JB 72-01

Entitled:

H2-Functions and Infinite Dimensional Realization Theory

Authors:

J.S. Baras and R.W. Brockett

Conference:

1972 IEEE Conference on Decision and Control (CDC) and 11^{th} Symposium on Adaptive Processes

CDC
PAPER No. TP3-1

H^2-FUNCTIONS AND INFINITE DIMENSIONAL REALIZATION THEORY*

J.S. Baras R.W. Brockett
Division of Engineering and Applied Physics
Harvard University
Cambridge, Massachusetts

Summary

In this paper the realization question for infinite dimensional linear systems is examined for both bounded and unbounded operators. In addition to obtaining realizability criteria covering the basic cases, we discuss the relationship between canonical realizations of the same system. What one finds is that the set of transfer functions which are realizable by triples (A,b,c) with A bounded is related in a close way, to the space of complex functions analytic and square integrable on the disk $|s| < 1$ and that the set of transfer functions which are realizable by triples (A,b,c) with A unbounded but generating a strongly continuous semigroup is related in a close way to functions analytic and square integrable on a half-plane.

1. Preliminaries and Notation

In this paper we study realization theory for a class of infinite dimensional linear systems. On one hand our motivation comes from a desire to understand engineering problems involving transmission lines, elastic deformations, moving fluids, and related matters, where the assumption of finite dimensionality is too restrictive; on the other hand we want to see the finite dimensional results themselves as part of a larger picture.

For the sake of definiteness we work in the most basic Hilbert space, $\ell_2(\mathbb{Z}^+)=\{a_i\}$ $i=1,2,3,\ldots$ such that the $\{a_i\}$ is a square summable sequence. This makes possible a fairly direct comparison with many well known results concerning the finite dimensional case. The problem is to express a given real function T defined on $[0,\infty)$ as $T(t)=\langle c,e^{At}b\rangle$ or to express its transform $\tilde{T}(s)$ as $\langle c,(Is-A)^{-1}b\rangle$. We consider several distinct, but related cases. The first centers around the existence of realizations (A,b,c) with A a bounded operator on $\ell_2(\mathbb{Z}^+)$ and b and c elements of $\ell_2(\mathbb{Z}^+)$. We call such triples <u>regular bounded realizations</u>. We call a triple (A,b,c) a <u>regular realization</u> if A is the infinitessimal generator of a strongly continuous semigroup $\{e^{At}\}$ defined on $\ell_2(\mathbb{Z}^+)$ and b and c are in $\ell_2(\mathbb{Z}^+)$. We also consider cases where A is the infinitesimal generator of a semigroup and b is restricted to belong to the domain of A (written $D_o(A)$) but **c** is merely required to map $D_o(A)$ into $\mathbb{R}$ with $|c(x)| < k(\|Ax\|+\|x\|)$ for all x in $D_o(A)$. Such realizations will be called <u>balanced realizations</u>. They have important properties not shared by regular realizations.

In order to describe what realizations realize what systems we need to introduce some notation. The open disk of radius ρ is denoted by $\mathbb{D}_\rho=\{s:|s|<\rho\}$. We write $\mathbb{D}$ for $\mathbb{D}_1$. The boundary of $\mathbb{D}^\rho$, the unit circle, is denoted by $\mathbb{T}$. By $H^2(\mathbb{D})$ we mean the set of complex functions which are analytic in $\mathbb{D}$ and have a Taylor series about zero with square summable coefficients. The space $H^2(\mathbb{D}_\rho)$ is defined by saying that $\psi(s)$ belongs to $H^2(\mathbb{D}_\rho)$ if and only if $\psi(s/\rho)$ belongs to $H^2(\mathbb{D})$. By $H^2(\mathbb{T})$ we mean the set of complex valued functions which are defined and square integrable, in the Lebesgue sense, on the unit circle. $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are related by the fact that for any $H^2(\mathbb{D})$ function the radial limits from within the disk

$$\lim_{r\to 1}\psi(re^{i\theta})=\phi(\theta)$$

exist for almost all θ and give an element ϕ of $H^2(\mathbb{T})$. This correspondence is, moreover, one to one and onto so that $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are closely related indeed. In fact, the Fourier coefficients of ϕ are the Taylor coefficients of ψ. In addition, $H^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle\psi_1,\psi_2\rangle=\int_{\mathbb{D}}\psi_1^*(s)\psi_2(s)d\mu(s)$$

where $\mu(s)$ is Lebesgue measure in the plane, normalized to give $\mathbb{D}$ measure 1. This makes $H^2(\mathbb{D})$ and $\ell_2(\mathbb{Z}^+)$ isomorphic as Hilbert spaces with the isomorphism defined by

$$(a_1,a_2,a_3,\ldots)\longmapsto\sum_{i=1}^{\infty}a_is^i$$

We denote by $H_0^2(\mathbb{D}_\rho)$ the subset of $H^2(\mathbb{D}_\rho)$ consisting of those functions which vanish at 0. We say that $\psi(s)$ belongs to $\tilde{H}^2(\mathbb{D}_\rho)$ if $\psi(1/s)$ belongs to $H^2(\mathbb{D}_\rho)$. By $\tilde{H}^2_\infty(\mathbb{D}_\rho)$ we mean the subset of $\tilde{H}^2(\mathbb{D}_\rho)$ consisting of those functions which vanish at ∞.

We denote by Π_ρ^+ the half-plane Re $s>\rho$. We understand by $H^2(\Pi^+)$ the functions which are analytic in Π_ρ^+ and square integrable along vertical lines in Π_ρ^+ such that

$$\sup_{x>\rho}\int_{-i\infty}^{i\infty}|\psi(x+iy)|^2dy\le M<\infty$$

The relationship between $H^2(\mathbb{D})$ and $H^2(\Pi^+)$ is this: $\phi(s)\in H^2(\Pi^+)$ if and only if ψ defined by

$$\psi(s)=\frac{1}{s-1}\,\phi\left(\frac{s+1}{s-1}\right)$$

belongs to $H^2(\mathbb{D})$. It may be recalled that $H^2(\Pi^+)$ plays an important role in the Paley-Wiener Theorem on the Fourier Integral.

The connections between realizability and H^2 functions is outlined in the following table. The time domain characterization is also given, and is to be regarded as an equality between the function $e^{-\rho t}T(t)$ and $\langle c,e^{At}b\rangle$ in the $L_2[0,\infty)$ sense.

Bounded	General
$\tilde{T}(s)\in\tilde{H}^2_\infty(\mathbb{D}_\rho)$	$\tilde{T}(s),(s\tilde{T}(s)-T(0))\in H^2(\Pi_\rho^+)$ (suf) $\tilde{T}(s)\in H^2(\Pi_\rho^+)\cap H^\infty(\Pi_\rho^+)$ (nec)
$\{\tilde{T}^{(n)}(0)/\rho^n\}\in\ell_2(\mathbb{Z}^+)$ T(t) real analytic	$\dot{T}(t)$ exists; exp order (suf) T(t) cont; exp order (nec)

As we were preparing this paper we received from Paul Fuhrmann a manuscript [13] which analyzes the bounded case and obtains a number of the results described here with certain small changes due to the fact that he works with discrete time systems. Helton [14] also investigates some questions of this type but emphasizes a different class of ideas.

*This work was supported by the U.S. Office of Naval Research under the Joint Services Electronics Program by Contract N00014-67-A-0298-0006.

2. Realizability Criteria, Bounded Case

Let $T:[0,\infty)\to \mathbb{R}^1$ be a continuous function of time. When can it be written as

$$T(t) = \langle c, e^{At} b\rangle$$

where b, $c \in \ell_2(\mathbb{Z}^+)$ and $A:\ell_2(\mathbb{Z}^+)\to\ell_2(\mathbb{Z}^+)$ is bounded? As is well known such a representation is possible for T with (A,b,c) all finite dimensional if and only if T is of exponential order and its transform

$$\tilde{T}(s) = \int_0^\infty e^{-st} T(t)dt \quad ; \quad \mathrm{Re}\ s > \sigma_o$$

is rational. In the present case A is bounded; $\{e^{At}\}$ defines a uniformly continuous semigroup of operators (see [1], page 626), and since b, and c belong to $\ell_2(\mathbb{Z}^+)$ we have

$$\langle c, e^{At} b\rangle \le ||b||\cdot||c||\cdot M\cdot e^{k\ t} \quad , t\in\mathbb{R}^1$$

where $||e^{At}|| \le Me^{k\ t}$ and the norms are $\ell_2(\mathbb{Z}^+)$ and induced $\ell_2(\mathbb{Z}^+)$ respectively. Thus the class we are looking for includes only functions of exponential order. Moreover since A is bounded, $\langle c,e^{At}b\rangle$ is an entire function.

Regarding $\langle c,e^{At}b\rangle$ as a function of a complex variable, we can expand it as

$$\langle c,e^{At}b\rangle = \langle c,b\rangle + t\langle c,Ab\rangle + (t^2/2)\langle c,A^2b\rangle + \ldots$$

Now if $||A|| < k$ then

$$|\langle c,A^i b\rangle| < ||c||\cdot||b||\cdot k^i$$

Thus $\langle c,e^{At/k}b\rangle$ has a power series expansion valid in the finite complex plane with the coefficients in the power series being square summable. The following two theorems characterize in the time and frequency domain the set of realizable input-output maps. (Compare with Fuhrmann, Theorem 2.6.)

Theorem 1: $T: [0,\infty)\to\mathbb{R}^1$ can be written as $T(t)=\langle c,e^{At}b\rangle$ if and only if T is an entire function such that if

$$T(t) = \sum_{n=0}^{\infty} c_n t^n$$

then $\{\frac{n!c_n}{k^n}\}_{n=0}^{\infty} \in \ell^2(\mathbb{Z}^+)$ for some positive finite k.

Proof: The necessity follows from the above. For the sufficiency take

$$A = k\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & 0 & 1 & \ddots \\ & & 0 & \ddots \end{bmatrix}$$

$$b = \{1,0,0,\ldots\}$$

$$c = \{c_o,\ \frac{c_1}{k},\ldots \frac{n!c_n}{k^n},\ldots\}$$

Using now Laplace-transform in the complex domain we pass from the equation $T(t)=\langle c,e^{At}b\rangle$ to the equation $\tilde{T}(s)=\langle c,(Is-A)^{-1}b\rangle$, for $\mathrm{Re}\ s > ||A||$. Moreover $\tilde{T}(s)$ is analytic in $\mathrm{Re}\ s > ||A||$. Since A is bounded using an elementary analytic continuation argument we see that $\tilde{T}(s)$ is analytic for $|s| \ge ||A||$ and also that $\tilde{T}(\infty) = 0$. Hence $\tilde{T}(s) = \langle c,b\rangle s^{-1} + \langle c,Ab\rangle s^{-2} + \langle c,A^2b\rangle s^{-3} + \ldots$ for $|s| > ||A||$. For any $k > ||A||$ then we have as before that the sequence $\langle c,A^i b\rangle/k^i$ is square summable.

Theorem 2: The function $T(t)=[0,\infty)\to\mathbb{R}^1$ can be written as $\langle c,e^{At}b\rangle$ if and only if the Laplace transform $\tilde{T}(s)$ of $T(t)$ is analytic outside a disk of finite radius k and vanishes at infinity, such that if

$$\tilde{T}(s) = \sum_{i=0}^{\infty} a_i s^{-(i+1)} \quad \text{for} \quad |s| > k$$

then $\{\frac{|a_i|}{k^i}\}_{i=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$.

Proof: The necessity follows clearly from the above. For the sufficiency take again

$$A = k\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & 0 & 1 & \ddots \\ & & 0 & \ddots \end{bmatrix}$$

$$b = \{1,0,\ldots\}$$

$$c = \{a_o, a_1/k, a_2/k^2,\ldots\}$$

It is clear from the above that the singularities of $\tilde{T}(s)$ must be contained in a disk of finite radius. Moreover since $\tilde{T}(s)$ is analytic at infinity all the branch cuts should be considered in the finite plane. An interesting question is how small k can be taken? It follows from a theorem in Widder's, "Laplace transform" that if σ_o is the exponential order (or type) of the entire function $T(t)$ then $\tilde{T}(s)$ will be analytic for $|s| > \sigma_o$ and will vanish at infinity, and conversely. Hence we have immediately that the k in Theorems 1 and 2 must satisfy always $k \ge \sigma_o$.

Corollary 2.1: A function $T(t):[0,\infty)\to\mathbb{R}^1$ can be written as $\langle c,e^{At}b\rangle$ if and only if $\tilde{T}(s)\in H^2_\infty(D_\rho)$ for some positive finite ρ.

The triplet (A,b,c) is called as usual a realization for the weighting pattern $T(\tau,\sigma)=T(\tau-\sigma)=T(t)$ iff $T(t)=\langle c,e^{At}b\rangle$. The system-theoretic interpretation of this equation in terms of "external" and "internal" descriptions of time-invariant linear systems with scalar input, scalar output is considered here to be well-known. The fact that we are using $\ell_2(\mathbb{Z}^+)$ as our state-space is not very restrictive, since any separable Hilbert space is isometrically isomorphic to $\ell_2(\mathbb{Z}^+)$. The correspondence between $\ell_2(\mathbb{Z}^+)$ and $H^2(\mathbb{T})$ has another merit. Namely the shift acting on $\ell_2(\mathbb{Z}^+)$ corresponds to multiplication by s in $H^2(\mathbb{T})$.

It is apparent from the above that if $T(t)$ has any realization, then it can be realized by a multiple of the unilateral shift in $\ell_2(\mathbb{Z}^+)$ or by a multiple of the bilateral shift in $\ell_2(\mathbb{Z})$. To see the latter take

$$b = \{\ldots,0,1,0,0,\ldots\},\quad c = \{\ldots,0,c_o,\ \frac{c_1}{k},\ldots \frac{n!c_n}{k^n},\ldots\}$$

and

$$A = k\begin{bmatrix} \ddots\ 0 & & & \\ \ddots\ 1 & 0 & & \\ \ddots\ 0 & 1 & 0 & \\ & 0 & 1 & \ddots \\ & & 0 & \ddots \end{bmatrix}$$

(the c_i's as in Theorem 1). This is not surprising in view of the fact that the shift can be considered as a "universal model" for bounded operators in Hilbert spaces (c.f. [9]).

3. Realizability Criteria, General Case

In this section we describe the functions $T(t)$ which can be written as $c[e^{At}b]$, where e^{At} denotes a strongly continuous semigroup of bounded operators on $\ell_2(\mathbb{Z}^+)$ with infinitesimal generator A, $b \in D_o(A)$, $|c[x]| \le k(||Ax|| + ||x||)$ for $x \in D_o(A)$. By the Hille-Yosida Theorem [18] a necessary and sufficient condition that a closed linear operator A with domain $D_o(A)$ dense, be the infinitesimal generator of a strongly continuous semigroup, is that there exist real numbers M and β such that for every real $\lambda > \beta$, λ is in the resolvent set of A and

$||(I\lambda-A)^{-n}|| \le \frac{M}{(\lambda-\beta)^n}$ (n=1,2,...). If these conditions hold for all $\lambda > \beta$ then $(Is-A)^{-1}$ exists for all complex s with Re s > β and is given by $(Is-A)^{-1}x=\int_0^\infty e^{-st}S(t)x\,dt$, for all $x\in\ell_2(\mathbb{Z}^+)$, $||(Is-A)^{-n}|| \le M/(\text{Re } s-\beta)^n$ for Re s > β, and $||S(t)|| \le Me^{\beta t}$.

Theorem 3: A necessary condition for T(t) to have a balanced realization is to be continuous and of exp. order. A sufficient condition is that $\dot{T}(t)$ exist and be of exp. order.

Proof: (Necessity) Let $T(t)=c[e^{At}b]$. Then by the definition of a balanced realization $b\in D_o(A)$ and $|T(t)|\le k(\|Ae^{At}b\|+\|e^{At}b\|)$. Since e^{At} is strongly continuous this proves continuity. By the Hille-Yosida Theorem $|T(t)| \le k\, Me^{\beta t}(\|Ab\|+\|b\|)$. (Sufficiency) Let $\dot{T}(t)$ be as in hypothesis. Then for large enough σ, $e^{-\sigma t}\dot{T}(t)\in L_2[0,\infty)$ Hence the function $e^{-\sigma t}T(t)$ is in $L_2[0,\infty)$, its derivative exists and is in $L_2[0,\infty)$. We will use the fact that the Laguerre functions satisfy the recurrence relation:

$$\dot{\psi}_k=-\frac{1}{2}\psi_k-\psi_{k-1}-\psi_{k-2}\cdots-\psi_o \quad ; \quad \psi_n(0) = 1$$

Consider the one parameter semigroup defined by:

$$S(t)=\begin{bmatrix} \psi_o(t) & \psi_1(t)-\psi_o(t) & \psi_2(t)-\psi_1(t) & \cdots \\ 0 & \psi_o(t) & \psi_1(t)-\psi_o(t) & \cdots \\ 0 & 0 & \psi_o(t) & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This clearly satisfies $S(t)S(\tau)=S(t+\tau)$ for $t, \tau \ge 0$ and is the identity at zero. We see from the above recurrence relation that

$$\dot{S}(t) = \begin{bmatrix} -1/2 & -1 & -1 & -1 & \cdots \\ 0 & -1/2 & -1 & -1 & \cdots \\ 0 & 0 & -1/2 & -1 & \cdots \\ 0 & 0 & 0 & -1/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} S(t)$$

Call the matrix appearing here A_o. If x is an $\ell_2(\mathbb{Z}^+)$ vector whose entries are the Laguerre coefficients of a function in $L_2[0,\infty)$, whose derivative is also in $L_2[0,\infty)$, then from the above recurrence relation we see that $A_o x$ is the vector of Laguerre coefficients for the derivative. Thus A_o is the differentiation operator and from [18] we know it is a closed operator with domain dense as described above, that it generates the semigroup of translations and that $\sigma(A_o)=\{s;\text{Re } s\le 0\}$. Now to complete the proof, we let c be the vector $\{1,1,1,\ldots\}$, and b be the vector of Laguerre coefficients of $e^{-\sigma t}T(t)$. For A we take $A_o+\sigma I$, the σI accounts for the multiplication by $e^{\sigma t}$.

Theorem 4: A necessary condition for $\tilde{T}(s)$ to have a balanced realization is to belong in $H^2(\Pi_\rho^+)\cap H^\infty(\Pi_\rho^+)$ for some $\rho > 0$. A sufficient condition is that $\tilde{T}(s)\in H^2(\Pi_\rho^+)$ and $(s\tilde{T}(s)-T(0))\in H^2(\Pi_\rho^+)$ for some $\rho > 0$.

Proof: This is an immediate consequence of Theorem 3, of the Paley-Wiener Theorem [7] and the Hille-Yosida Theorem.

Example: The delayed step whose transform is e^{-s}/s is not realizable whereas the delayed ramp e^{-s}/s^2 is realizable.

Remark: Let T(t) be continuous and of exponential order let b be the vector of Laguerre coefficients of $e^{-\sigma t}T(t)$, where σ is large enough. Let c_λ be the vector of Laguerre coefficients of $\frac{2}{\pi}\frac{\lambda}{\lambda^2+\tau^2} = h_\lambda(\tau)$. Then by Theorem 9.9 in [19] we have

$$e^{-\sigma t}T(t)=\lim_{\lambda\to 0} \langle c_\lambda,e^{A_o t}b\rangle =$$
$$= \lim_{\lambda\to 0}\int_0^\infty e^{-\sigma(t+\tau)}T(t+\tau)h_\lambda(\tau)d\tau$$

Hence $T(t) = \lim_{\lambda\to 0}\langle c_\lambda,e^{At}b\rangle$, where $A=\sigma I+A_o$. Hence T(t) can be written as the pointwise limit of a one parameter family of regular realizations.

4. Canonical Realizations

In the rest of this paper we will restrict our study to weighting patterns T(t) with bounded regular realization. Moreover we assume that T(t) satisfies the realizability criteria for k = 1 or equivalently can be realized with A such that $||A|| \le 1$. This does not harm the generality of the discussion, since using the corollary of Theorem 2 we can reduce the general case where $\sigma_o \le k$ (σ_o being the exp. order of T(t)), by a simple change of variable, to the above case. Indeed if we define $\tilde{T}_k(s) = \sum_{i=0}^\infty \frac{a_i}{k^i}s^{-(i+1)}=kT(ks)=\mathcal{L}[T(\frac{t}{k})]$, where $\mathcal{L}$ denotes Laplace transform, then corollary 2.1 says that, T(t) can be written as $\langle c,e^{At}b\rangle$ if and only if $\tilde{T}_k(1/s) \in H_o^2(\mathbb{D})$.

It is obvious that if a weighting pattern has one realization it has many. An element ϕ of a separable Hilbert space $\mathcal{H}$ is called a cyclic vector for a bounded operator A if and only if the linear span of $\phi, A\phi, A^2\phi, A^3\phi,\ldots$ is dense in $\mathcal{H}$. One calls a realization (A,b,c) canonical whenever b is a cyclic vector for A, and c is a cyclic vector for A*. However we askew the term "minimal" because many of the implications of this term are absent in the present setting. (Some authors prefer to call such a realization "controllable and observable" or "ε-controllable and ε-observable" or "weakly controllable.") If a weighting pattern T(t) has a finite dimensional realization, it has one with minimal dimension of the state space, which is called minimal. As is well known (see [2], pages 105-115) a finite dimensional system is minimal if and only if it is controllable and observable (canonical). Moreover any two minimal realizations differ by a change of basis in the state space and the spectral properties of A in any minimal realization are uniquely determined by the weighting pattern. Here the situation is much more complicated. It happens that a canonical realization is much more loosely specified by the weighting pattern.

We start with a construction of a canonical realization starting from a given one.

Theorem 5: Let $T(t) = \langle c,e^{At}b\rangle$. Let M be the closed linear span of c, A*c, $A^{*2}c$,... in $\mathcal{H}$ (a sep. Hilbert space.) Let P_M be the orthogonal projection on M.

Then (i) $T(t) = \langle c,e^{P_MAP_Mt}P_Mb\rangle$. Let now N be the closed linear span of $P_Mb,\ldots,(P_MAP_M)^iP_Mb,\ldots$ in M and let P_N be the orthogonal projection on N.

Then (ii) $T(t) = \langle P_Nc,e^{(P_NAP_N)t}P_Mb\rangle$. Moreover N is the closed linear span of $P_Mb,\ldots,(P_NAP_N)^iP_Mb\ldots$ and the closed linear span of $P_Nc,\ldots(P_NAP_N)^{*i}P_Nc$

Proof: It is obvious that M is the smallest closed subspace of $\mathcal{H}$ which contains c and is invariant under A*. Hence $M^\perp$ is invariant under A. Hence $A(I-P_M)x\in M^\perp$, $\forall x \in \mathcal{H}$. So $P_MA(I-P_M)x = 0$, $\forall x \in \mathcal{H}$. Hence

$$P_MA = P_MAP_M \qquad (1)$$

Using (1) we get $(P_MAP_M)^iP_Mb = P_MA^ib$.

Hence $<c, e^{P_M A P_M t} P_M b> = <c, P_M e^{At} b> = <c, e^{At} b> = T(t)$ and this proves (i). Similarly N is the smallest closed subspace of M which contains $P_M b$ and is invariant under $P_M A P_M$. Then for every $x \in \mathcal{H}$

$$(I-P_N)P_M A P_M P_N x = 0$$

So

$$P_N A P_N = P_N P_M A P_M P_N = P_M A P_M P_N = P_M A P_N \qquad (2)$$

Using (1), (2) we obtain

$$(P_M A P_M)^i P_M b = P_M A^i P_M b = (P_N A P_N)^i P_M b \qquad (3)$$

and this proves (ii) similarly as above.

The first assertion in the last statement is proved by (3). The second is an easy consequence of (2) and of the cyclicity of c for A^*.

Here, as we assumed in the beginning of this part, if T(t) is realizable it can be realized by the shift (unilateral or bilateral). Some important questions which arise naturally are the following. It is obvious that the realization given by Theorem 1 and 2 is controllable. Also we know that the spectrum of the unilateral shift is $\mathbb{D}$. Given a weighting pattern T(t), how simple can the spectrum of the infinitesimal generator A of a realization be? How small can the spectrum be? If we take a canonical realization (A,b,c) is the spectrum of A uniquely determined by T(t)? How are all canonical realizations of a given T(t) related to each other? When can we make the resolvent set of the infinitesimal generator A connected?

An immediate observation, which gives, however, some indication of the interplay of the notions described in these questions is the following: We can realize any such T(t) by the bilateral shift. Such a realization is obviously non-canonical. On the other hand since the spectrum of the bilateral shift is just $\mathbb{T}$, the spectrum can be considered as "simple." However the resolvent set is not connected; a property which has very important implications as far as the relationship to frequency response methods for system identification is concerned.

If we let $\sigma(\tilde{T}(s)) = \{s \in \mathbb{C} | \tilde{T}(s)$ is not analytic at $s\}$. We see immediately from the equation $\tilde{T}(s) = <c,(Is-A)^{-1}b>$ that for any realization (A,b,c) of T(t) we must have:

$$\sigma(\tilde{T}(s)) \subseteq \sigma_e(A)$$

This relation will be called in the sequel the "spectral inclusion property."

Given a weighting pattern T(t) we have the "shift realization" as described in Theorems 1 and 2

$$\frac{d}{dt} x(t) = Ux(t) + bu(t)$$
$$y(t) = <c, x(t)>$$

where $x(t) \in \ell_2(\mathbb{Z}^+)$ for all t, $b = \{1,0,0,\ldots\}$, U is the unilateral shift and $c = \{T(0), T^{(1)}(0), T^{(2)}(0), \ldots\}$. Here b is obviously a cyclic vector for U. It is immediately seen as a consequence of Theorem 5, that if we let M be the closed linear span of $c, U^*c, \ldots, U^{*i}c, \ldots$ and P_M the projection on M, then $(P_M U P_M, P_M b, c)$ is a canonical realization of T(t), with state space M. We can write the "shift realization" in terms of $H^2(\mathbb{T})$ functions as follows:

$$\frac{d}{dt} x(t,s) = sx(t,s) + u(t)$$
$$y(t) = \int_{\mathbb{T}} s\tilde{T}(s) x(t,s) d\mu(s)$$

where $x(\cdot, s) \in H^2(\mathbb{T})$. (Compare with [17] where similar equations are used.) Under the isomorphism between $\ell_2(\mathbb{Z}^+)$ and $H^2(\mathbb{T})$, c corresponds to $\frac{1}{s}\tilde{T}(\frac{1}{s})$ which equals with $\overline{s\tilde{T}(s)}$ on $\mathbb{T}$ (since $\tilde{T}(s)$ has real Taylor coefficients).

We need a few well-known facts from the theory of H^2 functions and Toeplitz operators. The reader is referred to [7], [8], and [10] for further details. A function $f \in H^2(\mathbb{T})$ is called inner if $|f(s)| = 1$ a.e. A function $f \in H^2(\mathbb{T})$ is called outer if it is a cyclic vector for the shift in $H^2(\mathbb{T})$. (i.e. the linear span of the functions f, sf, $s^2 f, \ldots$ is dense in $H^2(\mathbb{T})$. A Blaschke product is a function of the form

$$B(s) = s^k \prod_{j=1}^{\infty} \frac{a_j - s}{1 - \bar{a}_j s} \frac{\bar{a}_j}{|a_j|}$$

where k is nonnegative integer and the a_j are complex numbers (not necessarily distinct) such that $0 < |a_j| < 1$, $\sum_{j=1}^{\infty} (1-|a_j|) < \infty$. A singular function is a function of the form

$$S(s) = \exp\left(-\int \frac{e^{i\theta}+s}{e^{i\theta}-s} d\mu(\theta)\right)$$

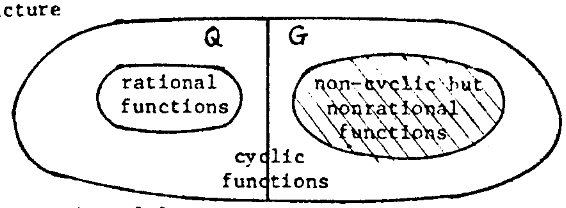
where μ is any positive finite measure on $[0,2\pi]$ which is singular with respect to the normalized Lebesque measure. Every $f \in H^2(\mathbb{T})$ has a factorization $f = \phi \cdot h$ where ϕ is inner and h is outer. The factors are unique up to constant factors of modulus one, B is a Blachke product, and S is a singular function. An inner function is normalized if we choose c = 1, or equivalently if we require the first non-zero Taylor coefficient to be real and positive. Beurling showed that to every closed subspace M of $H^2(\mathbb{T})$ which is invariant under the shift (i.e. under multiplication by s) there corresponds a unique normalized inner function Φ such that $M = \Phi H^2(\mathbb{T})$ and conversely. We have also the corresponding facts for $H^2(\mathbb{D})$.

A Laurent operator on $\ell_2(\mathbb{Z})$ has a matrix representative which is constant on diagonals (i.e. $\alpha_{ij} = a_{i-j}$) and corresponds to multiplication by $\phi(s) = \sum_{i=-\infty}^{\infty} a_i s^i$ on $L^2(\mathbb{T})$ (where $a_i = \alpha_{i+k,k}$). A Toeplitz operator A, on $\ell_2(\mathbb{Z}^+)$ has a similar matrix representative (which is infinite in one only direction) and if $P: L^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the associated projection; $\forall f \in H^2(\mathbb{T})$ we have

$$Af = P(\phi \cdot f) \quad .$$

The only way the "shift realization" can be canonical is if c is a cyclic vector for U^*, (i.e. for the backward shift) or equivalently if $\tilde{T}(1/s)/s$ is a cyclic vector for the backward shift on $H^2(\mathbb{T})$. (See also Fuhrmann Theorem 2.6). In [5] the authors studied cyclic vectors of the backward shift very extensively. We are going to use some of their results and we refer to [5] for further details. There exist many cyclic vectors for the backward shift on $H^2(\mathbb{T})$, as well as non-cyclic ones. The rational functions are non-cyclic. The authors give several ways of constructing cyclic vectors. Any H^2 function with isolated branch points on $\mathbb{T}$ is a cyclic vector and also any function with lacunary Taylor series and square summable Taylor coefficients is also a cyclic vector. Since $f(s) \in H^2(\mathbb{T})$ is a cyclic vector for the backward shift if and only if sf(s) is one, we have two cases to consider. Namely the case when $\tilde{T}(1/s)$ is a cyclic vector for the backward shift and the case when $\tilde{T}(1/s)$ is non-cyclic.

We would like to close this part with some important remarks about the cyclic and non-cyclic case. Let Q be the subset of the realizable transfer functions, for elements of which the set of reals k, such that $\tilde{T}(k/s)$ is in $H^2(\mathbb{D})$ is open. Let G be the complement of Q in the set of realizable functions. Theorem 2.2.4 in [5] reads as follows: "If f is holomorphic in $|s| < R$ for some $R > 1$, then f is either cyclic or a rational function." Since $k > \sigma_0$ for elements of Q, an immediate consequence of the above theorem is that the elements of Q are either cyclic or rational functions. Then in G we have either cyclic or non-cyclic but not rational functions. So we have the picture



Also from [5] we have that the set of cyclic vectors is dense in $H^2(\mathbb{T})$ as is the set of non-cyclic vectors. However the set of non-cyclic vectors is a set of the first category, whereas the set of cyclic vectors is not. Hence the non-cyclic vectors are somehow much more rare than the cyclic ones. Moreover an element of $H^2(\mathbb{T})$ is non-cyclic if and only if there exist a sequence of rational functions (satisfying special conditions see [5] Theorem 4.1.1) which converges to it in the $L^2(\mathbb{T})$ norm. A fact which indicates that the non-cyclic case is very much like the rational functions whereas the cyclic situation is new, harder, and potentially more interesting.

5. The Non-Cyclic Case

Now we consider the case where $\tilde{T}(1/s)$ is not a cyclic vector for the backward shift. This case is treated by Fuhrmann [13] in detail, however there are some additional facts given here about the spectrum of A.

To proceed we need the following theorem from [5], p. 56.

Theorem 6 ([5]): $f(s)\epsilon H^2(\mathbb{T})$ is non-cyclic if and only if there exists $g(s)\epsilon H^2(\mathbb{T})$ and an inner function ϕ such that $f(e^{i\theta})=e^{-i\theta}\overline{g(e^{i\theta})}\phi(e^{i\theta})$ a.e. on $\mathbb{T}$. Moreover if we require that ϕ be normalized and relatively prime to the inner factor of g, then ϕ and g are uniquely determined. In this case the closed subspace generated by $U^{*n}f$, $n=0,\ldots\infty$ is precisely $(\phi H^2(\mathbb{T}))^{\perp}$.

The normalized inner function ϕ thus uniquely associated with each non-cyclic (for the backward shift) vector f is called the "associated inner function" of f.

We see immediately that the subspace M of $\ell^2(\mathbb{Z}^+)$ which is the state space for the canonical realization (P_MUP_M, P_Mb, c) derived from the "shift realization" corresponds to the closed subspace of $H^2(\mathbb{T})$ generated by $\{U^{*n}\frac{1}{s}\tilde{T}(\frac{1}{s})\}_{n=0}^{\infty}$ which we call also M. Hence applying Theorem 6 we get that $M=(\phi H^2(\mathbb{T}))^{\perp}$ where $\phi(e^{i\theta})\overline{g(e^{i\theta})}=$ $=\tilde{T}(e^{-i\theta})=\overline{\tilde{T}(e^{i\theta})}$ a.e. on $\mathbb{T}$ (since $\tilde{T}(e^{i\theta})$ has real Fourier coefficients), and ϕ and g are uniquely determined by Theorem 6.

We need another theorem now from [6].

Theorem 7 ([6]): Let $K = \phi H^2(\mathbb{T})$ (i.e. K is a closed subspace of $H^2(\mathbb{T})$ invariant under the shift U). Let $M=(\phi H^2(\mathbb{T}))^{\perp}$. Then the spectrum of U restricted on M is the set s_ϕ which consists of

(i) all the points in $\mathbb{C}$ with $|\lambda|<1$, where $\phi(\lambda)=0$

(ii) all the points in $\mathbb{C}$ with $|\lambda|=1$, where $\phi(s)$ is not continuable analytically across the boundary $\mathbb{T}$ of $\mathbb{D}$ at λ.

Using theorems 6, 7 we see that the spectrum of the infinitesimal generator of the canonical realization (P_MUP_M, P_Mb, c) is uniquely determined by T(t). Namely the spectrum consists of the zeros of ϕ in $\mathbb{D}$, which coincide with the zeros of the Blaschke product part of ϕ, and the points of $\mathbb{T}$ through which ϕ is not continuable analytically outside the unit circle, which coincide with the union of the support of the measure on $\mathbb{T}$ which is associated with the singular part of ϕ and the set of points of $\mathbb{T}$ which are accumulation points of the sequence of zeros of ϕ. (See [7] p. 68-69)

When $\tilde{T}(s)$ has a meromorphic continuation in $\mathbb{D}$, it is easy to prove (using corollaries 3.1.8 and 3.1.10 p. 58-59 of [5]), that $s_\phi = \sigma(\tilde{T}(s))$.

So in this case we arrive at the conclusion that:

$$\sigma(\tilde{T}(s)) = \sigma(P_MUP_M)$$

We have thus proved the following:

Theorem 8: Let T(t) be a given weighting pattern which satisfies Theorem 1 (or Theorem 2) (with k=1) such that $\tilde{T}(1/s)$ is not a cyclic vector for the backward shift on $H^2(\mathbb{T})$. Then there exist a canonical realization of T(t) with the spectrum of the infinitesimal generator of the realization being exactly s_ϕ, where ϕ is the associated inner factor for $e^{-i\theta}\tilde{T}(e^{i\theta})$. If (U,b,c) is the "shift realization" for T(t) and M is the subspace of $\ell_2(\mathbb{Z}^+)$ defined in Theorem 5, the above realization is (P_MUP_M, P_Mb, c). Moreover if $\tilde{T}(s)$ has a meromorphic continuation in $\mathbb{D}$ this spectrum is just $\sigma(\tilde{T}(s))$.

We see that in the above case the "spectral inclusion property" becomes in fact an equality, i.e. the spectrum of the infinitesimal generator of the realization described in Theorem 6 is "minimal." This motivates the following definition:

Definition: A canonical realization (A,b,c) of a weighting pattern T(t) is called S-minimal (S from spectrum) if $\sigma(A) = \sigma(\tilde{T}(s))$ (multiplicities counted whenever possible).

These considerations lead us to a trivial corollary of Theorem 8:

Corollary 8.1: Any T(t) which has the "shift realization" and is such that $\tilde{T}(1/s)$ is not a cyclic vector of the backward shift on $H^2(\mathbb{T})$, and $\tilde{T}(s)$ has a meromorphic continuation in $\mathbb{D}$ has an S-minimal realization, with A having a connected resolvent set.

We do not have a complete picture for the relation between canonical (resp. S-minimal) realizations of the same weighting pattern T(t), in this case. However a partial analysis indicates that the non-cyclic case is very similar to the rational case.

6. The Cyclic-Case

The cyclic case, is very interesting since it reflects a number of physically interesting phenomena; e.g. transfer functions with branch points and branch cuts. Transfer functions like these arise in systems governed by partial differential equations; hence an understanding of the cyclic case should undoubtedly shed some light towards the realization problem for distributed systems.

This case is more difficult, since the associated inner factor of $\tilde{T}(s)$ which proves so crucial in the noncyclic case is now trivial. That is, the shift realization for cyclic transfer functions is already canonical. However the spectrum of this realization is far from being equal to $\sigma(\tilde{T}(s))$, unless we have a pathological transfer function with branch points on a dense subset of $\mathbb{T}$. Hence canonical by no means implies $\mathbf{S}$-minimal. (Again compare with Fuhrmann [13] Corollary 2.7 who observes the nonuniqueness of the spectrum.)

It is apparent from the spectral inclusion property that all the points on the branch cuts (if the transfer function has branch points) are included in the spectrum of any infinitesimal generator A which realizes the transfer function. Moreover the branch cuts are not uniquely defined. Hence the spectrum of $\tilde{T}(s)$ is not uniquely determined and consequently there is not a unique "minimal spectrum" for the infinitessimal generators of the realizations. A reasonable expectation is that the spectrum of an S-minimal realization (provided it exists) will be unique if there are no branch points, and otherwise will be unique modulo the branch cuts.

We conclude this part with an example of a realization for the Bessel function of zeroth order $\mathcal{J}_0(t)$ which achieves the spectrum of $\tilde{\mathcal{J}}_0(s)$ exactly. It is easy to verify that $\mathcal{J}_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (t/2)^{2m}}{m!m!}$ satisfies our realizability criteria. $\tilde{\mathcal{J}}_0(s) = \frac{1}{\sqrt{s^2+1}}$ has branch points at $\pm i$, hence $\tilde{\mathcal{J}}_0(s)$ is a cyclic vector. We must take the branch cut in the finite plane. We are after a realization whose infinitesimal generator has spectrum exactly the line between i and -i. Recalling that $(1/2)(s-1/s)t$ is a generating function for the Bessel functions of integral order, i.e. that

$$e^{(1/2)(s-1/s)t} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(t)s^n \quad ,$$

and using Laurent operators we see that for

$$A = \frac{1}{2}\begin{bmatrix} \ddots & -1 & & & 0 \\ \ddots & 0 & -1 & & \\ \ddots & 1 & 0 & -1 & \ddots \\ 0 & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t) & \mathcal{J}_{-2}(t) & \cdots \\ \cdot & \cdots & \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t) & \cdots \\ \cdot & \cdots & \mathcal{J}_2(t) & \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t) \cdots \\ & & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Hence the above A along with $b=c=\{\ldots 0\ 0\ 1\ 0\ 0\ \ldots\}$ give a realization for $\mathcal{J}_0(t)$, in $\ell_2(\mathbb{Z})$. That the spectrum of A is exactly $[i,-i]$ is a well known fact from [10].

We note also that the same trick will work for $\mathcal{J}_n(t)$, where $\tilde{\mathcal{J}}_n(s) = \frac{1}{\sqrt{s^2+1}} \left(\frac{1}{s+\sqrt{s^2+1}}\right)^n$ provided we keep A, b as above and take $c=\{\ldots 00100 \ldots\}$, with the 1 in the nth place.

References

1. Dunford and Schwartz, "Linear Operators," John Wiley, 1958.

2. R.W. Brockett, "Finite Dimensional Linear Systems," New York, Wiley, 1970.

3. A.V. Balakrishnan, "Introduction to Optimization Theory in a Hilbert Space," Springer-Verlag, 1970.

4. H.O. Fattorini, "On Complete Controllability of Linear Systems," Journal of Differential Equations 3, 391-402 (1967).

5. R.G. Douglas, H.S. Shapiro, A.L. Shields, "Cyclic Vectors and Invariant Subspaces for the Backward Shift Operator," Ann. Inst. Fourier, Grenoble 20, 1 (1971), 37-76.

6. J.W. Moeller, "On the Spectra of Some Translation Invariant Spaces," J. Math. Anal. Appl. 4 (1962) 276-296.

7. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice Hall, 1962.

8. H. Helson, "Lectures on Invariant Subspaces," Academic Press, 1964.

9. Gian-Carlo Rota, "On Models for Linear Operators," Comm. Pure and Appl. Math. 13 (1960), 468-472.

10. P. Halmos, "A Hilbert Space Problem Book," Van Nostrand, 1967.

11. Béla Sz-Nagy and C. Foias, "Harmonic Analysis of Operators on Hilbert Space," North Holland and American Elsevier, 1970.

12. B. Sz-Nagy and C. Foias, "Vecteurs cycliques et quasi-affinités," Studia Mathematica, 31 (1968) pp. 35-42.

13. P. Fuhrmann, "On Realization of Linear Systems and Applications to Some Questions of Stability," (to appear).

14. J.W. Helton, "Discrete time Systems, Operator Models, and Scattering Theory," (to appear).

15. B. Sz-Nagy and C. Foias, "Operateurs sans Multiplicité, Acta. Sci. Math. 30 (1969), pp. 1-18.

16. B. Sz-Nagy and C. Foias, "Modèle de Jordan pour une classe d'operateurs de l'espace de Hilbert," Acta Sci. Math. 31 (1970), 93-117.

17. R.W. Brockett and J.L. Willems, "Least Squares Optimization for Stationary Linear Partial Difference Equations," Proc. of the IFAC Symp. on The Control of Distributed Parameter Systems, Banff, Canada, 1971.

18. A. Friedman, "Partial Differential Equations," Holt, Rinehart, Winston, 1969.

19. W. Rudin, "Real and Complex Analysis," McGraw-Hill, 1966.