Detection of Markov Chain Models from Observed Data using Binary Hypothesis Testing and Consensus *

Aneesh Raghavan * John S. Baras **

* *Institute for Systems Research, University of Maryland, College Park, USA*
(e-mail: raghava@umd.edu).
** *Institute for Systems Research, University of Maryland, College Park, USA*
(e-mail: baras@umd.edu).

Abstract: In this paper we consider the problem of detecting which Markov chain model generates observed time series data. We consider two Markov chains. The state of the Markov chain cannot be observed directly, only a function of the state can be observed. Using these observations, the aim is to find which of the two Markov chains has generated the observations. We consider two observers. Each observer observes a function of the state of the Markov chains. We formulate a binary hypothesis testing problem for each observer. Each observer makes a decision on the hypothesis based on its observations. Then Observer 1 communicates its decision to Observer 2 and vice-versa. If the decisions are the same, then a consensus has been achieved. If their decisions are different then the binary hypothesis testing problem is continued. This process is repeated until consensus has been achieved. We solve the binary hypothesis testing problem and prove the convergence of the consensus algorithm. The "value" of the information gained through 1-bit communication is discussed along with simulation results.

Keywords: Hidden Markov Models, Hypothesis testing, Consensus , Value of Information.

1. INTRODUCTION

Hidden Markov Models (HMM) are models in which the state of the Markov chain cannot be observed directly, instead only a function of the state can be observed. These models are used in speech recognition, econometrics, computational biology and computer vision and many other fields (Cappé et al. (2005)). The hypothesis testing problems have been well studied in literature, one of the standard assumptions being that the observations are i.i.d. Hidden Markov models are instances of models in which observations have memory and hence are not i.i.d. Chen and Willett (2000) have formulated the problem of quickest detection of transient signals using hidden Markov models. They develop a procedure analogous to Pages test for dependent observations which can be applied to the detection of a change in hidden Markov modeled observations, i.e., a switch from one HMM to another. Lalitha et al. (2014) consider the problem where individual nodes in a network receive noisy observations whose distributions depend on the hypotheses. They analyze an update rule (for the belief of hypotheses), where each agent performs a Bayesian update based on local observations and a linear consensus among its neighbors. They prove that the belief of any agent in any incorrect hypotheses converges to zero exponentially fast.
Alanyali et al. (2004) address the problem where N sensors are observing an event and obtain noisy observations. The sensor network is modeled by a graph and the sensors are restricted to exchange messages alone. They characterize conditions under which the N sensors achieve consensus and derive conditions under which the consensus converges to the centralized MAP

estimate. Nayyar and Teneketzis (2011) consider the problem of sequential decentralized detection where each sensor makes repeated noisy observations of a binary hypothesis. At each time the peripheral sensors need to decide whether to continue making costly observations or to send a final decision to the the fusion center. The fusion center is also faced with a stopping problem and needs to take into account the decision of the peripheral sensors. They provide parametric characterization of the optimal policies for the peripheral sensors and fusion center. In this paper, we consider two Markov chains and two observers. Under the alternate hypothesis, each observer observes a different function of the state of the first Markov chain. Under the null hypothesis, each observer observes a different function of the state of the second Markov chain. Thus each observer has its own sequence of observations. Given two sequences of observations (one for each observer), the objective is to find if the sequences were generated under the null hypothesis or under the alternate hypothesis.

An example of this scenario would be when there are 2 cameras observing an environment/scene and have different perspectives / views of the scene. The elementary events in sample space could be defined based on the environment. Consider the problem where the environment has two states . The manner in which the scene or the environment changes in each state with time is Markovian. The images (or the observations in the present example) obtained by the cameras are functions of the states of the environment. Given the images we would like to arrive at a consensus on the state of environment.

For both observers , the hypothesis testing problem is formulated and solved as partially observed stochastic control problem. Thus both observers make individual decisions on the hypothesis. Then they communicate their decisions. If they
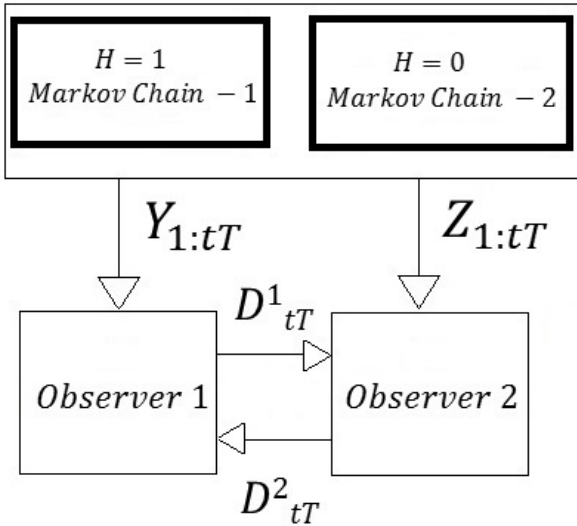
Fig. 1. Proposed Framework

have arrived at the same decision, then they have arrived at a consensus on the hypothesis though it could be wrong. If their decisions are different, then they collect more observations and repeat the hypothesis testing problem. This algorithm is repeated until consensus has been achieved. The convergence of this consensus algorithm has been proven. Figure 1 depicts the proposed framework.

To understand as to what was gained by the use of 2 observers and the 1 bit communications , the notion of value of information has been introduced. We define the value of information and perform simulations to obtain the value of information.

2. PROBLEM FORMULATION

2.1 System Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two systems are considered whose dynamics are described as follows : State of system 1 is described by a finite-state, homogeneous, discrete time Markov chain $X_k^1, k \in \mathbb{N}$. The distribution of X_0^1 is assumed to be known. State of system 2 is also described by a finite-state, homogeneous, discrete time Markov chain $X_k^2, k \in \mathbb{N}$ and the distribution of X_0^2 is assumed to be known. The state space of X_k^1 and X_k^2 is assumed to be have N_s elements and is identified by the set

$$S_X = \{e_1, ..., e_{N_s}\},$$

where e_i are unit vectors in $\mathbb{R}^{N_s}$ with unity as ith element and zeros elsewhere. Let $\mathcal{F}_k^1$ be the complete σ algebra generated by $\{X_0^1, ..., X_k^1\}$ and $\mathcal{F}_k^2$ be the complete σ algebra generated by $\{X_0^2, ..., X_k^2\}$. The Markov property implies that :

$$\mathbb{P}(X_{k+1}^1 = e_j | \mathcal{F}_k) = \mathbb{P}(X_{k+1}^1 = e_j | X_k^1),$$
$$\mathbb{P}(X_{k+1}^2 = e_j | \mathcal{F}_k) = \mathbb{P}(X_{k+1}^2 = e_j | X_k^2).$$

The transition matrices for the Markov chains can be defined as:

$$a_{ji}^1 = \mathbb{P}(X_{k+1}^1 = e_j | X_k^1 = e_i), \ A^1 = (a_{ji}^1) \in \mathbb{R}^{N_s \times N_s},$$
$$a_{ji}^2 = \mathbb{P}(X_{k+1}^2 = e_j | X_k^2 = e_i), \ A^2 = (a_{ji}^2) \in \mathbb{R}^{N_s \times N_s}.$$

Thus the Markov property also implies,

$$\mathbb{E}[X_{k+1}^1 | \mathcal{F}_k^1] = \mathbb{E}[X_{k+1}^1 | X_k^1] = A^1 X_k^1,$$
$$\mathbb{E}[X_{k+1}^2 | \mathcal{F}_k^2] = \mathbb{E}[X_{k+1}^2 | X_k^2] = A^2 X_k^2.$$

Define:

$$W_{k+1}^1 = X_{k+1}^1 - A^1 X_k^1, \ W_{k+1}^2 = X_{k+1}^2 - A^2 X_k^2.$$

So that

$$X_{k+1}^1 = A^1 X_k^1 + W_{k+1}^1, \ X_{k+1}^2 = A^2 X_k^2 + W_{k+1}^2.$$

H (signifying the hypothesis) is a Bernoulli random variable such that

$$\mathbb{P}(H = 1) = \bar{p}_1, \ \mathbb{P}(H = 0) = \bar{p}_0 = 1 - \bar{p}_1.$$

It is assumed that H, X_0^1 and X_0^2 are independent random variables. Let $\mathcal{F}_k = \sigma\{H, X_0^1, ..., X_k^1, X_0^2, ..., X_k^2\}$ denote the complete σ algebra generated by $X_0^1, ..., X_k^1, X_0^2, ..., X_k^2$. It is also assumed that:

$$\mathbb{E}[X_{k+1}^1 | \mathcal{F}_k] = A^1 X_k^1, \ \mathbb{E}[X_{k+1}^2 | \mathcal{F}_k] = A^2 X_k^2.$$

The state processes for these systems are not observed directly. Consider Observer 1, under $H = 1$, it observes a function $c^1(.,.)$ (with finite range) of X_k^1:

$$Y_{k+1}^1 = c^1(X_k^1, v_{k+1}^1), \ k \geq 0, \qquad (1)$$

where v_k^1 is a sequence of independent, identically distributed random variables. It is assumed that $\{v_k^1\}_{k \geq 1}$ are independent of $H, X_0^1, X_0^2, \{W_k^1\}_{k \geq 1}$ and $\{W_k^2\}_{k \geq 1}$. Similarly under $H = 0$, it observes a function $c^2(.,.)$ (with finite range) of X_k^2:

$$Y_{k+1}^2 = c^2(X_k^2, v_{k+1}^2), \ k \geq 0, \qquad (2)$$

where v_k^2 is a sequence of independent, identically distributed random variables. It is assumed that $\{v_k^2\}_{k \geq 1}$ are independent of $H, X_0^1, X_0^2, \{W_k^1\}_{k \geq 1}, \{W_k^2\}_{k \geq 1}$ and $\{\bar{v}_k^1\}_{k \geq 1}$. Let $\mathcal{G}_k^1$ denote the complete σ algebra generated by $H, X_0^1, ..., X_k^1, X_0^2, ..., X_k^2, Y_1^1, ..., Y_k^1, Y_1^2, ..., Y_k^2$. Without loss of generality, we can assume that range of $c^1(.,.)$ and $c^2(.,.)$ consists of M_1 points and identify it with set of unit vectors

$$S_Y = \{f_1^1, ..., f_{M_1}^1\},$$

where f_j^1 are unit vectors in $\mathbb{R}^{M_1}$ with unity as jth element and zeros elsewhere. (1) and (2) imply

$$\mathbb{P}(Y_{k+1}^1 = f_j^1 | \mathcal{G}_k^1) = \mathbb{P}(Y_{k+1}^1 = f_j^1 | X_k^1),$$
$$\mathbb{P}(Y_{k+1}^2 = f_j^1 | \mathcal{G}_k^1) = \mathbb{P}(Y_{k+1}^2 = f_j^1 | X_k^2).$$

The state to output transition matrices are defined as:

$$c_{ji}^1 = \mathbb{P}(Y_{k+1}^1 = f_j^1 | X_k^1 = e_i), \ C^1 = (c_{ji}^1) \in \mathbb{R}^{M_1 \times N_s},$$
$$c_{ji}^2 = \mathbb{P}(Y_{k+1}^2 = f_j^1 | X_k^2 = e_i), \ C^2 = (c_{ji}^2) \in \mathbb{R}^{M_1 \times N_s}.$$

Thus,

$$\mathbb{E}[Y_{k+1}^1 | \mathcal{G}_k^1] = \mathbb{E}[Y_{k+1}^1 | X_k^1] = C^1 X_k^1,$$
$$\mathbb{E}[Y_{k+1}^2 | \mathcal{G}_k^1] = \mathbb{E}[Y_{k+1}^2 | X_k^2] = C^2 X_k^2.$$

Define:

$$V_{k+1}^1 = Y_{k+1}^1 - C^1 X_k^1, \ V_{k+1}^2 = Y_{k+1}^2 - C^2 X_k^2.$$
$$Y_{k+1}^1 = C^1 X_k^1 + V_{k+1}^1, \ Y_{k+1}^2 = C^2 X_k^2 + V_{k+1}^2.$$

Hence when $H = 1$, Observer 1 is a discrete Hidden Markov Model (HMM) (under $\mathbb{P}$) and is defined by the state space equations :

$$X_{k+1}^1 = A^1 X_k^1 + W_{k+1}^1,$$
$$Y_{k+1}^1 = C^1 X_k^1 + V_{k+1}^1.$$

and when $H = 0$, it is again a discrete HMM (under $\mathbb{P}$) and is defined by the state space equations :

$$X_{k+1}^2 = A^2 X_k^2 + W_{k+1}^2,$$
$$Y_{k+1}^2 = C^2 X_k^2 + V_{k+1}^2.$$

Hence the observation equation for Observer 1, is given by :

$$Y_{k+1} = \left[(C^1 X_k^1 + V_{k+1}^1)H + (C^2 X_k^2 + V_{k+1}^2)(1 - H)\right],$$

where $X_k^1, X_k^2 \in S_X$ and $V_k^1, V_k^2 \in S_Y$, A^1, A^2, C^1, C^2 are matrices of transition probabilities. The entries satisfy

$$\sum_{j=1}^{N_s} a_{ji}^1 = 1, \quad \sum_{j=1}^{N_s} a_{ji}^2 = 1,$$

$$\sum_{j=1}^{M_1} c_{ji}^1 = 1, \quad \sum_{j=1}^{M_1} c_{ji}^2 = 1, \ c_{ji}^1 > 0, \ c_{ji}^2 > 0.$$

W_k^1, W_k^2 and V_k^1, V_k^2 are martingale increments satisfying

$$\mathbb{E}[W_{k+1}^1 | \mathcal{F}_k^1] = \mathbb{E}[V_{k+1}^1 | \mathcal{G}_k^1] = 0,$$
$$\mathbb{E}[W_{k+1}^2 | \mathcal{F}_k^2] = \mathbb{E}[V_{k+1}^2 | \mathcal{G}_k^1] = 0.$$

Observer 2, under $H = 1$ observes a function $d^1(.,.)$ (with finite range) of X_k^1:

$$Z_{k+1}^1 = d^1(X_k^1, u_{k+1}^1), \ k \geq 0, \qquad (3)$$

where u_k^1 is a sequence of independent, identically distributed random variables. It is assumed that $\{u_k^1\}_{k \geq 1}$ are independent of H, X_0^1, X_0^2, $\{W_k^1\}_{k \geq 1}$ and $\{W_k^2\}_{k \geq 1}$. Under $H = 0$, it observes a function $d^2(.,.)$ (with finite range) of X_k^2:

$$Z_{k+1}^2 = d^2(X_k^2, u_{k+1}^2), \ k \geq 0, \qquad (4)$$

where u_k^2 is a sequence of independent, identically distributed random variables. It is assumed that $\{u_k^2\}_{k \geq 1}$ are independent of H, X_0^1, X_0^2, $\{W_k^1\}_{k \geq 1}$, $\{W_k^2\}_{k \geq 1}$ and $\{u_k^1\}_{k \geq 1}$. Each of $d^1(.,.)$ and $d^2(.,.)$ are assumed to have M_2 points in their range and the points are identified with set of unit vectors

$$S_Z = \{f_1^2, ..., f_{M_2}^2\},$$

where f_j^2 are unit vectors in $\mathbb{R}^{M_2}$ with unity as jth element and zeros elsewhere. Following the procedure which was used to derive the observation equation for Observer 1, it can be shown that the observation equation for the Observer 2 is given by :

$$Z_{k+1} = \left[(D^1 X_k^1 + U_{k+1}^1) H + (D^2 X_k^2 + U_{k+1}^2)(1 - H) \right],$$

where D^1, D^2 are matrices of transition probabilities and the entries satisfy

$$\sum_{j=1}^{M_2} d_{ji}^1 = 1, \quad \sum_{j=1}^{M_2} d_{ji}^2 = 1, \ d_{ji}^1 > 0, \ d_{ji}^2 > 0.$$

Notation :

(1) $< a, b >$ denotes inner product in Euclidean space. Hence $< a, b >= a^T b$.
(2) Let a and b be real numbers . Then $a \wedge b = \min(a,b)$.
(3) $Y_k^{j,(l)} = < Y_k^j, f_l^1 >$ so that $Y_k^j = (Y_k^{j,(1)}, ..., Y_k^{j,(M_1)})^T$. For each $k \in \mathbb{N}$, exactly one component =1, the reminder being 0. $Y_k^{(l)} = Y_k^{1,(l)} H + Y_k^{2,(l)}(1 - H)$. Index j corresponds to hypothesis and index l corresponds to the component. Thus $j = 1, 2$ and $l = 1, ..., M_1$. $Z_k^{j,(l)}$, $Z_k^{(l)}$ are defined similarly.
(4) $c_{k+1}^{j,(l)} = \mathbb{E}[Y_{k+1}^{j,(l)} | \mathcal{G}_k^1] = \sum_{m=1}^{N_s} c_{lm}^j < X_k^j, e_m >$. Thus $c_{k+1}^j = \mathbb{E}[Y_{k+1}^j | \mathcal{G}_k^1] = C^j X_k^j$ and $c_{k+1} = \mathbb{E}[Y_{k+1} | \mathcal{G}_k^1] = C^1 X_k^1 H + C^2 X_k^2 (1 - H) = c_{k+1}^1 H + c_{k+1}^2 (1 - H)$. Define $c_{k+1}^{(l)} = c_{k+1}^{1,(l)} H + c_{k+1}^{2,(l)}(1 - H)$. Again, index j corresponds to hypothesis and index l corresponds to the component. Hence, $j = 1, 2$ and $l = 1, ..., M_1$. $d_{k+1}^{j,(l)}$, d_{k+1}^j, $d_{k+1}^{(l)}$, d_{k+1} are defined similarly.
(5) $\sigma(Y_k)$ denotes the smallest complete σ algebra generated by the random variable Y_k.

(6) if $\mathcal{H}_1$ and $\mathcal{H}_2$ are 2 sub σ algebras of $\mathcal{F}$, then $\sigma(\mathcal{H}_1 \cup \mathcal{H}_2)$ denotes the smallest complete σ algebra generated by the sets in $\mathcal{H}_1$ and $\mathcal{H}_2$.

2.2 Hypothesis Testing Problem

We consider the 2 observer problem given by :

$$Under \ \ H = 1 \ : \ X_{k+1}^1 = A^1 X_k^1 + W_{k+1}^1,$$
$$Under \ \ H = 0 \ : \ X_{k+1}^2 = A^2 X_k^2 + W_{k+1}^2,$$
$$Observer \ \ O1 \ :$$
$$Y_{k+1} = \left[(C^1 X_k^1 + V_{k+1}^1) H + (C^2 X_k^2 + V_{k+1}^2)(1 - H) \right],$$
$$Observer \ \ O2 \ :$$
$$Z_{k+1} = \left[(D^1 X_k^1 + U_{k+1}^1) H + (D^2 X_k^2 + U_{k+1}^2)(1 - H) \right].$$

Let $\mathcal{Y}_k$ denote the complete σ algebra generated by $Y_1, ..., Y_k$ and $\mathcal{Z}_k$ denote the complete σ algebra generated by $Z_1, ..., Z_k$. In this paper, we consider the block testing problem with fixed number of samples, T. $t - 1$ denotes the number of times the block testing problem has been performed. Hence when the block testing problem is performed for the tth time, tT number of observations have been collected.

For Observer 1, the aim is to find $D_t^1 \in \{0, 1\}$ which is $\mathcal{Y}_{tT}$ measurable, such that the following cost is minimized:

$$J^1(D_t^1) = \mathbb{E} \left[C_{10}^1 H (1 - D_t^1) + C_{01}^1 (1 - H) D_t^1 \right],$$

where C_{10}^1 and C_{01}^1 are positive real numbers.
For Observer 2, the aim is to find $D_t^2 \in \{0, 1\}$ which is $\mathcal{Z}_{tT}$ measurable, such that the following cost is minimized:

$$J^2(D_t^2) = \mathbb{E} \left[C_{10}^2 H (1 - D_t^2) + C_{01}^2 (1 - H) D_t^2 \right],$$

where C_{10}^2 and C_{01}^2 are positive real numbers.

2.3 Consensus

Let the optimal decisions at t for Observer 1 and Observer 2 be denoted by $D_t^{1,*}$ and $D_t^{2,*}$ respectively.
while $D_t^{1,*} \neq D_t^{2,*}$
Repeat binary hypothesis testing by taking T more samples and finding $D_{t+1}^{1,*}$ and $D_{t+1}^{2,*}$.

3. SOLUTION

3.1 Hypothesis Testing Problem

First we discuss the solution to the binary hypothesis testing problem. We present the solution for Observer 1. An identical procedure can be used to find the solution for Observer 2.

Theorem 3.1. Let π_k^1 (the information state) be defined as :

$$\pi_k^1 = \mathbb{E}_{\mathbb{P}}[H | \mathcal{Y}_k]. \qquad (5)$$

The optimal decision $D_t^{1,*}$ is given by :

$$D_t^{1,*} = 0 \ \ if \ \ C_{01}^1 (1 - \pi_{tT}^1) \geq C_{10}^1 \pi_{tT}^1,$$
$$= 1 \ \ Otherwise. \qquad (6)$$

Also, π_k^1 can be calculated recursively as follows :

$$\pi_k^1 = \frac{Num(k)}{Num(k) + Den(k)}, \tag{7}$$

$$Num(k) = \sum_{r=1}^{N_s} q_k(e_r),$$

$$q_{k+1}(e_r) = M_1 \sum_{j=1}^{N_s} q_k(e_j) a_{rj}^1 \prod_{i=1}^{M_1} (c_{ij}^1)^{Y_{k+1}^{(i)}},$$

$$q_1(e_r) = M_1 \times \bar{p}_1 \times [\sum_{l=1}^{N_s} \prod_{i=1}^{M_1} (c_{il}^1)^{Y_1^{(i)}} (\mathbb{P}(X_0^1 = e_l)) a_{rl}^1],$$

$$Den(k) = \sum_{r=1}^{N_s} p_k(e_r),$$

$$p_{k+1}(e_r) = M_1 \sum_{j=1}^{N_s} p_k(e_j) a_{rj}^2 \prod_{i=1}^{M_1} (c_{ij}^2)^{Y_{k+1}^{(i)}},$$

$$p_1(e_r) = M_1 \times \bar{p}_0 \times [\sum_{l=1}^{N_s} \prod_{i=1}^{M_1} (c_{il}^2)^{Y_1^{(i)}} (\mathbb{P}(X_0^2 = e_l)) a_{rl}^2]. \tag{8}$$

Proof: From the tower law of conditional expectation, the cost function can be written as :

$$= \mathbb{E}[\mathbb{E}[C_{10}^1 H(1 - D_t^1) + C_{01}^1 (1 - H)D_t^1]|\mathcal{Y}_{tT}].$$

Since D_t^1 is $\mathcal{Y}_{tT}$ measurable and $\pi_{tT}^1 = \mathbb{E}_{\mathbb{P}}[H|\mathcal{Y}_{tT}]$, it follows that the cost function can be written as

$$\mathbb{E}[(C_{10}^1 \pi_{tT}^1) \times (1 - D_t^1) + (C_{01}^1 (1 - \pi_{tT}^1)) \times D_t^1].$$

From monotonicity of expectation, it follows that :

$$D_t^{1,*} = 0 \ \ if \ \ C_{01}^1 (1 - \pi_{tT}^1) \geq C_{10}^1 \pi_{tT}^1,$$
$$= 1 \ \ Otherwise.$$

and the optimal cost is given by :

$$J^1(D_t^{1,*}) = \mathbb{E}_{\mathbb{P}}[[C_{01}^1 (1 - \pi_{tT}^1)] \wedge [C_{10}^1 \pi_{tT}^1]]. \tag{9}$$

$\square$

For the derivation of the recursion equations for the information state we refer to Appendix A. From the above theorem, it follows that after collecting a new observation the information state can be updated using the same. At specific time instants $(k = T, 2T, ..., tT)$, the updated information state can be used to find the optimal decision using (6), which is a threshold based policy.

3.2 Convergence to Consensus

Theorem 3.2. $(\pi_k^1, \mathcal{Y}_k)_{k \in \mathbb{N}}$ and $(\pi_k^2, \mathcal{Z}_k)_{k \in \mathbb{N}}$ for $i = 1, 2$ are right-closable martingales. Also,

$$\lim_{k \to \infty} \pi_k^i = H \ \mathbb{P} \ a.s, \ i = 1, 2, \tag{10}$$

$$\lim_{k \to \infty} \mathbb{E}[\pi_k^i] = p_1, \ i = 1, 2, \tag{11}$$

$$\lim_{t \to \infty} J^i(D_t^{i,*}) = 0, \ i = 1, 2, \tag{12}$$

$$\inf_{t \in \mathbb{N}} J^i(D_t^{i,*}) = 0, \ i = 1, 2. \tag{13}$$

Proof: The proof is mentioned for Observer 1. The same proof can be extended for Observer 2 as well. $\mathbb{E}[\pi_{k+1}^1|\mathcal{Y}_k] = \mathbb{E}[\mathbb{E}_{\mathbb{P}}[H|\mathcal{Y}_{k+1}]|\mathcal{Y}_k] = \mathbb{E}_{\mathbb{P}}[H|\mathcal{Y}_k] = \pi_k^1$. Thus $(\pi_k^1, \mathcal{Y}_k)_{k \in \mathbb{N}}$ is a martingale. Since $\exists$ random variable $\pi_\infty^1 = H$ such that

$$\pi_k^1 = \mathbb{E}[\pi_\infty^1|\mathcal{Y}_k] \ \forall \ k,$$

it follows that $(\pi_k^1, \mathcal{Y}_k)_{k \in \mathbb{N}}$, is a right-closable martingale. By Doob's theorem (Koralov and Sinai (2007)) for the convergence of right closable martingles (10) follows. Since $(\pi_k^1, \mathcal{Y}_k)_{k \in \mathbb{N}}$ is a martingale, it follows that $\mathbb{E}[\pi_k^1] = p_1 \ \forall \ k$. Hence (11) follows. (10) implies that :

$$\lim_{k \to \infty} [C_{01}^1 (1 - \pi_k^1)] \wedge [C_{10}^1 \pi_k^1] = 0 \ \mathbb{P} \ a.s.$$

Also note that $|[C_{01}^1 (1 - \pi_k^1)] \wedge [C_{10}^1 \pi_k^1]| \leq C_{10}^1 + C_{01}^1, \forall \ \omega \in \Omega, k$. By the Lebesgue dominated convergence theorem, (12) follows. Consider,

$$[C_{01}^1 (1 - \pi_{tT}^1)] \wedge [C_{10}^1 \pi_{tT}^1]$$
$$= \frac{C_{01}^1 + \pi_{tT}^1 (C_{10}^1 - C_{01}^1) - |C_{01}^1 - \pi_{tT}^1 (C_{01}^1 + C_{10}^1)|}{2}.$$

Since $(\pi_k^1, \mathcal{Y}_k)_{k \in \mathbb{N}}$ is a martingale, it follows that $(C_{01}^1 + \pi_{tT}^1 (C_{10}^1 - C_{01}^1), \mathcal{Y}_{tT})_{t \in \mathbb{N}}$ and $(C_{01}^1 - \pi_{tT}^1 (C_{10}^1 + C_{01}^1), \mathcal{Y}_{tT})_{t \in \mathbb{N}}$ are martingales. As $\Phi(x) = |x|$ is convex, from the conditional Jensen's inequality, it follows that $(|C_{01}^1 - \pi_{tT}^1 (C_{01}^1 + C_{10}^1)|, \mathcal{Y}_{tT})_{t \in \mathbb{N}}$ is a submartingale. Hence $([C_{01}^1 (1 - \pi_{tT}^1)] \wedge [C_{10}^1 \pi_{tT}^1], \mathcal{Y}_{tT})_{t \in \mathbb{N}}$ is a supermartingale. Hence,

$$J^1(D_{t+1}^{1,*}) \leq J^1(D_t^{1,*}) \ \forall \ t.$$

Hence by the monotone convergence theorem, (13) follows.

$\square$

The main result of the above theorem is that, the information state converges to the true hypothesis. This result is used in proving the convergence of the consensus algorithm which is done in the following theorem.

Theorem 3.3. $\forall \ \omega \in \Omega$, $\exists \ \hat{t}(\omega) \in \mathbb{N}$ such that

$$D_{\hat{t}(\omega)}^{1,*}(\omega) = D_{\hat{t}(\omega)}^{2,*}(\omega) = H(\omega) \tag{14}$$

Proof: Fix $\omega \in \Omega$. From (10), it follows that $\forall \ \epsilon > 0$, $\exists \ N^i(\epsilon, \omega)$ such that

$$|\pi_k^i(\omega) - H(\omega)| < \epsilon \ \forall \ k \geq N^i(\epsilon, \omega), \ i = 1, 2.$$

Suppose $H(\omega) = 1$, then let $\epsilon_1^i = 1 - \frac{C_{01}^i}{C_{10}^i + C_{01}^i}$. Then, $\forall \ k \geq \max(N^1(\epsilon_1^1, \omega), N^2(\epsilon_1^2, \omega))$,

$$\pi_k^i(\omega) > \frac{C_{01}^i}{C_{10}^i + C_{01}^i}, \ i = 1, 2.$$

Thus $\forall \ \hat{t}(\omega) > \lceil \frac{\max(N^1(\epsilon_1^1, \omega), N^2(\epsilon_1^2, \omega))}{T} \rceil$,

$$D_{\hat{t}(\omega)}^{1,*}(\omega) = D_{\hat{t}(\omega)}^{2,*}(\omega) = H(\omega) = 1.$$

Suppose $H(\omega) = 0$, then let $\epsilon_2^i = \frac{C_{01}^i}{C_{10}^i + C_{01}^i}$. Then, $\forall \ k \geq \max(N^1(\epsilon_2^1, \omega), N^2(\epsilon_2^2, \omega))$,

$$\pi_k^i(\omega) < \frac{C_{01}^i}{C_{10}^i + C_{01}^i}, \ i = 1, 2.$$

Thus $\forall \ \hat{t}(\omega) > \lceil \frac{\max(N^1(\epsilon_2^1, \omega), N^2(\epsilon_2^2, \omega))}{T} \rceil$

$$D_{\hat{t}(\omega)}^{1,*}(\omega) = D_{\hat{t}(\omega)}^{2,*}(\omega) = H(\omega) = 0.$$

This completes the proof of (14). Hence convergence is guaranteed.

$\square$

The above result states that, for every sample path, there is a index $\hat{t}$ such that the optimal decision of both the observers is the same and is equal to the true hypothesis. Since the result is

an asymptotic result, in practice it is possible that the observers arrive at a consensus to the wrong hypothesis even before reaching the index $\hat{t}$.

Another special case to consider would be when $A^1 = A^2$. Though $A^1 = A^2$, it should be noted that $C^1 \neq C^2$ and $D^1 \neq D^2$ as the noise seen by the observers is not the same under either hypothesis. In this case also, convergence is guaranteed as none of the results mentioned in the previous section involve conditions on A^1 and A^2.

4. SIMULATION RESULTS

We are also interested in understanding the "*value of information*" associated with the repeated 1 bit communication. So through simulations we would like to understand whether through the 1 bit communications, the number of false alarms and number of misses reduced. A heuristic way to calculate the value of information for this specific problem would be : Calculate the average reduction in detection error as :

α = Number of simulations in which consensus occurs to correct hypothesis after one iteration

β = Number of simulations in which consensus occurs to wrong hypothesis while the decision for either observers after the first iteration was equal to true hypothesis.

γ = Total number of bits communicated in all the simulations

$total$ = Total number of simulations

$$Value\ of\ information = \frac{\dfrac{\alpha - \beta}{total}}{\dfrac{\gamma}{total}} = \frac{\alpha - \beta}{\gamma} \qquad (15)$$

The simulations were performed with two three state Markov chains. The transition matrices for the two Markov chains were chosen as:

$$A^1 = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.3 & 0.35 & 0.6 \\ 0.5 & 0.25 & 0.2 \end{bmatrix}, \ A^2 = \begin{bmatrix} 0.6 & 0.25 & 0.25 \\ 0.15 & 0.5 & 0.35 \\ 0.25 & 0.25 & 0.4 \end{bmatrix}.$$

Observer 1 and Observer 2 were considered to have two and four outputs respectively. The state to output transition matrices were chosen as :

$$C^1 = \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.3 & 0.5 & 0.6 \end{bmatrix}, \ C^2 = \begin{bmatrix} 0.35 & 0.5 & 0.55 \\ 0.65 & 0.5 & 0.45 \end{bmatrix},$$

$$D^1 = \begin{bmatrix} 0.25 & 0.1 & 0.35 \\ 0.15 & 0.15 & 0.5 \\ 0.2 & 0.5 & 0.05 \\ 0.4 & 0.25 & 0.1 \end{bmatrix}, \ D^2 = \begin{bmatrix} 0.5 & 0.1 & 0.15 \\ 0.2 & 0.5 & 0.05 \\ 0.15 & 0.1 & 0.50 \\ 0.15 & 0.3 & 0.3 \end{bmatrix}.$$

The convergence of the information state to the true hypothesis for a particular sample path has been shown in figure 2. The costs were assigned the values $C^1_{10} = 8$, $C^1_{01} = 5$, $C^2_{10} = 11$, $C^1_{01} = 9$. T was set 50 samples. $\bar{p}_1$ was set to 0.6. The number of simulations was varied from 10 to 10^5. The value of information was calculated in each case and has been tabulated (Table 1).

5. CONCLUSION AND FUTURE WORK

The binary hypothesis testing problem with observations generated by Markov chains and two communicating observers has

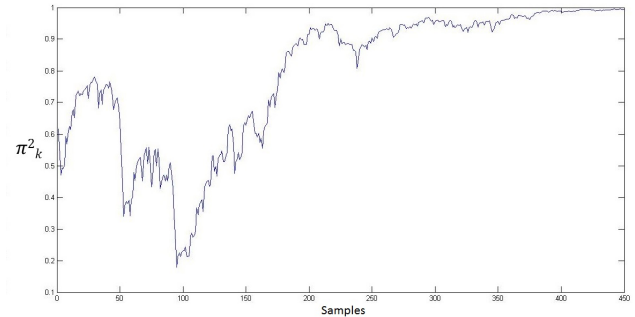| Number of Simulations	Value of Information
10	0.2181
100	0.2115
1000	0.2197
10000	0.2212
100000	0.2198

Table 1. Value of Information

Fig. 2. The Information state for Observer 2 converging to the true hypothesis = 1.

been solved by formulating the problem as a partially observed stochastic control problem. Convergence of the information state to the true hypothesis and optimal cost to zero has been studied. The convergence of the consensus algorithm has been proven. To understand the value of the 1 bit communication used to achieve consensus, simulations were performed. It was observed there was a reduction in miss and false detection. On an average, if the observers exchanged their decisions 3 times it led to reduction in a miss or false detection.

In the present work, communication between the observers was assumed to be perfect but in practice communication errors could occur as well. Hence we could study the problem by modeling the communication channel using a binary symmetric channel with error probability ϵ. Also we have restricted ourselves to one bit communication. The problem could be studied in the framework where the observers not only share their decisions but also share their respective probabilities of a miss and false detection.

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Appendix A. INFORMATION STATE RECURSION EQUATIONS

Due to space restrictions some of the details have been skipped. To prove the recursions mentioned in equations (7) and (8), we consider a change of measure. Define :

$$\alpha_l = \prod_{i=1}^{M_1} \left(\frac{M_1^{-1}}{c_l^{(i)}}\right)^{Y_l^{(i)}}, \ \Gamma_k = \prod_{l=1}^{k} \alpha_l.$$

It can be shown that $(\Gamma_k, \mathcal{G}_k^1)_{k \in \mathbb{N}}$ is a martingale. We now define a new probability measure $\bar{\mathbb{P}}$ on $(\Omega, \cup_{l=1}^{\infty} \mathcal{G}_l^1)$ by restricting the Radon- Nikodym derivative, $d\bar{\mathbb{P}}/d\mathbb{P}$ to the σ algebra $\mathcal{G}_k^1$ equal to Γ_k. Thus $\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{G}_k^1} = \Gamma_k \Rightarrow \bar{\mathbb{P}}(B) = \int_B \Gamma_k d\mathbb{P} \ \forall B \in \mathcal{G}_k^1.$

The existence of such a measure $\bar{\mathbb{P}}$ follows from *Kolmogorov's Extension Theorem* [Elliott et al. (2008)].

(1) Under, $\bar{\mathbb{P}}$, $\{Y_k\}$, $k \in \mathbb{N}$, is a sequence of i.i.d random variables each having uniform distribution that assigns probability $\frac{1}{M_1}$ to each point f_i^1 , $1 \leq i \leq M_1$, in its range space. $\bar{\mathbb{P}}(Y_{k+1}^{(i)} = 1|\mathcal{G}_k^1) = \frac{1}{M_1}$.

(2) Under $\bar{\mathbb{P}}$, X_k^1 and X_k^2 remain Markov chains with transition matrices A^1 and A^2 respectively .

Given probability measure $\bar{\mathbb{P}}$ on $(\Omega, \cup_{l=1}^{\infty} \mathcal{G}_l^1)$ such that (1) and (2) (as mentioned above) hold true and matrices $\hat{C}^1$ and $\hat{C}^2$, we construct a measure $\hat{\mathbb{P}}$ as follows: Let $\hat{c}_{k+1} = \hat{C}^1 X_k^1 H + \hat{C}^2 X_k^2 (1 - H)$. Define:

$$\bar{\alpha}_l = \prod_{i=1}^{M_1} (M_1 \hat{c}_l^{(i)})^{Y_l^{(i)}}, \ \bar{\Gamma}_k = \prod_{l=1}^{k} \bar{\alpha}_l, \ \left.\frac{d\hat{\mathbb{P}}}{d\bar{\mathbb{P}}}\right|_{\mathcal{G}_k^1} = \bar{\Gamma}_k.$$

Again, the existence of such a measure $\hat{\mathbb{P}}$ follows from *Kolmogorov's Extension Theorem* (Elliott et al. (2008)). With the above definitions, the following are true:

(1) $\mathbb{E}_{\bar{\mathbb{P}}}[\bar{\alpha}_{k+1}|\mathcal{G}_k^1] = 1$. Thus $(\bar{\Gamma}_k, \mathcal{G}_k^1)_{k \in \mathbb{N}}$ is a martingale.
(2) $\mathbb{E}_{\hat{\mathbb{P}}}[Y_{k+1}|\mathcal{G}_k^1] = \hat{C}^1 X_k^1 H + \hat{C}^2 X_k^2 (1 - H)$

If $C^1 = \hat{C}^1$ and $C^2 = \hat{C}^2$ then it follows that $\hat{\mathbb{P}} = \mathbb{P}$ on $(\Omega, \cup_{l=1}^{\infty} \mathcal{G}_l^1)$. Thus by letting $C^1 = \hat{C}^1$ and $C^2 = \hat{C}^2$, we obtain, $\mathbb{E}_{\mathbb{P}}[H|\mathcal{Y}_k] = \frac{\mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H|\mathcal{Y}_k]}{\mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k|\mathcal{Y}_k]}$. Define:

$Num(k) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H|\mathcal{Y}_k], \ Den(k) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k(1 - H)|\mathcal{Y}_k],$

$q_k(e_r) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H < X_k^1, e_r > |\mathcal{Y}_k],$

$p_k(e_r) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k(1 - H) < X_k^2, e_r > |\mathcal{Y}_k].$

It follows that :

$$Num(k) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H \sum_{r=1}^{N_s} < X_k^1, e_r > |\mathcal{Y}_k] = \sum_{r=1}^{N_s} q_k(e_r),$$

$$Den(k) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k(1 - H) \sum_{r=1}^{N_s} < X_k^2, e_r > |\mathcal{Y}_k] = \sum_{r=1}^{N_s} p_k(e_r),$$

$\mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k|\mathcal{Y}_k] = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k[H + (1 - H)]|\mathcal{Y}_k] = Num(k) + Den(k).$

$$\Rightarrow \pi_k^1 = \mathbb{E}_{\mathbb{P}}[H|\mathcal{Y}_k] = \frac{Num(k)}{Num(k) + Den(k)}$$

We now prove the recursion for $q_k(e_r)$:

$$q_{k+1}(e_r) = \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H(\prod_{i=1}^{M_1} (M_1(c_{k+1}^{1,(i)} H + c_{k+1}^{2,(i)}(1 - H)))^{Y_{k+1}^{(i)}})$$
$$< X_{k+1}^1, e_r > |\mathcal{Y}_{k+1}]$$
$$= M_1 \times \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H(\prod_{i=1}^{M_1} (c_{k+1}^{1,(i)} H)^{Y_{k+1}^{(i)}})$$
$$< A^1 X_k^1, e_r > |\mathcal{Y}_{k+1}]$$
$$+ M_1 \times \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H(\prod_{i=1}^{M_1} (c_{k+1}^{1,(i)} H)^{Y_{k+1}^{(i)}})$$
$$< W_{k+1}^1, e_r > |\mathcal{Y}_{k+1}]$$

Since under $\bar{\mathbb{P}}$, $\mathbb{E}_{\bar{\mathbb{P}}}[< W_{k+1}^1, e_r > |\sigma(\mathcal{G}_k^1 \cup \sigma(Y_{k+1}))] = 0$, the second term in the summation equals zero.

$$q_{k+1}(e_r) = M_1 \times [\sum_{j=1}^{N_s} \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H(< X_k^1, e_j >)|\mathcal{Y}_{k+1}]$$
$$a_{rj}^1 (\prod_{i=1}^{M_1} (c_{ij}^1)^{Y_{k+1}^{(i)}})]$$

Since under $\bar{\mathbb{P}}$, $\sigma(\sigma(H) \cup \sigma(X_k^1) \cup \mathcal{Y}_k)$ is independent of $\sigma(Y_{k+1})$,

$$= M_1 \times [\sum_{j=1}^{N_s} \mathbb{E}_{\bar{\mathbb{P}}}[\bar{\Gamma}_k H(< X_k^1, e_j >)|\mathcal{Y}_k] a_{rj}^1 (\prod_{i=1}^{M_1} (c_{ij}^1)^{Y_{k+1}^{(i)}}]$$
$$= M_1 \times [\sum_{j=1}^{N_s} q_k(e_j) a_{rj}^1 (\prod_{i=1}^{M_1} (c_{ij}^1)^{Y_{k+1}^{(i)}})].$$

The initial condition , $q_1(e_r)$:

$$q_1(e_r) = M_1 \times \mathbb{E}_{\bar{\mathbb{P}}}[\prod_{i=1}^{M_1} (c_1^{1,(i)} H)^{Y_1^{(i)}} H(< X_1^1, e_r >)|\mathcal{Y}_1]$$
$$= M_1 \times \mathbb{E}_{\bar{\mathbb{P}}}[\prod_{i=1}^{M_1} (c_1^{1,(i)} H)^{Y_1^{(i)}} H(< A^1 X_0^1, e_r >)|\mathcal{Y}_1]$$
$$= M_1 \times [\sum_{l=1}^{N_s} \prod_{i=1}^{M_1} (c_{il}^1)^{Y_1^{(i)}} \mathbb{E}_{\bar{\mathbb{P}}}[H(< X_0^1, e_l >)|\mathcal{Y}_1] a_{rl}^1]$$
$$= M_1 \times [\sum_{l=1}^{N_s} \prod_{i=1}^{M_1} (c_{il}^1)^{Y_1^{(i)}} \mathbb{E}_{\bar{\mathbb{P}}}[H(< X_0^1, e_l >)] a_{rl}^1]$$

the last equality is true since under $\bar{\mathbb{P}}$, $\sigma(H, X_0^1)$ is independent of $\mathcal{Y}_1$. Since $\mathbb{E}_{\bar{\mathbb{P}}}[\alpha_1|\sigma(H, X_0^1)] = 1$, it follows that:

$$\mathbb{E}_{\bar{\mathbb{P}}}[H(< X_0^1, e_l >)] = \mathbb{E}_{\mathbb{P}}[\alpha_1 H(< X_0^1, e_l >)]$$
$$= \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\alpha_1 H(< X_0^1, e_l >)|\sigma(H, X_0^1)]]$$
$$= \mathbb{E}_{\mathbb{P}}[H(< X_0^1, e_l >)\mathbb{E}_{\mathbb{P}}[\alpha_1|\sigma(H, X_0^1)]]$$
$$= \mathbb{E}_{\mathbb{P}}[H(< X_0^1, e_l >)]$$
$$= \mathbb{E}_{\mathbb{P}}[H]\mathbb{E}_{\mathbb{P}}[< X_0^1, e_l >]$$
$$= \bar{p}_1 \times (\mathbb{P}(X_0^1 = e_l))$$
$$\Rightarrow q_1(e_r) = M_1 \times [\sum_{l=1}^{N_s} \prod_{i=1}^{M_1} (c_{il}^1)^{Y_1^{(i)}} \bar{p}_1 \times (\mathbb{P}(X_0^1 = e_l)) a_{rl}^1].$$

The recursion for p_{k+1} and p_1 can found by the exact same procedure. This completes the proof.