

Event-Triggered Output-Feedback H_∞ Control with Minimum Directed Information

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Abstract—In this paper, the H_∞ control problem with minimum information exchange between a plant and a remote controller is considered. For transmission of the measured outputs, an event-triggering mechanism is proposed. We measure the transferred information satisfying the causality condition by the directed information, and measure the H_∞ control performance by the L_2 -gain. We achieve the minimum directed information required for a guaranteed level of control performance. In particular, corresponding to the worst case disturbances, we develop a filter with a recursive variable, and show that the optimal control policy is a linear function of this recursive variable. Finally, we define the value of information, and determine the structure of the optimal sampling policy based on the value of information.

I. INTRODUCTION

In this paper, the H_∞ control synthesis problem with minimum information exchange between a plant and a remote controller is considered. This study has a broad range of applications in cyber-physical systems in which privacy preservation and communication constraints matter. Examples include surveillance and reconnaissance, planetary explorations, wireless wearables, and teleoperation.

The optimal H_∞ control problem in the frequency domain was originally formulated by Zames [1] for sensitivity reduction in linear plants. Later, Glover and Doyle [2] developed the optimal H_∞ control synthesis in state space for linear systems based on the solutions of two coupled Riccati equations. The connection with the risk sensitive problem was made by Whittle [3]. Then, Başar and Bernhard [4] studied the optimal H_∞ control synthesis using dynamic games, and James and Baras [5] covered the H_∞ output-feedback control problem for general nonlinear partially observed dynamic games. In the dynamic game interpretation, the optimal H_∞ control problem is seen as a min-max optimization problem where the controller is viewed as the minimizing player and the disturbance as the maximizing player. For problems with incomplete information, the certainty equivalence principle (see e.g., [6] and [5]) states that at each time one can compute the estimate of the state based on the worst disturbance or on the information state, and then use it in the optimal

state-feedback policy, obtained as the saddle point of a full information dynamic zero-sum game.

We measure the information between the plant and the remote controller by directed information. First, Massey in [7] proposed directed information as a natural counterpart of mutual information for characterizing causality in information and control systems, in a more general manner than the Granger causality [8]. Then, Kramer [9] extended the use of directed information to discrete memoryless networks with feedback. In order to reduce the transferred information from the plant to the controller, we employ an event-triggering mechanism. Åström and Bernhardsson [10] showed that for a stochastic scalar linear system under an average sampling rate constraint event-driven sampling outperforms uniform sampling. This result has received much attention leading to the development of different event-triggered sampling policies for estimation and control problems with communication costs or constraints, see e.g., [11], [12], [13]. Recently, it was shown in [14] (see also [15], [16], [17]) that for the LQG control problem the optimal sampling policy, without presuming a priori any structure, samples a measurement whenever the value of information exceeds a threshold.

In this paper, we develop a framework for linear partially observable H_∞ control with minimum directed information. We achieve the minimum directed information required for a guaranteed level of control performance. In particular, corresponding to the worst case disturbances, we develop a filter with a recursive variable, and show that the optimal control policy is a linear function of this recursive variable. Finally, we define the value of information, and determine the structure of the optimal sampling policy based on the value of information.

The outline of the paper is as follows. After some preliminaries, we provide the system model and formulate the problem in Section II. In Section III, we derive the optimal H_∞ control and sampling policies. Finally, concluding remarks are made in Section IV.

II. PROBLEM FORMULATION

A. Preliminaries

In the sequel, we write a^T to indicate the transpose of the vector a . We represent an n dimensional vector at time k with a_k and the stack vector of all a_ℓ , $\ell = 0, \dots, k$ with \mathbf{a}_k . The weighted norm of the vector a with respect to the matrix A is denoted by $\|a\|_A^2$. The identity matrix with dimension n is denoted by I_n . For matrices A and B , we write $A \succ 0$ and $B \succeq 0$ to denote that A and B

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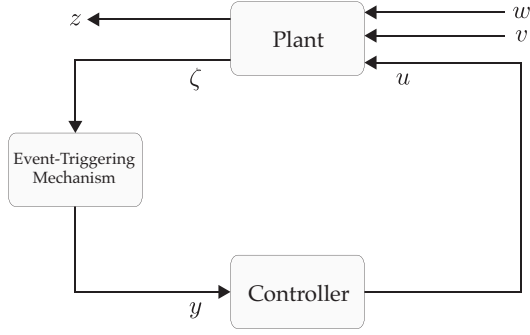


Fig. 1. General system interconnection in H_∞ control under event-triggered sampling.

are positive definite and positive semi-definite, respectively. The probability distribution of the stochastic variable a is denoted by $\mathbb{P}(a)$. The expected value and the covariance of a are denoted by $\mathbb{E}[a]$ and $\text{cov}[a]$, respectively.

For the zero-sum game with state x and cost function J , the upper value \bar{J} and the lower value \underline{J} are defined as

$$\begin{aligned}\bar{J} &= \inf_{u_1(x)} \sup_{u_2(x)} J(u_1, u_2), \\ \underline{J} &= \sup_{u_2(x)} \inf_{u_1(x)} J(u_1, u_2).\end{aligned}$$

The game is said to have a saddle point (u'_1, u'_2) if the following inequalities are satisfied

$$J(u'_1, u_2) \leq J(u'_1, u'_2) \leq J(u_1, u'_2).$$

for any u_1 and u_2 . Then, we have $\bar{J} = \underline{J}$.

B. System Model

Consider a plant with discrete-time stochastic dynamics given by the following linear state system:

$$\begin{aligned}x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, \\ \zeta_k &= C_2 x_k + D_{21} v_k + D_{22} u_k,\end{aligned}$$

for $k \geq 0$ and with initial condition x_0 , where $x_k \in \mathbb{R}^n$ is the state of the system, $w_k \in \mathbb{R}^{m_w}$ and $v_k \in \mathbb{R}^{m_v}$ are disturbances, $u_k \in \mathbb{R}^{m_u}$ is the control input to be decided by a remote controller, $z_k \in \mathbb{R}^{n_z}$ is the controlled output, and $\zeta_k \in \mathbb{R}^p$ is the measured output. We assume that the system is controllable and observable. For abbreviation, we use $Q_1 = C_1^T C_1$, $Q_2 = D_{12}^T D_{12}$, $R_1 = B_1 B_1^T$, and $R_2 = D_{21} D_{21}^T$. We assume that D_{12} and D_{21} have full column rank and full row rank, respectively. One can transform this system to an equivalent one such that

$$\begin{aligned}D_{11} &= 0, & D_{22} &= 0, \\ D_{12} D_{12}^T &= I, & D_{21}^T D_{21} &= I.\end{aligned}\quad (1)$$

In addition, without loss of generality we can make the following standard assumptions:

$$D_{12}^T C_1 = 0, \quad D_{21} B_1^T = 0. \quad (2)$$

Under the above assumptions, we can write the dynamics of the plant by

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k, \quad (3a)$$

$$z_k = C_1 x_k + D_{12} u_k, \quad (3b)$$

$$\zeta_k = C_2 x_k + D_{21} v_k. \quad (3c)$$

The measured outputs ζ_k should be transmitted to the remote controller. In order to reduce the transferred information from the plant to the controller, we employ an event-triggering mechanism that samples and transmits the measured outputs only at specific times $k \in \mathcal{S}$ where \mathcal{S} is the set of all sampling times. The decisions are represented by a sampling variable δ_k defined as

$$\delta_k = \begin{cases} 1, & \text{if } k \in \mathcal{S}, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The transmitted measured output are denoted by $y_k \in \mathbb{R}^p$ such that

$$y_k = \begin{cases} \zeta_k, & \text{if } \delta_k = 1, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (5)$$

We assume that the transmitted measurements are available at the controller by one-step delay. Therefore, at time k the information available at the event-triggering mechanism and at the controller are $\mathcal{I}_k = \{\zeta_k, \delta_{k-1}, \mathbf{u}_{k-1}\}$ and $\tilde{\mathcal{I}}_k = \{\mathbf{y}_{k-1}, \delta_{k-1}, \mathbf{u}_{k-1}\}$, respectively.

Next, we will specify the measure of information and the measure of performance. We measure the transferred information satisfying the Granger's causality condition from the plant to the controller by the directed information [18].

Definition 1: We define the causally conditioned directed information from the plant to the controller over the time horizon N by

$$I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) = \sum_{k=0}^N I(\mathbf{x}_k; y_k | \tilde{\mathcal{I}}_k), \quad (6)$$

where $I(\mathbf{x}_k; y_k | \tilde{\mathcal{I}}_{k-1})$ represents the mutual information between \mathbf{x}_k and y_k conditioned on $\tilde{\mathcal{I}}_{k-1}$.

Let us consider the initial condition x_0 as a part of the disturbance, define the disturbance till time k by $\omega_k = (\mathbf{w}_k, \mathbf{v}_k, x_0)$, and show the whole disturbance by $\omega = \omega_N$.

We measure the H_∞ control performance by the L_2 -gain of the system [19]:

$$\sup_{\omega} \frac{\|(\mathbf{z}_N, x_{N+1})\|}{\|(\mathbf{w}_N, \mathbf{v}_N, x_0)\|}, \quad (7)$$

where

$$\begin{aligned}\|(\mathbf{z}_N, x_{N+1})\| &= (\|\mathbf{z}_N\|_2^2 + \|x_{N+1}\|_{Q_3}^2)^{\frac{1}{2}}, \\ \|(\mathbf{w}_N, \mathbf{v}_N, x_0)\| &= (\|\mathbf{w}_N\|_2^2 + \|\mathbf{v}_N\|_2^2 + \|x_0\|_{Q_0}^2)^{\frac{1}{2}}.\end{aligned}$$

We are seeking for a class of stabilizing controllers that achieve a desired level of performance γ .

Definition 2: We define the γ -feasible H_∞ controller class as the set of all stabilizing controllers that satisfy

$$\sup_{\omega} J_\gamma \leq 0, \quad (8)$$

where

$$J_\gamma = -\gamma^2 \|x_0\|_{Q_0}^2 + \|x_{N+1}\|_{Q_3}^2 + \sum_{k=0}^N \|z_k\|^2 - \gamma^2 \|w_k\|^2 - \gamma^2 \|v_k\|^2. \quad (9)$$

To achieve the minimum directed information required for a guaranteed level of control performance γ , we need to solve the following problem:

$$\inf_{\pi \in \mathcal{P}, \mu \in \mathcal{M}} \sup_{\omega} I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N), \quad (10)$$

where π is the sampling policy defined in the admissible set \mathcal{P} and μ is the control policy defined in the admissible set \mathcal{M} . We can recast the above problem as

$$\inf_{\pi \in \mathcal{P}, \mu \in \mathcal{M}} \left\{ I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) + \lambda \sup_{\omega} J_\gamma \right\}, \quad (11)$$

where $\lambda \geq 0$. In the sequel, we study this problem.

III. MAIN RESULTS

A. Directed Information and Least-Square Estimator

To measure the information flow in the system, we need first to specify the distribution of the disturbances. In the sequel, we assume that w_k and v_k are white noises with standard normal distributions, x_0 is normal with mean m and covariance R_0 , and w_k , v_k , and x_0 are mutually independent. In addition, we need the following assumption for the results of this section.

Assumption 1: We assume that due to privacy the structure of the event-triggering mechanism, i.e., the set of admissible sampling policies \mathcal{P} , is not revealed to the controller. In other words, the controller will not make any inference based on the structure of the event-triggering mechanism.

The following lemma gives the optimal least-square estimator which can be used at the controller.

Lemma 1: Let $\hat{x}_k = \mathbb{E}[x_k | \tilde{\mathcal{I}}_k]$ and $P_k = \text{cov}[x_k | \tilde{\mathcal{I}}_k]$. The following estimator adopted by the controller minimizes the mean-square error:

$$\hat{x}_{k+1} = A\hat{x}_k + B_2 u_k + \delta_k K_k (y_k - C_2 \hat{x}_k), \quad (12a)$$

$$P_{k+1} = AP_k A^T + R_1 - \delta_k K_k C_2 P_k A^T, \quad (12b)$$

where

$$K_k = AP_k C_2^T (C_2 P_k C_2^T + R_2)^{-1}. \quad (12c)$$

with initial conditions $\hat{x}_0 = m_0$ and $P_0 = R_0$.

Proof: Following the Kolmogorov forward equation [20], the estimate and its covariance are propagated as

$$\hat{x}_{k+1} = A\hat{x}_{k+} + B_2 u_k, \quad (13a)$$

$$P_{k+1} = AP_{k+} A^T + R_1, \quad (13b)$$

where k^+ denotes time k just after the update of the estimate and its covariance. From Bayes' rule [20], the estimate and

its covariance are updated when $\delta_k = 1$ as

$$\hat{x}_{k^+} = \hat{x}_k + P_k C_2^T (C_2 P_k C_2^T + R_2)^{-1} (y_k - C_2 \hat{x}_k); \quad k \in \mathcal{S}, \quad (14a)$$

$$P_{k^+} = \left(I_n - P_k C_2^T (C_2 P_k C_2^T + R_2)^{-1} C_2 \right) P_k; \quad k \in \mathcal{S}. \quad (14b)$$

Following Assumption 1, the controller cannot make any inference when $\delta_k = 0$. We obtain (12) by substituting (14) in (13). ■

The following lemma expresses the causally conditioned directed information as a function of δ_k .

Lemma 2: The causally conditioned directed information for the system is equal to

$$I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) = - \sum_{k=0}^N \frac{1}{2} \ln \det (I_n - \delta_k K_k C_2). \quad (15)$$

Proof: We can write:

$$\begin{aligned} I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) &= \sum_{k=0}^N I(\mathbf{x}_k; y_k | \tilde{\mathcal{I}}_{k-1}) \\ &= \sum_{k=0}^N I(x_k; y_k | \tilde{\mathcal{I}}_{k-1}) + I(\mathbf{x}_{k-1}; y_k | \tilde{\mathcal{I}}_{k-1}, x_k) \\ &= \sum_{k=0}^N I(x_k; y_k | \tilde{\mathcal{I}}_{k-1}) \\ &= \sum_{k=0}^N h(x_k | \tilde{\mathcal{I}}_{k-1}) - h(x_k | \tilde{\mathcal{I}}_{k-1}, y_k), \end{aligned}$$

where the second equality is due to the property of the mutual information, and third equality is because \mathbf{x}_{k-1} is independent of y_k given x_k . Following Assumption 1, the conditional distributions $\mathbb{P}(x_k | \tilde{\mathcal{I}}_k)$ and $\mathbb{P}(x_k | \tilde{\mathcal{I}}_k, y_k)$ remain Gaussian. Hence, the differential entropies are:

$$h(x_k | \tilde{\mathcal{I}}_k) = \frac{1}{2} \ln ((2\pi e)^n \det P_k),$$

$$h(x_k | \tilde{\mathcal{I}}_k, y_k) = \frac{1}{2} \ln ((2\pi e)^n \det P_{k^+}).$$

Therefore, we can obtain:

$$\begin{aligned} I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) &= \sum_{k=0}^N \frac{1}{2} \ln \det P_k \\ &\quad - \sum_{k=0}^N \frac{1}{2} \ln \det ((I_n - \delta_k K_k C_2) P_k) \\ &= - \sum_{k=0}^N \frac{1}{2} \ln \det (I_n - \delta_k K_k C_2). \end{aligned}$$

This completes the proof. ■

B. Recursive Filter and Optimal H_∞ Control Policy

According to Lemma 2, $I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N)$ does not depend on the control policy. Hence, we can write the optimal value

of the optimization problem in (11) as

$$\inf_{\pi \in \mathcal{P}} \left\{ I_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) + \lambda \inf_{\mu \in \mathcal{M}} \sup_{\omega} J_\gamma \right\}. \quad (17)$$

Now, consider the following zero-sum dynamic game problem:

$$\inf_{\mu \in \mathcal{M}} \sup_{\omega} J_\gamma. \quad (18)$$

In the following, we employ forward-backward dynamic programming [21] to solve this problem.

Definition 3: We define the value function of the zero-sum dynamic game problem in (18) with full state feedback as

$$V_k = \inf_{\mu \in \mathcal{M}'} \sup_{\omega} \left\{ \|x_{N+1}\|_{Q_3}^2 + \sum_{\ell=k}^N \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 \right\}. \quad (19)$$

where \mathcal{M}' is the set of admissible state-feedback policies.

For the zero-sum dynamic game in (18) with full state feedback, the disturbance v_k can be set to zero, because measured outputs are not influential. Moreover, we do not add $-\gamma^2 \|x_0\|_{Q_0}^2$ to the value function.

Lemma 3: The value function V_k satisfies the dynamic programming recursion, i.e.,

$$V_k = \inf_{u_k \in \mathcal{M}'} \sup_{w_k} \left\{ \|z_k\|^2 - \gamma^2 \|w_k\|^2 + V_{k+1} \right\}, \quad (20)$$

with boundary condition $V_{N+1} = \|x_{N+1}\|_{Q_3}^2$.

Proof: The proof is straightforward by induction. ■

Lemma 4: The value function V_k is of quadratic form

$$V_k = x_k^T M_k x_k, \quad (21)$$

where M_k is given by the following Riccati equation

$$M_k = A^T (M_{k+1}^{-1} + B_2 B_2^T - \gamma^2 R_1)^{-1} A + Q_1, \quad (22)$$

with boundary condition $M_{N+1} = Q_{N+1}$.

Proof: The proof follows from the dynamic programming recursion in (20). ■

Lemma 5: The saddle point $(\mathbf{u}'_N, \mathbf{w}'_N)$ of the zero-sum dynamic game problem in (14) with full state feedback exists if the Riccati equation in (22) admits a solution. Then,

$$u'_k = -B_2^T (M_{k+1}^{-1} + B_2 B_2^T - \gamma^2 R_1)^{-1} A x_k, \quad (23)$$

$$w'_k = \gamma^2 B_1^T (M_{k+1}^{-1} + B_2 B_2^T - \gamma^2 R_1)^{-1} A x_k. \quad (24)$$

Proof: Using Lemma 3 and Lemma 4, we can write:

$$\begin{aligned} V_k &= \inf_{u_k} \sup_{w_k} \left\{ \|z_k\|^2 - \gamma^2 \|w_k\|^2 + \|x_{k+1}\|_{M_{k+1}}^2 \right\} \\ &= \inf_{u_k} \sup_{w_k} \left\{ \|x_k\|_{Q_1}^2 + \|u_k\|_{Q_2}^2 - \gamma^2 \|w_k\|^2 \right. \\ &\quad \left. + \|Ax_k + B_1 w_k + B_2 u_k\|_{M_{k+1}}^2 \right\}. \end{aligned}$$

Taking derivatives from the terms inside the brace with respect to u_k and w_k and setting them to zero, we obtain (23) and (24). ■

In the following, we assume that controls in the future are selected based on the optimal state-feedback policy. At time k , the transmitted measured outputs \mathbf{y}_{k-1}^o , the sampling variables δ_{k-1}^o , and the controls \mathbf{u}_{k-1}^o are fixed. We will

obtain the worst case disturbance $\tilde{\omega}_{k-1} = (\tilde{\mathbf{w}}_{k-1}, \tilde{\mathbf{v}}_{k-1}, \tilde{x}_0)$ that maximizes the total cost and is compatible with all the information available at the controller. Then, we will show that the worst case disturbance defines a state trajectory that can be used in the optimal control policy.

Let us define the disturbance constraint set as

$$\Omega_k = \left\{ \omega_{k'} \left| \begin{array}{l} k' \geq k, \ell \leq k, \\ x_{\ell+1} = Ax_\ell + B_1 w_\ell + B_2 u_\ell^o, \\ y_\ell^o = C_2 x_\ell + D_{21} v_\ell, \text{ if } \delta_\ell^o = 1 \end{array} \right. \right\}, \quad (25)$$

and the auxiliary performance index as

$$\begin{aligned} G_k &= -\gamma^2 \|x_0\|_{Q_0}^2 + \sum_{\ell=0}^k \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 \\ &\quad + V_{k+1}(x_{k+1}). \end{aligned} \quad (26)$$

The worst case disturbance under event-triggered sampling is obtained by solving the following problem:

$$\sup_{\omega_{k-1} \in \Omega_{k-1}} G_{k-1}, \quad (27)$$

given \mathbf{y}_{k-1}^o , \mathbf{u}_{k-1}^o , and δ_{k-1}^o .

Lemma 6: Suppose that there exists a unique maximum to the problem in (27). Then, the following two-point boundary value problem admits a unique solution satisfying the worse case disturbance and worst case state trajectory:

$$\begin{aligned} \tilde{x}_{\ell+1} &= A\tilde{x}_\ell + \gamma^{-2} R_1 \lambda_{\ell+1} + B_2 u_\ell^o, \\ \lambda_{\ell+} &= A^T \lambda_{\ell+1} + Q_1 \tilde{x}_\ell; \lambda_k = M_k \tilde{x}_k, \\ \lambda_\ell &= \lambda_{\ell+} + \gamma^2 C_2^T R_2^{-1} (y_\ell^o - C_2 \tilde{x}_\ell); \ell \in \mathcal{S}, \\ \tilde{w}_\ell &= \gamma^{-2} B_1^T \lambda_\ell, \\ \tilde{v}_\ell &= D_{21}^{-1} (y_\ell^o - C_2 \tilde{x}_\ell), \\ \tilde{x}_0 &= \gamma^{-2} Q_0^{-1} \lambda_0. \end{aligned}$$

where ℓ^+ denotes time ℓ^+ before λ_ℓ is updated.

Proof: Let us introduce the compact dynamics for $\zeta_{k-1}^o, \mathbf{z}_{k-1}, x_k$ as

$$\zeta_{k-1}^o = \phi_1 \mathbf{u}_{k-1}^o + \phi_2 \mathbf{w}_{k-1} + \phi_3 x_0 + \phi_4 \mathbf{v}_{k-1}, \quad (28)$$

$$\mathbf{z}_{k-1} = \theta_1 \mathbf{u}_{k-1}^o + \theta_2 \mathbf{w}_{k-1} + \theta_3 x_0, \quad (29)$$

$$x_k = \psi_1 \mathbf{u}_{k-1}^o + \psi_2 \mathbf{w}_{k-1} + \psi_3 x_0. \quad (30)$$

Following the above compact dynamics and Lemma 4, we can rewrite the problem in (27) as

$$\begin{aligned} \text{maximize } G_{k-1} &= \|\psi_1 \mathbf{u}_{k-1}^o + \psi_2 \mathbf{w}_{k-1} + \psi_3 x_0\|_{M_k}^2 \\ &\quad + \|\theta_1 \mathbf{u}_{k-1}^o + \theta_2 \mathbf{w}_{k-1} + \theta_3 x_0\|^2 \\ &\quad - \gamma^2 (\|\mathbf{w}_{k-1}\|^2 + \|\mathbf{v}_{k-1}\|^2 + \|x_0\|_{Q_0}^2) \\ \text{subject to } T \zeta_{k-1}^o &= T(\phi_1 \mathbf{u}_{k-1}^o + \phi_2 \mathbf{w}_{k-1} + \phi_3 x_0 + \phi_4 \mathbf{v}_{k-1}), \end{aligned}$$

over $\mathbf{w}_{k-1}, \mathbf{v}_{k-1}, x_0$ where T is a diagonal matrix of appropriate dimension with $T_{\ell,\ell} = 0$ if $\ell \notin \mathcal{S}$ and $T_{\ell,\ell} = 1$ otherwise. We form the Lagrangian with the Lagrange

multiplier $2\mathbf{p}_{N-1}$ as

$$\begin{aligned} L = & \|\psi_1 \mathbf{u}_{k-1}^o + \psi_2 \mathbf{w}_{k-1} + \psi_3 x_0\|_{M_k}^2 \\ & + \|\theta_1 \mathbf{u}_{k-1}^o + \theta_2 \mathbf{w}_{k-1} + \theta_3 x_0\|^2 \\ & - \gamma^2 (\|\mathbf{w}_{k-1}\|^2 + \|\mathbf{v}_{k-1}\|^2 + \|x_0\|_{Q_0}^2) \\ & + 2\mathbf{p}_{N-1}^T (\phi_1 \mathbf{u}_{k-1}^o + \phi_2 \mathbf{w}_{k-1} + \phi_3 x_0 + \phi_4 \mathbf{v}_{k-1}) \\ & - 2\mathbf{p}_{N-1}^T \mathbf{y}_{k-1}^o. \end{aligned}$$

Notice that $p_\ell = 0$ if $\ell \notin \mathcal{S}$. Let $\tilde{\omega}_{k-1} = (\tilde{\mathbf{w}}_{k-1}, \tilde{\mathbf{v}}_{k-1}, \tilde{x}_0)$ be the solution. Taking derivatives with respect to $\tilde{\mathbf{w}}_{k-1}$, $\tilde{\mathbf{v}}_{k-1}$, and \tilde{x}_0 , and setting them to zero we obtain

$$\begin{aligned} \phi_2^T \mathbf{p}_{N-1} + \theta_2^T \mathbf{q}_{N-1} + \psi_2^T \lambda_N &= \gamma^2 \tilde{\mathbf{w}}_{k-1}, \\ \phi_4^T \mathbf{p}_{N-1} &= \gamma^2 \tilde{\mathbf{v}}_{k-1}, \\ \phi_3^T \mathbf{p}_{N-1} + \theta_3^T \mathbf{q}_{N-1} + \psi_3^T \lambda_N &= \gamma^2 Q_0 \tilde{x}_0, \end{aligned}$$

where

$$\begin{aligned} \mathbf{q}_{N-1} &= \theta_1 \mathbf{u}_{k-1}^o + \theta_2 \tilde{\mathbf{w}}_{k-1} + \theta_3 \tilde{x}_0, \\ \lambda_N &= M_k (\psi_1 \mathbf{u}_{k-1}^o + \psi_2 \tilde{\mathbf{w}}_{k-1} + \psi_3 \tilde{x}_0). \end{aligned}$$

From Lemma 8 in the Appendix, we can show that the following internal representation of the worst case disturbances is admitted:

$$\tilde{w}_\ell = \gamma^{-2} B_1^T \lambda_\ell, \quad (31)$$

$$\tilde{v}_\ell = \gamma^{-2} D_{21} p_\ell, \quad (32)$$

$$\tilde{x}_0 = \gamma^{-2} Q_0^{-1} \lambda_0, \quad (33)$$

where $p_\ell = \gamma^2 (D_{21} D_{21}^T)^{-1} (y_\ell^o - C_2 \tilde{x}_\ell)$ for $\ell \in \mathcal{S}$ obtained by substituting the worst case disturbance into the constraint. After few modifications, we get

$$\begin{aligned} \tilde{x}_{\ell+1} &= A \tilde{x}_\ell + \gamma^{-2} B_1 B_1^T \lambda_{\ell+1} + B_2 u_\ell^o, \\ \lambda_{\ell+1} &= A^T \lambda_\ell + C_1^T C_1 \tilde{x}_\ell, \\ \lambda_\ell &= \lambda_{\ell+1} + \gamma^2 C_2^T (D_{21} D_{21}^T)^{-1} (y_\ell^o - C_2 \tilde{x}_\ell); \ell \in \mathcal{S}. \end{aligned}$$

This completes the proof. \blacksquare

The results in Lemma 6 are not very useful from a computational point of view. Rather, we would like to construct a recursive equation for the worst case state trajectory. The next proposition gives us such a conversion.

Proposition 1: The two-point boundary value problem in Lemma 6 can be converted into an initial value problem in terms of new variables \check{x}_k and Σ_k :

$$\check{x}_{k+1} = \bar{A} \check{x}_k + B_2 u_k + \delta_k \bar{K}_k (y_k - C_2 \check{x}_k), \quad (34a)$$

$$\Sigma_{k+1} = A (\Sigma_k^{-1} + \delta_k C_2^T R_2^{-1} C_2 - \gamma^{-2} Q_1)^{-1} A^T + R_1, \quad (34b)$$

where

$$\bar{A} = A + A (\Sigma_k^{-1} - \gamma^{-2} Q_1)^{-1} \gamma^{-2} Q_1, \quad (34c)$$

$$\bar{K}_k = \bar{A} (\Sigma_k^{-1} + C_2^T R_2^{-1} C_2)^{-1} C_2^T R_2^{-1}, \quad (34d)$$

with initial conditions $\check{x}_0 = 0$ and $\Sigma_0 = Q_0^{-1}$.

Proof: Consider the solution Σ_k of the following Riccati equation:

$$\Sigma_{k+1} = A (\Sigma_{k+}^{-1} - \gamma^{-2} Q_1)^{-1} A^T + R_1,$$

$$\Sigma_{k+} = (\Sigma_k^{-1} + C_2^T R_2^{-1} C_2)^{-1}; \quad k \in \mathcal{S},$$

and define the recursive variable $\check{x}_k = \tilde{x}_k - \gamma^{-2} \Sigma_k \lambda_k$. We can write:

$$\check{x}_{k+1} = \bar{A} \check{x}_k + B_2 u_k, \quad (35)$$

$$\check{x}_{k+} = \check{x}_k + \Sigma_k + C_2^T R_2^{-1} (y_k - C_2 \check{x}_k); \quad k \in \mathcal{S}. \quad (36)$$

From the definition of the sampling variable δ_k , we obtain recursive equations in (34). \blacksquare

Lemma 7: Assume the transmitted outputs \mathbf{y}_{k-1} , the sampling variables δ_{k-1} , and the controls \mathbf{u}_{k-1} are fixed. Then,

$$\sup_{\omega_{k-1} \in \Omega_{k-1}} G_{k-1} = \inf_{u_k} \sup_{\omega_k \in \Omega_{k-1}} G_k, \quad (37)$$

and the optimal $\tilde{\omega}_{k-1}$ of both problems coincide.

Proof: We note that the disturbances w_k and v_k are unconstrained. Following the definition of G_k and in view of Lemma 3, we have:

$$\begin{aligned} \inf_{u_k} \sup_{\omega_k \in \Omega_{k-1}} G_k &= \inf_{u_k} \sup_{\omega_k \in \Omega_{k-1}} \left\{ -\gamma^2 \|x_0\|_{Q_0}^2 \right. \\ &\quad \left. + \sum_{\ell=0}^k \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 + V_{k+1}(x_{k+1}) \right\} \\ &= \sup_{\omega_{k-1} \in \Omega_{k-1}} \left\{ -\gamma^2 \|x_0\|_{Q_0}^2 + \sum_{\ell=0}^{k-1} \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 \right. \\ &\quad \left. + \inf_{u_k} \sup_{w_k, v_k} \left\{ \|z_k\|^2 - \gamma^2 \|w_k\|^2 - \gamma^2 \|v_k\|^2 + V_{k+1}(x_{k+1}) \right\} \right\} \\ &= \sup_{\omega_{k-1} \in \Omega_{k-1}} \left\{ -\gamma^2 \|x_0\|_{Q_0}^2 + \sum_{\ell=0}^{k-1} \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 \right. \\ &\quad \left. + V_k(x_k) \right\} = \sup_{\omega_{k-1} \in \Omega_{k-1}} G_{k-1}. \end{aligned}$$

This completes the proof. \blacksquare

The following theorem shows the optimal value of the game based on the auxiliary performance index G_k .

Theorem 1: The optimal value of the zero-sum dynamic game problem in (18) is given by

$$\inf_{\mu \in \mathcal{M}} \sup_{\omega} J_\gamma = \sup_{x_0} G_0. \quad (38)$$

Proof: Given $\omega_{k-1} \in \Omega_{k-1}$, we have

$$\begin{aligned} F_k &= \sup_{\omega_{k-1} \in \Omega_{k-1}} G_{k-1} = \inf_{u_k} \sup_{\omega_k \in \Omega_{k-1}} G_k \\ &= \sup_{\omega_k \in \Omega_{k-1}} G_k(u_k^*). \end{aligned}$$

Using u_k^* , we can write

$$F_{k+1} = \sup_{\omega_k \in \Omega_k} G_k(u_k^*).$$

Since $\Omega_k \subset \Omega_{k+1}$, we can show that F_k for a control policy $\tilde{\mu}$ is a decreasing function with time k , i.e., $F_{k+1} \leq F_k$. Moreover, $G_{N+1} = J_\gamma(\tilde{\mu}, \omega)$. Therefore,

$$J_\gamma(\tilde{\mu}, \omega) \leq \sup_{x_0} G_0. \quad (39)$$

Furthermore, we can choose a disturbance $\tilde{\omega}$ based on the state-feedback saddle-point policy such that for any μ ,

$$J_\gamma(\mu, \tilde{\omega}) \geq \sup_{x_0} G_0. \quad (40)$$

Hence, we establish the result. \blacksquare

Now, we can obtain the optimal control policy in (18) based on the worst case state trajectory.

Proposition 2: The optimal control policy in the problem (18) is a worst case certainty equivalence control given by

$$u_k^* = -L_k(I_n - \gamma^{-2}\Sigma_k M_k)\tilde{x}_k, \quad (41a)$$

where

$$L_k = B_2^T(M_{k+1}^{-1} + B_2 B_2^T - \gamma^{-2}R_1)^{-1}A. \quad (41b)$$

and \tilde{x}_k is the recursive variable given by (34).

Proof: From Lemma 7, we have

$$\begin{aligned} & \inf_{u_k} \sup_{\omega_k \in \Omega_{k-1}} G_k \\ &= \sup_{\omega_{k-1} \in \Omega_{k-1}} \left\{ -\gamma^2 \|x_0\|_{Q_0}^2 + \sum_{\ell=0}^{k-1} \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 \right. \\ & \quad \left. + \inf_{u_k} \sup_{\omega_k, v_k} \{ \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 + V_{k+1}(x_{k+1}) \} \right\}. \end{aligned}$$

Assume that the minimum and maximum exist. From Lemma 4, we have

$$u_k^*(x_k) = \operatorname{argmin}_{u_k} \max_{\omega_k} \{ \|z_\ell\|^2 - \gamma^2 \|w_\ell\|^2 - \gamma^2 \|v_\ell\|^2 + x_{k+1}^T M_{k+1} x_{k+1} \}.$$

From Theorem 1, this gives us the optimal control. In addition, $x_k = \tilde{x}_k$ is specified at time k following Lemma 6. From Lemma 5 and using the following transformation

$$\tilde{x}_k = (I_n - \gamma^{-2}\Sigma_k M_k)\tilde{x}_k, \quad (42)$$

we obtain (41). \blacksquare

Remark 1: Notice that the control policy in (41) is of form $u_k = u_k(\delta_{k-1})$, and it resembles the one in the problem analyzed by Witsenhausen [22]. Here, we showed that the control policy in (41) is optimal, because in addition to the transmitted output measurement y_{k-1} , the sampling variable δ_{k-1} is available to the controller at time k .

Remark 2: The disturbances corresponding to the worst case certainty equivalence control in (41) are

$$w_k^* = -G_k(I_n - \gamma^{-2}\Sigma_k M_k)\tilde{x}_k, \quad (43)$$

$$v_k^* = 0. \quad (44)$$

where

$$G_k = \gamma^2 B_1^T (M_{k+1}^{-1} + B_2 B_2^T - \gamma^{-2}R_1)^{-1}A. \quad (45)$$

C. Optimal Sampling Based on Value of Information

Now, we come back to the problem in (17). We have,

$$\begin{aligned} & \inf_{\pi \in \mathcal{P}} \left\{ J_c(\mathbf{x}_N \rightarrow \mathbf{y}_N) + \lambda \inf_{\mu \in \mathcal{M}} \sup_{\omega} J_\gamma \right\} \\ &= \inf_{\pi \in \mathcal{P}} \left\{ \sum_{k=0}^N I(\mathbf{x}_k; y_k | \tilde{\mathcal{I}}_k) \right. \\ & \quad - \lambda \gamma^2 \|x_0^*\|_{Q_0}^2 + \lambda \|x_{N+1}\|_{Q_3}^2 \\ & \quad \left. + \sum_{k=0}^N \lambda \|x_k\|_{Q_1}^2 + \lambda \|u_k^*\|_{Q_2}^2 - \lambda \gamma^2 \|w_k^*\|^2 - \lambda \gamma^2 \|v_k^*\|^2 \right\}, \end{aligned}$$

where the terms including x_0 do not depend on π and the conditional mutual information is calculated in (15). Using the worst case certainty equivalence control, we consider the worst case state \tilde{x}_k as the actual state of the system. Therefore, we can write

$$\begin{aligned} & \inf_{\pi \in \mathcal{P}} \left\{ -\sum_{k=0}^N \frac{1}{2} \ln \det(I_n - \delta_k K_k C_2) \right. \\ & \quad \left. + \lambda \|\tilde{x}_{N+1}\|_{Q_3}^2 \right. \\ & \quad \left. + \lambda \sum_{k=1}^N \|\tilde{x}_k\|_{Q_1}^2 + \|\tilde{x}\|_{L_k^T Q_2 L_k}^2 - \|\tilde{x}_k\|_{\gamma^2 G_k^T G_k}^2 \right\} \\ &= \inf_{\pi \in \mathcal{P}} \left\{ -\sum_{k=0}^N \frac{1}{2} \ln \det(I_n - \delta_k K_k C_2) + \lambda \sum_{k=1}^{N+1} \|\tilde{x}_k\|_{E_k}^2 \right\}, \end{aligned}$$

where

$$E_k = (I_n - \gamma^{-2}\Sigma_k M_k)^T (Q_1 + L_k^T Q_2 L_k - \gamma^2 G_k^T G_k) \times (I_n - \gamma^{-2}\Sigma_k M_k); \quad k = 1, \dots, N,$$

$$E_{N+1} = (I_n - \gamma^{-2}\Sigma_{N+1} M_{N+1})^T Q_3 (I_n - \gamma^{-2}\Sigma_{N+1} M_{N+1}).$$

Define W_k as

$$W_k = -\sum_{\ell=k}^N \frac{1}{2} \ln \det(I_n - \delta_\ell K_\ell C_2) + \sum_{\ell=k}^{N+1} \lambda \|\tilde{x}_\ell\|_{E_\ell}^2, \quad (46)$$

with final condition $W_{N+1} = \lambda \|\tilde{x}_{N+1}\|_{E_{N+1}}^2$. Following dynamic programming, we can show:

$$W_k = \min_{\delta_k} \left\{ -\frac{1}{2} \ln \det(I_n - \delta_k K_k C_2) + \lambda \|\tilde{x}_k\|_{E_k}^2 + W_{k+1} \right\}. \quad (47)$$

Definition 4: We define the value of information α_k at time k as

$$\alpha_k = W_{k+1}(0) - W_{k+1}(1), \quad (48)$$

where $W_{k+1}(\delta_k)$ is defined in (47).

The following proposition gives the optimal sampling policy based on the value of information.

Proposition 3: The optimal sampling policy in the optimization problem in (17) is given by

$$\delta_k^* = \begin{cases} 1, & \text{if } \alpha_k \geq -\frac{1}{2} \ln \det(I_n - K_k C_2), \\ 0, & \text{otherwise.} \end{cases} \quad (49)$$

Proof: The proof follows directly from (47) and (48). \blacksquare

Remark 3: The function W_k in (46) can be calculated using techniques in approximate dynamic programming [21].

IV. CONCLUSION

In this work, we studied the optimal H_∞ control problem where for transmission of the measurements, an event-triggering mechanism based on the value of the information is proposed. We specified the distribution of the disturbances, and employed the directed information to measure the transmitted information between the plant and the controller satisfying the causality condition. We showed that the optimal control can be written as a linear function of a recursive

variable obtained by solving an initial value problem. Finally, we showed that the optimal sampling policy can be written in terms of the value of information.

V. APPENDIX

Lemma 8: Suppose that the system in (3) has the following compact dynamics

$$\begin{aligned} \mathbf{y}_{k-1} &= \phi_1 \mathbf{u}_{k-1} + \phi_2 \mathbf{w}_{k-1} + \phi_3 x_0 + \phi_4 \mathbf{v}_{k-1}, \\ \mathbf{z}_{k-1} &= \theta_1 \mathbf{u}_{k-1} + \theta_2 \mathbf{w}_{k-1} + \theta_3 x_0, \\ \hat{x}_k &= \psi_1 \mathbf{u}_{k-1} + \psi_2 \mathbf{w}_{k-1} + \psi_3 x_0, \end{aligned}$$

with the corresponding dual operator:

$$\begin{aligned} \alpha_{N-1} &= \phi_1^T \mathbf{p}_{N-1} + \theta_1^T \mathbf{q}_{N-1} + \psi_1^T \lambda_N, \\ \beta_{N-1} &= \phi_2^T \mathbf{p}_{N-1} + \theta_2^T \mathbf{q}_{N-1} + \psi_2^T \lambda_N, \\ \lambda_0 &= \phi_3^T \mathbf{p}_{N-1} + \theta_3^T \mathbf{q}_{N-1} + \psi_3^T \lambda_N, \\ \epsilon &= \phi_4^T \mathbf{p}_{N-1}, \end{aligned}$$

with $p_k = 0$ for $k \in \mathcal{S}$. Then, the following internal dynamics is admitted

$$\begin{aligned} \alpha_k &= B_2^T \lambda_k + D_{12}^T q_k, \\ \beta_k &= B_1^T \lambda_k, \\ \lambda_{k^+} &= A^T \lambda_{k+1} + C_1^T q_k, \\ \lambda_k &= \lambda_{k^+} + C_2^T p_k; \quad k \in \mathcal{S} \\ \epsilon_k &= D_{21} p_k. \end{aligned}$$

where k^+ denotes time k^+ before λ_k is updated.

Proof: Substitute x_{k+1} and λ_k in the identity

$$\begin{aligned} \lambda_N x_N - \lambda_0 x_0 &= \sum_{k=1}^{N-1} (\lambda_{k+1} x_{k+1} - \lambda_k x_k) + \sum_{k \in \mathcal{S}} (\lambda_{k^+} - \lambda_k) x_k. \quad (53) \end{aligned}$$

Then, we have

$$\begin{aligned} \lambda_N x_N + \sum_{k=1}^N (q_k^T z_k) + \sum_{k \in \mathcal{S}} (p_k y_k) &= \lambda_0 x_0 + \sum_{k \in \mathcal{S}} (\pi_k w_k) + \sum_{k=1}^N (\beta_k v_k) + \sum_{k=1}^N (\alpha_k u_k). \quad (54) \end{aligned}$$

This proves the lemma. \blacksquare

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