quantum filtering theory, without the special assumptions of [1] - [3]. We briefly describe the necessary quantum mechanical framework and we refer to [6] for further details. We are only interested in quantum statistical mechanics. Let $\mathcal{K}$ be a complex Hilbert space with inner product $< \cdot \, , \, \cdot >$. The <u>state</u> of the quantum system is a self-adjoint, non-negative trace class operator of trace one, usually denoted by $\rho$. We let $\mathcal{T}_s(\mathcal{K})$ denote the set of all self-adjoint trace class operators on $\mathcal{K}$ with the trace norm

$$||A||_{tr} = tr(A^*A)^{1/2} = \sum_{n=1}^{\infty} \lambda_n \qquad (1)$$

where $\lambda_n$ are the eignevalues of the compact operator $(A^*A)^{1/2}$. This is a Banach space with positive cone $\mathcal{T}_s(\mathcal{K})^+$. Customarily a measurement is represented by a self-adjoint operator on $\mathcal{K}$, V, or with the associated projection - valued measure $E_V$. If we let v denote the measurement outcome then the probability distribution function of v is given by

$$F_v(\xi) = tr[\rho \, E_V(-\infty, \xi)] \qquad (2)$$

for $\xi \in R$. Since this concept of a measurement proved to be restrictive for estimation problems [1] - [4] the more general concept of a measure-

ment represented by a positive operator valued measure was introduced.
A p.o.m. M is a map from the $\sigma$-algebra of Borel sets $\mathcal{B}^n$ of $R^n$, to the
algebra $\mathcal{L}(\mathcal{K})$ of all bounded operators on $\mathcal{K}$, such that

    i)    M(B) is self-adjoint and $\geq 0$ for every $B \in \mathcal{B}^n$

    ii)   if $\{B_i\} \subseteq \mathcal{B}^n$ is a partition of $R^n$ then $\sum_i M(B_i) = I$ (weakly).     (3)

This a necessary concept for effectively handling vector signal processes
[2] - [4]. For more details and relations to Naimark's theorem as well
as interpretations in terms of approximate joint measurement of non-
compatible variables we refer to [4] and in particular to [7, ch. 3].
The measurement outcome v is in $R^n$ now, and its statistics are de-
scribed by the probability measure

$$\mu_v(B) = tr[\rho M(B)], \text{ for } B \in \mathcal{B}^n. \tag{4}$$

In [1] - [3] we assumed that $\rho$, the state of the quantum field, depended
on a stochastic process x(t), thought as the information carrying signal.
To avoid complications, and in particular the time evolution of $\rho$, it was
assumed that $\rho$ does not depend explicitly on time. It was also assumed
that $x(t)$ is a discrete time stochastic process. At discrete instants of
time $t_i$, a measurement was performed on the system and gave outcome
$v(t_i)$. Both cases where $x(t_i)$, $v(t_i) \in R$ or $R^n$ were considered. Again, in
order to avoid complications arising from the time evolution of $\rho$ and the
well-known interaction between the state of a quantum system and a
measurement [6] [7], it was assumed that the measurement outcomes
$v(t_i)$ conditioned upon the signal sequence $x(t_i)$ are independent from time
to time. This assumption leads to a factorization of the probability
distribution function:

$$F_{v(t_1) \cdots v(t_k) | x(t_1) \cdots x(t_k)} = \prod_{i=1}^{k} F_{v(t_i) | x(t_i)}. \tag{5}$$

A filtered estimator for $x(t_k)$ was formed

$$\hat{x}(t_k) = \sum_{i=0}^{k} C_i(k) v(t_i) \tag{6}$$

and the problem of selecting the nxn matrices $C_i(k)$ and p.o.m. $M_k$ in
order to minimize the mean square error

$$M.S.E = E\{\|x(t_k) - \hat{x}(t_k)\|_{R^n}\} \tag{7}$$

was solved. This is an example of a discrete time linear filtering prob-
lem with quantum measurements. The results obtained depend critically
on the assumptions made above.

    In this paper we want to formulate a general filtering problem, with-
out making use of the assumptions used in [1] - [3]. In particular we
want to bring into the formulation the time evolution of $\rho$ and the inter-
action between state propagation in quantum systems and quantum
measurements. We shall denote by $\mathcal{L}_s(\mathcal{K})$ the space of all self-adjoint
bounded operators on $\mathcal{K}$ with positive cone $\mathcal{L}_s(\mathcal{K})^+$.

## 2. STOCHASTIC EVOLUTION OF STATES

If H is the Hamiltonian operator of a closed quantum system [7], it

is a consequence of the Schrödinger equation that the state $\rho(t)$ evolves according to the evolution equation in $\mathcal{T}_s(\mathcal{H})$:

$$\frac{\partial \rho(t)}{\partial t} = - i[H, \rho(t)] \tag{8}$$

or in integrated form

$$\rho_t = T_t(\rho) = e^{-iHt} \rho e^{iHt} , \tag{9}$$

where we assumed H is time-independent. The interpretation of (8) depends on the particular representation of the Hilbert space $\mathcal{H}$, but usually H is an unbounded self-adjoint operator. $T_t$ in (9) is a strongly continuous group of isometries and for more details on this subject we refer to [7]. Quite often, and, as we shall see shortly, in particular for us, it is necessary to consider equations, like (8) where H depends on a stochastic process x(t). Then (8) becomes a stochastic evolution equation which is quite difficult and yet not known how to handle in this generality. For "nice" states we have shown in [8] that (8) reduces to a stochastic partial differential equation, typically of the Fokker-Planck type, which can be handled quite satisfactorily. In particular we singled out the case where H depends linearly on a stochastic process, since this is frequently the case for common modulation schemes of laser beams [8] [9]. For example electrooptic amplitude modulation of a single mode laser can be modeled by the interaction Hamiltonian

$$H_I = - i g x(t) [a - a^*] , \tag{10}$$

while electrooptic phase modulation can be modeled by

$$H_I = g x(t) a^* a , \tag{11}$$

where $a$, $a^*$ are the photon annihilation and creation operators of the mode [6], and g a coupling constant. Then the total Hamiltonian of the laser mode is

$$H = \omega a^* a + H_I . \tag{12}$$

where $\omega$ is the carrier frequency and is clearly linear in the signal process x(t). For stochastic excitations x(t) with piecewise continuously differentiable sample paths equations like (8) can be handled rather satisfactorily whenever H is linear in the process x(t). In particular if

$$H(t) = H + x(t)B \tag{13}$$

where H is self-adjoint, unbounded and generates a unitary group, while $B \in \mathcal{L}_s(H)$ (8) has been studied in [13], with stochastic inputs x(t). This case however exludes hamiltonians arising from Boson quantum fields (such as from lasers), because then the operator B is unbounded. Indeed since $a, a^*$ are unbounded for Boson fields (10) and (11) provide such examples. Fortunately in the most usual cases B is quadratic in a and $a^*$ and then one can still study (8) quite satisfactorily, see [8] and [12] [13]. The same is expected to be true for unbounded operators B which are polynomials in $a, a^*$. These are the only cases we consider here. Clearly in the case of a vector stochastic signal the term x(t)B in (13)

should be replaced by a term $\sum_{i=1}^{n} x_i(t) B_i$, where the component processes $x_i(t)$ and operators $B_i$ have similar properties. In general therefore the modulation process (i.e. the injection of the signal to the quantum field) is described by the stochastic evolution equation

$$\frac{\partial \rho(t)}{\partial t} = - i[H, \rho(t)] - i \sum_{i=1}^{n} x_i(t)[B_i, \rho(t)] \tag{14}$$

Via (14) $\rho(t)$ becomes an operator-valued stochastic process.

After modulation the quantum field is transmitted, and therefore interacts with the external world. We model this by a reservoir with hamiltonian $H_R$ and an interaction Hamiltonian $gH_I$, where g is again a coupling constant. Let $H_\gamma(t)$ be the Hamiltonian in (14), then the total system is represented on $\mathcal{H} \otimes \mathcal{J}$ ($\mathcal{J}$ the Hilbert space of the reservoir) by the total state $\rho_1$ which satisfies the equation

$$\frac{\partial \rho_1(t)}{\partial t} = - i [H_1(t), \rho_1(t)] \tag{15}$$

where

$$H_1(t) = H_\gamma(t) \otimes I + I \otimes H_R + gH_I \tag{16}$$

Then one employes Master equations methods (i.e. tracing over the reservoir variables) [7] to obtain the stochastic evolution of the state $\rho$ of the quantum field including transmission effects.

## 3. QUANTUM MEASUREMENT PROCESSES

After modulation and transmission the information carrying quantum field reaches the receiver where measurements are made. Suppose we first make a measurement represented by the self-adjoint operator Y, or the projection valued measure $E_Y$ which is discrete (i.e. Y has discrete spectrum). If $y_i$ is an eigenvalue of Y, it is conventionally supposed that a measurement which gives the value $y_i$ transforms the original state $\rho$ to the state

$$\frac{P_i \rho P_i}{tr[\rho P_i]} \tag{17}$$

where $P_i$ is the corresponding eigen projection [7]. For convenience one does not consider the normalization factors and constructs the transformation

$$\rho \longmapsto \sum_{i=1}^{\infty} P_i \rho P_i \tag{18}$$

which describes the effect on $\rho$ when Y is measured. This is a conditional expectation in the sense of Umegaki [15] which cannot be generalized however to non discrete observables. It is strongly related to von Neuman's repeatability hypothesis and the related difficulties with continuum spectrum. It is this interaction between states and measurement that was bypassed by the various assumptions in [1] - [3]. In [16] this interaction is considered when measurements are made in dis-

crete instants of time and the underlying quantum states are Gaussian. Here we follow and extend the general approach of Davies [7], [10] - [14] who considered quantum stochastic processes which are the appropriate generalization of marked point-processes. If B is a self-adjoint operator with spectral measure $E_B$, one defines in the same spirit as (18) the observable conditioned by the measurement of Y by

$$E_{B|Y}(F) = \sum_{i=1}^{\infty} P_i E_B(F) P_i, \text{ for } F \in \mathcal{B}, \tag{19}$$

Note that $E_{B|Y}$ is a p.o.m and not a projection-valued measure. Moreover the joint measurement of Y and B is characterized by the p.o.m

$$E_{Y,B}(F \times E) = \sum_{y_i \in E} P_i E_B(F) P_i . \tag{20}$$

In [14] Davies and Lewis defined the proper generalizations of these ideas by introducing the concept of an <u>instrument.</u> Let $\mathcal{Y}$ be a set and $\mathcal{B}$ a $\sigma$-algebra of subsets of $\mathcal{Y}$. An instrument with values in $\mathcal{Y}$ is a map $\mathcal{E}$ from $\mathcal{B}$ into $\mathcal{L}^+(\mathcal{V})$ (the space of bounded positive linear maps of $\mathcal{V}$ into $\mathcal{V}$ where $\mathcal{V} = \mathcal{J}_s(\mathcal{K})$)) such that

i) $\mathcal{E}(E) \geq \mathcal{E}(\phi) = 0$, for $E \in \mathcal{B}$

ii) if $\{E_i\} \subseteq \mathcal{B}$ is a countable collection of disjoint sets in $\mathcal{B}$ then

$$\mathcal{E}(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathcal{E}(E_i) \text{ (strongly.)} \tag{21}$$

iii) $\text{tr}[\mathcal{E}(\mathcal{Y})\rho] = \text{tr}[\rho]$

for all $\rho \in \mathcal{J}_s(\mathcal{K})$ .

This corresponds to the idea that an instrument accepts a state $\rho$ and produces a measurement outcome and a new state conditional on the measurement outcome. Typically for us $\mathcal{Y}$ will be either a finite set, or $R^n$. By employing the fact that $\mathcal{L}_s(\mathcal{K})$ is the dual of $\mathcal{J}_s(\mathcal{K})$ we have [7][14] that to every instrument on $(\mathcal{Y}, \mathcal{B})$ there corresponds a unique p.o.m. M (3) such that

$$\text{tr}[\mathcal{E}(E)\rho] = \text{tr}[M(E)\rho] \text{ for } \rho \in \mathcal{J}_s(\mathcal{K}) . \tag{22}$$

We call M the <u>measurement performed by the instrument</u> $\mathcal{E}$. Conversely given a measurement M on $(\mathcal{Y}, \mathcal{B})$, choose a partition $\{E_i\}$ of $\mathcal{Y}$ and a sequence of states $\rho_i$, with $\text{tr}[\rho_i] = 1$. Then the formula

$$\mathcal{E}(E)\rho = \sum_{i=1}^{\infty} \text{tr}[M(E \cap E_i)\rho] \rho_i \tag{23}$$

defines an instrument on $(\mathcal{Y}, \mathcal{B})$ which performs the measurement M. Suppose $\mathcal{E}^1, \mathcal{E}^2$ are instruments on $(\mathcal{Y}^1, \mathcal{B}^1)$, $(\mathcal{Y}^2, \mathcal{B}^2)$ with $\mathcal{Y}^1, \mathcal{Y}^2$ complete separable metric spaces, then there is a unique instrument $\mathcal{E}^{1,2}$, called the <u>composition of $\mathcal{E}^2$ following $\mathcal{E}^1$</u>, on $\mathcal{Y}^1 \times \mathcal{Y}^2$ such that

$$\mathcal{E}^{1,2}(E_1 \times E_2) \rho = \mathcal{E}^2(E_2) \mathcal{E}^1(E_1)\rho \tag{24}$$

for all $E_1 \in \mathcal{B}^1$, $E_2 \in \mathcal{B}^2$, $\rho \in \mathcal{J}_s(\mathcal{K})$. Then if $M_1, M_2$ are the measurements

performed by $\mathcal{E}^1, \mathcal{E}^2$, the p.o.m.

$$M_{2|1}(F) = \mathcal{E}^1(\mathcal{Y}^1)^* M_2(F), \quad F \in \mathcal{B}^2 \tag{25}$$

is the measurement $M_2$ conditioned by the measurement $M_1$. Moreover the joint statistics of the two measurements are given by the probability measure

$$\mu_{1,2}(A) = tr[\mathcal{E}^{1,2}_\nu(A)\rho] / tr[\rho], \quad A \in \mathcal{B}^1 \times \mathcal{B}^2. \tag{26}$$

For our purposes we need a more general concept. Namely a one parameter family of instruments: a "quantum process". Davies in [7] and [10-12] developed such a concept, but for measurements with outcomes forming a point process. As a generalization we introduce a quantum measurement process. Let $\mathcal{Y}$ a complete separable metric space and $\mathcal{B}$ the Borel $\sigma$-algebra of $\mathcal{Y}$. Let $\mathcal{Y}^t$ be the set of all measurable functions from $[0,t]$ into $\mathcal{Y}$ with the usual $\sigma$-algebra. Given $s, t \geq 0$ there is a one-one Borel map $c$ of $\mathcal{Y}^s \times \mathcal{Y}^t$ onto $\mathcal{Y}^{s+t}$ defined by concatenation

$$c(\omega^s, \omega^t)(\tau) = (\omega^s \circ \omega^t)(\tau) = \begin{cases} \omega^s(\tau) & , 0 \leq \tau < s \\ \omega^t(\tau-s), & s \leq \tau \leq t+s \end{cases} . \tag{27}$$

A quantum measurement process is a family of instruments $\mathcal{E}^t$ on $\mathcal{Y}^t$ for $t \geq 0$ such that

i) $\lim_{t \geq 0} \mathcal{E}^t(\mathcal{Y}^t) \rho = \rho$, for $\rho \in \mathcal{T}_s(\mathcal{H})$

ii) for $\rho \in \mathcal{T}_s(\mathcal{H})$ and $s, t \geq 0$

$$\mathcal{E}^t(F)\mathcal{E}^s(E) \rho = \mathcal{E}^{s+t}(c(E \times F)) \rho \tag{28}$$

for $F \in \sigma$-algebra of $\mathcal{Y}^t$, $E \in \sigma$-algebra of $\mathcal{Y}^s$

According to the desirable properties of the measurement outcome process, additional assumptions on $\mathcal{Y}^t$ can be made which will give (28) more structure (such as : continuous, square integrable, mark point processes). In this generality we do not know as yet how to characterize the process. Davies has studied extensively such processes when $\mathcal{Y}^t = \{(y_i, t_i): 1 \leq i \leq n,$ $n$ arbitrary$\}$. If we let $z$ denote the empty sample path, he analyzed processes with bounded interaction rate in the sense that

$$tr[\mathcal{E}^t(\mathcal{Y}^t - \{z\})\rho] \leq Kt\, tr[\rho]. \tag{29}$$

Defining the two semigroups on $\mathcal{T}_s(\mathcal{H})$

$$T_t(\rho) = \mathcal{E}^t(\mathcal{Y}^t)\rho \tag{30}$$

$$S_t(\rho) = \mathcal{E}^t(\{z\})\rho \tag{31}$$

and assuming that $S_t$ has the form

$$S_t(\rho) = B_t \rho B_t^* \tag{32}$$

where $B_t$ a strongly continuous contraction semigroup on $\mathcal{H}$, be showed

that they are uniquely characterized by the infinitesimal generator $\Psi$ of $B_t$ and a positive map valued measure on $\mathcal{Y}$, $\mathcal{g}$ (i.e. which satisfies (21) with iii) replaced by $\text{tr}[\mathcal{g}(\mathcal{Y})\rho) \leq K \text{ tr }[\rho])$ and are related by: 

$$\text{tr}[\mathcal{g}(\mathcal{Y})|b\rangle\langle b|] = -2\,\text{Re}\langle \Psi b, b\rangle \tag{33}$$

for all $b \in \mathcal{D}(\Psi)$. An important role in the anlysis is played by the total interaction rate of the process R defined by

$$\text{tr}[R\rho] = \text{tr}[\mathcal{g}(\mathcal{Y})\rho] \ . \tag{34}$$

Then it can be shown that if H(t) is the Hamiltonian of the quantum system before interacting with measurements the state obeys the differential equation

$$\frac{\partial \rho(t)}{\partial t} = -i[H(t), \rho] + \mathcal{g}(\mathcal{Y})\rho \tag{35}$$

$$-\frac{1}{2}\{R\rho + \rho R\}$$

see [7] [13] for details. This has also been established for Boson fields with photon counters as measurements in [12] [13], which are processes with unbounded interaction rates.

## 4. QUANTUM FILTERING

We are now ready to formulate the general quantum filtering problem. Let $\rho(t, x^t)$ be the state of the quantum system at time $t$, which depends on the sample path $x^t$ of the signal up to time t. This is obtained from sections 2 and 3 above assuming a "modulating" interaction Hamiltonian linear in x(t). More complicated dependence, including memory is clearly possible. Given a class C of quantum measurement processes we have available from each the classical process y(t) which denotes the measurement out comes. The general filtering problem is: Choose the quantum measurement process $\mathcal{E}^t$ from C and functionals $f_t$ measurable with respect to $\sigma(y_s, s\leq t)$ such that

$$\hat{x}(t) = f_t(y_s, \ s \leq t) \ . \tag{36}$$

is the minimum variance estimator of x(t).

Motivated by the work of Davies [13] we consider the question: Does there exist a family of quantum measurement processes in C, parametrized by the sample paths $x^t$ of the signal process on $\mathcal{Y}^t$

$$\mathcal{E}^t(x^t, E)\rho(o) \tag{37}$$

such that

$$\rho(t) = \mathcal{E}^t(x^t, \mathcal{Y}^t)\rho(o) \tag{38}$$

satisfies an equation like (35)? In what sense? Once this is established further progress can be made, since then the probability measure

$$\mu_t(x^t, E) = \text{tr}[\mathcal{E}^t(x^t, E)\rho(o)] \tag{39}$$

characterizes the statistics of the classical measurement process y(t) given the classical signal process x(t). We now give two simple examples to illustrate the approach.

The first is similar to that of Davies [13]. We consider an amplitude modulated single mode laser with the class C consisting of photon counters. The Hamiltonian before measurement is given by (10) (12). We let $\mathcal{K}$ be the Boson Fock space for a single mode [7] [8]. The coherent states $|\alpha><\alpha|$ with $|\alpha>$ an eigenvector of the annihilation operator a play an important role in the sequel since they provide good models for lasers above threshold and their linear span is dense in $\mathcal{J}_s(\mathcal{K})$. Here $\mathcal{Y}$ collapses to a single point. If we start at a state $|\alpha><\alpha|$ a measurement of a photon will reduce the number of photons by 1 and therefore the new state will be a $|\alpha><\alpha|a^*$. So the instrument is described by

$$\mathcal{E}\rho = a\rho a^* = \mathcal{J}\rho.$$

(40)

From (34) $R = a^*a$. So (35) gives

$$\frac{\partial\rho(t)}{\partial t} = -i\omega[a^*a, \rho(t)] - gx(t)[a - a^*, \rho(t)]$$

$$-\frac{1}{2}[a^*a\,\rho(t) + \rho(t)a^*a] + a\rho(t)a^*$$

(41)

If we work with classical states, i.e. $\rho = \int |\alpha><\alpha|P(\alpha)d^2\alpha$

(42)

we can translate (41) into a p.d.e. for P [8]. We obtain

$$\frac{\partial P(\alpha, t)}{\partial t} = -i\omega\left(\frac{\partial}{\partial\bar{\alpha}}\bar{\alpha} - \frac{\partial}{\partial\alpha}\alpha\right)P(\alpha, t)$$

$$- gx(t)\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\bar{\alpha}}\right)P(\alpha, t)$$

$$-\frac{1}{2}\left[2\bar{\alpha}\alpha - \frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\bar{\alpha}}\bar{\alpha}\right]P(\alpha, t) + \alpha\bar{\alpha}P(\alpha, t),$$

(43)

or if $\alpha = \zeta + i\xi$ we have the Fokker-Planck equation

$$\frac{\partial P(\zeta, \xi, t)}{\partial t} = -\omega\left(\xi\frac{\partial}{\partial\zeta} - \zeta\frac{\partial}{\partial\xi}\right)P(\zeta, \xi, t)$$

$$- gx(t)\frac{\partial}{\partial\zeta}P(\zeta, \xi, t)$$

$$+\frac{1}{2}\left(\zeta\frac{\partial}{\partial\zeta} + \xi\frac{\partial}{\partial\xi} + 2\right)P(\zeta, \xi, t).$$

(44)

This is easily seen to correspond to the linear stochastic differential equation:

$$\begin{bmatrix} d\zeta(t) \\ d\xi(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \omega \\ -\omega & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \zeta(t) \\ \xi(t) \end{bmatrix} + gx(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(45)

Again since (44) is a degenerate Fokker-Planck equation we see in the same fashion as in [8] that if we start at a coherent state

$$\rho(0) = |\alpha_0 > < \alpha_0|$$

we will end at the coherent state

$$\rho(t) = |\alpha(t) > < \alpha(t)| \tag{46}$$

where $\alpha(t) = \zeta(t) + i\xi(t)$ is computed from (45) or from

$$\frac{d\alpha(t)}{dt} = \left(-i\omega - \frac{1}{2}\right)\alpha(t) + g\,x(t) \tag{47}$$

$$\alpha(0) = \alpha_o .$$

Now in this case the classical measurement process is a point process $y(t)$. Let $N_t$ denote the corresponding counting process [18]. The most relevant quantity is the rate process

$$\lambda_t(N_t; w_1; \ldots; w_{N_t}; x(s),\ s \leq t) = \tag{48}$$

$$= \lim_{\Delta t \to 0} \Delta t^{-1}\ \Pr(N_{t,\,t+\Delta t} = 1 | N_t; w_1; \ldots; w_{N_t}; x^t)$$

where $w_1, \ldots w_{N_t}$ are the occurence times. It can be seen through the construction of $^t \mathcal{E}^t$ [13] that in this case

$$\lambda_t = tr[\mathcal{I}\rho(t)] = tr\,[a\,|\alpha(t) > < \alpha(t)|\,a^*] \tag{49}$$

$$= |\alpha(t)|^2 = \left|g \int_{s=0}^{t} e^{-(i\omega + \frac{1}{2})(t-s)} x(s)ds\right|^2$$

This has now been reduced to a classical problem which can be solved. Notice that no measurement optimization was involved here.

In the second example we consider a monochromatic laser beam operating above threshold which carries a two dimensional real signal as its in phase and quadrature amplitudes $x_1(t)$, $x_2(t)$. The received field's state has a P-representation with

$$P(\alpha, t) = \frac{1}{\pi \bar{n}}\ \exp\left\{-\frac{\left|\alpha - x_1(t) + i x_2(t)\right|^2}{\bar{n}}\right\} \tag{50}$$

[2]. We assume the signals are generated according to a linear stochastic differential equation

$$\begin{bmatrix} \dfrac{d\,x_1(t)}{dt} \\[2mm] \dfrac{d\,x_2(t)}{dt} \end{bmatrix} = A(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + B(t)\begin{bmatrix} dw_1(t) \\ dw_2t \end{bmatrix} \tag{51}$$

where $w_1(t)$, $w_2(t)$ are Wiener processes. We conjecture that the optimal quantum measurement process is one which performs the measurement

$$M(A) = \frac{1}{\pi} \int_A |\alpha > < \alpha|\,d\alpha$$

for all times. Then the classical measurement process can be repre-

sented by

$$y(t) = x(t) + v(t) \qquad\qquad (52)$$

where v(t) is a white noise process with mean zero and variance $\frac{n+1}{2}$ .
(51) and (52) define a classical filtering problem.

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