

Emergent Behaviors over Signed Random Dynamical Networks: Relative-State-Flipping Model

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Abstract—We study asymptotic dynamical patterns that emerge among a set of nodes interacting in a dynamically evolving signed random network, where positive links carry out standard consensus and negative links induce relative-state flipping. A sequence of deterministic signed graphs defines potential node interactions that take place independently. Each node receives a positive recommendation consistent with the standard consensus algorithm from its positive neighbors, and a negative recommendation defined by relative-state flipping from its negative neighbors. After receiving these recommendations, each node puts a deterministic weight to each recommendation, and then encodes these weighted recommendations in its state update through stochastic attentions defined by two Bernoulli random variables. We establish a number of conditions regarding almost sure convergence and divergence of the node states. We also propose a condition for almost sure state clustering for essentially weakly balanced graphs, with the help of several martingale convergence lemmas. Some fundamental differences on the impact of the deterministic weights and stochastic attentions to the node state evolution are highlighted between the current relative-state-flipping model and the state-flipping model considered in Shi *et al.*, *IEEE Transaction on Control of Network Systems*, 2015.

Keywords. Random graphs, Signed networks, Consensus dynamics, Belief clustering

I. INTRODUCTION

The emergent behaviors, such as consensus, swarming, clustering, and learning, of the dynamics evolving over a large complex network of interconnected nodes have attracted a significant amount of research attention in the past decades [2]–[6]. In most cases node interactions are collaborative, reflected by that their state updates obey the same rule which is spontaneous or artificially designed aiming for some particular collective task. This however might not always be true since nodes take on different, or even opposing, roles, where examples arise in biology [8], [9], social science [10]–[12], and engineering [13].

Consensus problems aim to compute a weighted average of the initial values held by a collection of nodes, in a distributed manner. The DeGroot’s model [2], as a standard

consensus algorithm, described how opinions evolve in a network of agents, and showed that a simple deterministic opinion update based on the mutual trust and the differences in belief between interacting agents could lead to global convergence of the beliefs. Consensus dynamics have since then been widely adopted for describing opinion dynamics in social networks, e.g., [6], [7], [14]. In engineering sciences, a huge amount of literature has studied these algorithms for distributed averaging, formation forming and load balancing between collaborative agents under fixed or time-varying interaction networks [15]–[22]. Randomized consensus seeking has also been widely studied, motivated by the random nature of interactions and updates in real complex networks [23]–[30].

This paper aims to study consensus dynamics with both collaborative and non-collaborative node interactions. A convenient framework for modeling different roles and relationships between agents is to use signed graphs introduced in the classical work by Heider in 1946 [10]. Each link is associated with a sign, either positive or negative, indicating collaborative or non-collaborative relationships. In [34], a model for consensus over signed graphs was introduced for continuous-time dynamics, where a node flips the sign of its true state to a negative (antagonistic) node during the interaction. The author of [34] showed that state polarization (clustering) of the signed consensus model is closely related to the so-called structural balance in classical social signed graph theory [37]. In [35], the authors proposed a model for investigating the transition between agreement and disagreement when each link randomly takes three types of interactions: attraction, repulsion, and neglect, which was further generalized to a signed-graph setting in [36].

We assume a sequence of deterministic signed graphs that defines the interactions of the network. Random node interactions take place under independent, but not necessarily identically distributed, random sampling of the environment. Once interaction relations have been realized, each node receives a positive recommendation consistent with the standard consensus algorithm from its positive neighbors. Nodes receive negative recommendations from its negative neighbors. After receiving these recommendations, each node puts a (deterministic) weight to each recommendation, and then encodes these weighted recommendations in its state update through stochastic attentions defined by two Bernoulli random variables. In [1], we studied almost sure convergence, divergence, and clustering under the definition of Altafini [34] for negative interactions, for which we referred to as a state-

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flipping model.

In this paper, we further investigate this random consensus model for signed networks under a relative-state-flipping setting, where instead of taking negative feedback of the relative state in standard consensus algorithms [2], [4], a positive feedback takes place along every interaction arc of a negative sign. This relative-state flipping formulation is consistent with the models in [35], [36], and can be viewed as a natural opposite of the DeGroot's type of node interactions. For the proposed relative-state-flipping model, we establish a number of conditions regarding almost sure convergence and divergence of the node states. We also propose a condition for almost sure node state clustering for essentially weakly balanced graphs, with the help of several martingale convergence lemmas. Some fundamental differences on the impact of the deterministic weights and stochastic attentions to the node state evolution are highlighted between the current relative-state-flipping model and the state-flipping model.

The remainder of the paper is organized as follows. Section II presents the network dynamics and the node update rules, and specifies the information-level difference between the relative-state-flipping and state-flipping models. Section III presents our main results; the detailed proofs are given in Section IV. Finally some concluding remarks are drawn in Section V.

Graph Theory, Notations and Terminologies

A simple directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set \mathcal{V} of nodes and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where $e = (i, j) \in \mathcal{E}$ denotes an *arc* from node $i \in \mathcal{V}$ to $j \in \mathcal{V}$ with $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. We call node j *reachable* from node i if there is a directed path from i to j . In particular every node is supposed to be reachable from itself. A node v from which every node in \mathcal{V} is reachable is called a *center node* (root). A digraph \mathcal{G} is *strongly connected* if every two nodes are mutually reachable; \mathcal{G} has a spanning tree if it has a center node; \mathcal{G} is *weakly connected* if every two nodes are reachable from each other after removing all the directions of the arcs in \mathcal{E} . A subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is a graph on the same node set \mathcal{V} whose arc set is a subset of \mathcal{E} . The induced graph of $\mathcal{V}_i \subseteq \mathcal{V}$ on \mathcal{G} , denoted $\mathcal{G}|_{\mathcal{V}_i}$, is the graph $(\mathcal{V}_i, \mathcal{E}_i)$ with $\mathcal{E}_i = (\mathcal{V}_i \times \mathcal{V}_i) \cap \mathcal{E}$. A weakly connected component of \mathcal{G} is a maximal weakly connected induced graph of \mathcal{G} . If each arc $(i, j) \in \mathcal{E}$ is associated uniquely with a sign, either '+' or '-', \mathcal{G} is called a signed graph and the sign of $(i, j) \in \mathcal{E}$ is denoted as σ_{ij} . The positive and negative subgraphs containing the positive and negative arcs of \mathcal{G} , are denoted as $\mathcal{G}^+ = (\mathcal{V}, \mathcal{E}^+)$ and $\mathcal{G}^- = (\mathcal{V}, \mathcal{E}^-)$, respectively.

Depending on the argument, $|\cdot|$ stands for the absolute value of a real number, the Euclidean norm of a vector or the cardinality of a set. The σ -algebra of a random variable is denoted as $\sigma(\cdot)$. We use $\mathbb{P}(\cdot)$ to denote the probability and $\mathbb{E}\{\cdot\}$ the expectation of their arguments, respectively.

II. RANDOM NETWORK MODEL AND NODE UPDATES

In this section, we present the considered random network model and specify individual node dynamics. We use the same

definition of random signed networks as introduced in [1], where each link is associated with a sign indicating cooperative or antagonistic relations. In the current work we study relative-state-flipping dynamics along each negative arcs, in contrast with the state-flipping dynamics studied in [1]. The main difference of the information patterns between the two models will also be carefully explained.

A. Signed Random Dynamical Networks

Consider a network with a set $\mathcal{V} = \{1, \dots, n\}$ of n nodes, with $n \geq 3$. Time is slotted for $t = 0, 1, \dots$. Let $\{\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)\}_{t=0}^{\infty}$ be a sequence of (deterministic) signed directed graphs over node set \mathcal{V} . We denote by $\sigma_{ij}(t) \in \{+, -\}$ the sign of arc $(i, j) \in \mathcal{E}_t$. The positive and negative subgraphs containing the positive and negative arcs of \mathcal{G}_t , are denoted by $\mathcal{G}_t^+ = (\mathcal{V}, \mathcal{E}_t^+)$ and $\mathcal{G}_t^- = (\mathcal{V}, \mathcal{E}_t^-)$, respectively. We say that the sequence of graphs $\{\mathcal{G}_t\}_{t \geq 0}$ is *sign consistent* if the sign of any arc (i, j) does not evolve over time, i.e., if for any $s, t \geq 0$,

$$(i, j) \in \mathcal{E}_s \text{ and } (i, j) \in \mathcal{E}_t \implies \sigma_{ij}(s) = \sigma_{ij}(t).$$

We also define $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ with $\mathcal{E}_* = \bigcup_{t=0}^{\infty} \mathcal{E}_t$ as the total graph of the network. If $\{\mathcal{G}_t\}_{t \geq 0}$ is sign consistent, then the sign of each arc \mathcal{E}_* never changes and in that case, $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ is a well-defined signed graph. The notion of positive cluster in a signed directed graph is defined as follows.

Definition 1: Let \mathcal{G} be a signed digraph with positive subgraph \mathcal{G}^+ . A subset \mathcal{V}_* of the set of nodes \mathcal{V} is a positive cluster if \mathcal{V}_* constitutes a weakly connected component of \mathcal{G}^+ . A positive cluster partition of \mathcal{G} is a partition of \mathcal{V} into $\mathcal{V} = \bigcup_{i=1}^{T_p} \mathcal{V}_i$ for some $T_p \geq 1$, where for all $i = 1, \dots, T_p$, \mathcal{V}_i is a positive cluster.

Note that \mathcal{G} admitting a positive-cluster partition is a generalization of the classical definition of weakly structural balanced graph for which negative links are strictly forbidden inside each positive cluster [38]. Given any node i , there is a unique weakly connected component of \mathcal{G}^+ that contains the node i (particularly, if all arcs pointing to and leaving from node i are negative, node i by itself is a weakly connected component of \mathcal{G}^+ from the definition). Therefore, it is clear that for any signed graph \mathcal{G} , there is a unique positive cluster partition $\mathcal{V} = \bigcup_{i=1}^{T_p} \mathcal{V}_i$ of \mathcal{G} , where T_p is the number of positive clusters covering the entire set \mathcal{V} of nodes.

Each node randomly interacts with her neighboring nodes in \mathcal{G}_t at time t . We present a general model on the random node interactions at a given time t . At time t , some pairs of nodes are randomly selected for interaction. We denote by $E_t \subset \mathcal{E}_t$ the random subset of arcs corresponding to interacting node pairs at time t . To be precise, E_t is sampled from the distribution μ_t defined over the set Ω_t of all subsets of arcs in \mathcal{E}_t . We assume that E_0, E_1, \dots form a sequence of independent sets of arcs. Formally, we introduce the probability space $(\Theta, \mathcal{F}, \mathbb{P})$ obtained by taking the product of the probability spaces $(\Omega_t, \mathcal{S}_t, \mu_t)$, where \mathcal{S}_t is the discrete σ -algebra on Ω_t ; $\Theta = \prod_{t \geq 0} \Omega_t$, \mathcal{F} is the product of σ -algebras \mathcal{S}_t , $t \geq 0$, and \mathbb{P} is the product probability measure of μ_t , $t \geq 0$. We denote

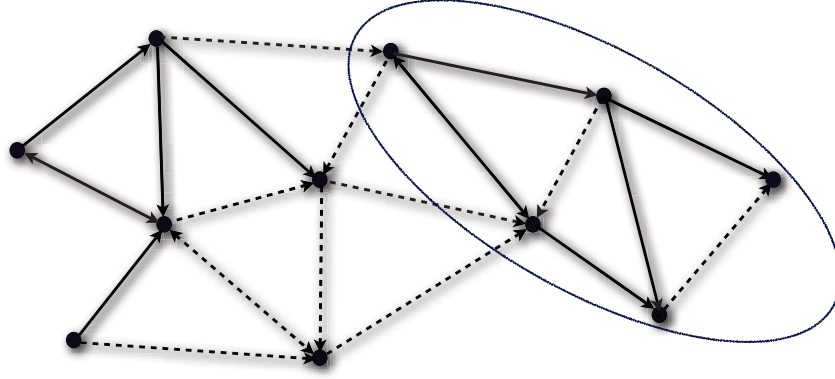


Figure 1. A signed network and its three positive clusters. The positive arcs are solid, and the negative arcs are dashed. Note that negative arcs are allowed within positive clusters.

by $G_t = (\mathcal{V}, E_t)$ the random subgraph of \mathcal{G}_t corresponding to the random set E_t of arcs. The disjoint sets E_t^+ and E_t^- denote the positive and negative arc set of E_t , respectively. Finally, we split the random set of nodes interacting with node i at time t depending on the sign of the corresponding arc: for node i , the set of positive neighbors is defined as $N_i^+(t) := \{j : (j, i) \in E_t^+\}$, whereas similarly, the set of negative neighbors is $N_i^-(t) := \{j : (j, i) \in E_t^-\}$.

B. Node updates

Each node i holds a state $s_i(t) \in \mathbb{R}$ at $t = 0, 1, \dots$. To update her state at time t , node i considers recommendations received from her positive and negative neighbors:

- (i) The positive recommendation node i receives at time t is

$$h_i^+(t) := - \sum_{j \in N_i^+(t)} (s_i(t) - s_j(t));$$

- (ii) The negative recommendations node i receives at time t is defined as:

$$h_i^-(t) := \sum_{j \in N_i^-(t)} (s_i(t) - s_j(t)).$$

In the above expressions, we use the convention that summing over empty sets yields a recommendation equal to zero, e.g., when node i has no positive neighbors, then $h_i^+(t) = 0$. In view of the definition of $h_i^-(t)$ in contrast to $h_i^+(t)$, the model is referred to as the relative-state-flipping model.

Remark 1: In [1], we have considered another notion of negative recommendations, namely the state-flipping model introduced in [34], defined as $h_i^-(t) := - \sum_{j \in N_i^-(t)} (s_i(t) + s_j(t))$. We remark that for the relative-state-flipping model, the network does not require a central global coordinate system and nodes can interact based on relative state only. As has been pointed in [1], in the state-flipping model, the network nodes are necessary to share a common knowledge of the origin of the state space.

Remark 2: The two definitions of negative recommendations, the relative-state-flipping model considered in the

current paper, and the state-flipping model studied in [1], [34], have different physical interpretations and make different assumptions on the knowledge that nodes possess about their neighbor relationships. In the state-flipping model, naturally it is the *head* node along each negative arc, that possesses the knowledge of sign of that arc. In the relative-state-flipping model, on the other hand, it is the *tail* node knows the sign of each directed arc so that nodes know if a specific neighbor is positive or negative to implement the state updates that cause the repulsive influence from its negative neighbors.

Let $\{B_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$ be two sequences of independent Bernoulli random variables. We assume that $\{B_t\}_{t \geq 0}$, $\{D_t\}_{t \geq 0}$, and $\{G_t\}_{t \geq 0}$ define independent processes. For any $t \geq 0$, define $b_t = \mathbb{E}\{B_t\}$ and $d_t = \mathbb{E}\{D_t\}$. The processes $\{B_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$ represent how much attention node i pays to the positive and negative recommendations, respectively. Node i updates her state as follows:

$$s_i(t+1) = s_i(t) + \alpha B_t h_i^+(t) + \beta D_t h_i^-(t), \quad (1)$$

where $\alpha, \beta > 0$ are two positive constants marking the weight each node put on the positive and negative recommendations, respectively. For the value of α , we impose the following assumption.

A1. (i) There holds $\alpha \in (0, (n-1)^{-1})$; (ii) There is a constant $p_* \in (0, 1)$ such that for all $t \geq 0$ and $i, j \in \mathcal{V}$, $\mathbb{P}((i, j) \in E_t) \geq p_*$ if $(i, j) \in \mathcal{E}_t$.

The Assumption A1.(i) is just to ensure that the node update rule (1) falls to standard consensus algorithms with positive recommendations only, which has to be made a bit conservative since we aim to build the results on general random dynamical networks. This bound for α can certainly be relaxed without affecting any of our results if the random dynamical network takes certain particular forms, e.g., it will only require $\alpha \in (0, 1)$ for the upcoming analysis if gossiping processes [24] are considered. The Assumption A1.(ii) is a regularity condition to the arc existence in different time slots, which is in fact quite general and holds for many existing random network models, e.g., [23], [25], [26]. The Assumption

A1 will be adopted throughout this paper without specific further mention.

Let $s(t) = (s_1(t) \dots s_n(t))^T$ be the random vector representing the network state at time t . The main objective of this paper is to analyze the behavior of the stochastic process $\{s(t)\}_{t \geq 0}$. In the following, we denote by \mathbb{P} the probability measure capturing all random components driving the evolution of $s(t)$.

In the remainder of the paper, we establish the asymptotic properties of the network state evolution under relative-state-flipping model. As will be shown in the following, the state-flipping and relative-state-flipping models share some common nature, e.g., almost sure state convergence/divergence, no-survivor property, etc. In the mean time these common properties can be driven by fundamentally different parameters regarding network connectivity and recommendation weights and attentions. We introduce a few assumptions on the random graph process and dynamical environment.

A2. $\{\mathcal{G}_t\}_{t \geq 0}$ is sign consistent admitting a total graph \mathcal{G}_* .

A3. The events $\{(i, j) \in E_t\}$, $i, j \in \mathcal{V}$, $t = 0, 1, \dots$ are independent and there is a constant $p^* \in (0, 1)$ such that for all $t \geq 0$ and $i, j \in \mathcal{V}$, $\mathbb{P}((i, j) \in E_t) \leq p^*$ if $(i, j) \in \mathcal{E}_t$.

The Assumption A2 ensures that although the arc sets of \mathcal{G}_t^+ and \mathcal{G}_t^- can be time-varying, there is no arc that varies its sign over time. This means that the nature of the relationship (positive or negative) between any two nodes is fixed while the underlying interaction structure is changing. The Assumption A3 is a technical assumption, which again can be verified for a large class of random network models studied in the literature, e.g., [23], [25], [26].

III. MAIN RESULTS

In this section, we present the main results for the asymptotic behaviors of the random process defined by the considered relative-state-flipping model.

A. General Conditions

First of all, the following theorem provides general conditions for convergence and divergence.

Theorem 1: Assume that for any $t \geq 0$, $\mathcal{G}_t \equiv \mathcal{G}$ for some digraph \mathcal{G} , and that each positive cluster of \mathcal{G} admits a spanning tree in \mathcal{G}^+ . For Algorithm (1) under the relative-state-flipping model, we have:

- (i) If $\sum_{t=0}^{\infty} d_t < \infty$, then $\mathbb{P}(\lim_{t \rightarrow \infty} s_i(t) \text{ exists}) = 1$ for all node $i \in \mathcal{V}$ and all initial states $s(0)$;
- (ii) If $\sum_{t=0}^{\infty} d_t = \infty$, \mathcal{G} has two positive clusters with no negative links in each cluster, and there is a negative arc between any two nodes from different clusters, then there exist an infinite number of initial states $s(0)$ such that

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} \max_{i, j \in \mathcal{V}} |s_i(t) - s_j(t)| = \infty\right) = 1. \quad (2)$$

Note that by saying $\lim_{t \rightarrow \infty} s_i(t)$ exists, we mean $s_i(t)$ converges to a finite limit. The first part of the above theorem

indicates that when the environment is frozen, and when positive clusters are properly connected, then irrespective of the mean of the positive attentions $\{b_t\}_0^\infty$, the system states converge if the attention each node puts in her negative neighbors decays sufficiently fast over time. The second part of the theorem states that when this attention does not decay, divergence can be observed.

Remark 3: Theorem 1 shows that well-structured positive arcs and asymptotically decaying attention guarantee state convergence for relative-state-flipping model. The essential reason is that when $\sum_{t=0}^{\infty} d_t < \infty$, the first Borel-Cantelli lemma (cf. Theorem 2.3.1, [32]) ensures that along almost every sample path, negative interactions happen only for a finite number of time instants. The positive interactions continue to guide the network states to a finite limit under suitable connectivity. It is then clear that the same condition can also guarantee state-convergence for the state-flipping model considered in [1].

In fact, for the state convergence property of the state-flipping model, a much stronger conclusion regarding state convergence was shown (Theorem 1 in [1]) indicating that each positive/negative arc contributes to the state convergence under constant attention $\{b_t\}$ and $\{d_t\}$, as long as $\alpha + \beta \leq (n-1)^{-1}$. We can easily build examples showing that it is a completely different story on this matter for the relative-state-flipping model considered in the current paper.

Remark 4: The divergence statement in Theorem 1 is not true for the state-flipping model [1], where almost sure state divergence always requires sufficiently large β .

B. Deviation Consensus

We define almost sure deviation consensus as follows.

Definition 2: Algorithm (1) achieves almost sure deviation consensus if

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} \max_{i, j \in \mathcal{V}} |s_i(t) - s_j(t)| = 0\right) = 1.$$

Note that almost sure deviation consensus means that the distances among the node states converge to zero, but convergence of each node state is not required. We need the following condition on the positive graphs, which assumes a directed spanning tree in the union graph of the environment over certain bounded time intervals [19].

A4. There is an integer $K \geq 1$ such that the union graph $\mathcal{G}^+([t, t+K]) = (\mathcal{V}_i, \bigcup_{\tau \in [t, t+K-1]} \mathcal{E}_\tau^+)$ has a spanning tree for all $t \geq 0$.

Theorem 2: Assume that A4 holds. Denote $K_0 = (2n-3)K$ and $\rho_* = \min\{\alpha, 1 - (n-1)\alpha\}$. Define

$$X_m = \frac{p_*^{n-1} \rho_*^{K_0}}{2} \prod_{t=mK_0}^{(m+1)K_0-1} (b_t(1-d_t)),$$

and

$$Y_m = (1 + 2\beta(n-1))^{K_0} \left(1 - \prod_{t=mK_0}^{(m+1)K_0-1} (1-d_t)\right).$$

Then under the relative-state-flipping model, if $0 \leq X_m - Y_m \leq 1$ for all $m \geq 0$ and $\sum_{m=0}^{\infty} (X_m - Y_m) = \infty$, Algorithm (1) achieves almost sure deviation consensus for all initial states.

The following corollary is a direct consequence from Theorem 2, obtained by straightforward verification of the parameter conditions in Theorem 2.

Corollary 1: Assume that A4 holds. Let $b_t \equiv b$ and $d_t \equiv d$ with $b, d \in (0, 1)$. Then there exists $d_* > 0$ such that deviation consensus is achieved almost surely whenever $d < d_*$.

Note that, deviation consensus does not necessarily guarantee the convergence of the state of each node. In fact, simple examples can be constructed with arbitrarily small β such that under the relative-state-flipping model, the state of each node grows arbitrarily large while deviation consensus still holds. This fact is in sharp contrast with the result for the state-flipping model: the condition $\alpha + \beta < (n-1)^{-1}$ prevents the state of individual nodes to diverge (i.e., Theorem 1 in [1]).

C. Almost Sure Divergence

We continue to provide conditions under which the maximal gap between the states of two nodes grows large almost surely. We introduce a new technical condition on the negative graph, which assumes that the negative arcs span the entire network leading to a weakly connected graph over certain bounded time intervals.

A5. There is an integer $K \geq 1$ such that the union graph $\mathcal{G}^-([t, t+K]) = (\mathcal{V}, \bigcup_{\tau \in [t, t+K-1]} \mathcal{E}_\tau^-)$ is weakly connected for all $t \geq 0$.

Theorem 3: Assume that A3 and A5 hold. Let $b_t \equiv b$ and $d_t \equiv d$ for some constants $b, d \in (0, 1)$. In addition we require $\alpha \in [0, (n-1)^{-1}/2)$. Then for Algorithm (1) under the relative-state-flipping model, there is $b_* > 0$ such that whenever $b < b_*$, we have $\mathbb{P}(\lim_{t \rightarrow \infty} \max_{i,j \in \mathcal{V}} |s_i(t) - s_j(t)| = \infty) = 1$ for almost all initial states (under the standard Lebesgue measure).

Remark 5: Theorem 3 indicates that in relative-state-flipping model, almost sure relative-state divergence can be achieved as long as negative interactions happen sufficiently more often than the positive interactions. As explained in above remarks, for state-flipping model, state divergence necessarily require sufficiently large weight on negative recommendations.

D. State Clustering

Finally, we investigate the clustering of states of nodes within each positive cluster. The following assumption ensures a minimum level of connectivity inside each positive cluster as the environment changes.

A6. Assume that A2 holds and let $\mathcal{V} = \bigcup_{i=1}^{T_p} \mathcal{V}_i$ be a positive-cluster partition of the total graph \mathcal{G}_* . There is an integer $K \geq 1$ such that the union graph $\mathcal{G}^+([t, t+K])|_{\mathcal{V}_i} = (\mathcal{V}_i, \bigcup_{\tau \in [t, t+K-1]} \mathcal{E}_\tau^+|_{\mathcal{V}_i})$ has a spanning tree for all $t \geq 0$.

Theorem 4: Assume that A2 and A6 hold and let $\mathcal{V} = \bigcup_{i=1}^{T_p} \mathcal{V}_i$ be a positive-cluster partition of \mathcal{G}_* . Define $J(m) = \prod_{t=m}^{(m+1)K_0-1} b_t$ and $W(m) = \sum_{t=m}^{(m+1)K_0-1} d_t$ with $K_0 = (2n-3)K$. Further assume that $\sum_{m=0}^{\infty} J(m) = \infty$, $\sum_{t=0}^{\infty} d_t < \infty$, and $\lim_{m \rightarrow \infty} W(m)/J(m) = 0$. Then under the relative-state-flipping model, for any initial state $s(0)$, Algorithm (1) achieves a.s. state clustering in the sense that there are T_p real-valued random variables, $w_1^*, \dots, w_{T_p}^*$, such that

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} s_i(t) = w_j^*, i \in \mathcal{V}_j, j = 1, \dots, T_p\right) = 1.$$

Theorem 4 shows the possibility of state clustering for every positive cluster, whose proof is based on a martingale convergence lemma.

IV. PROOFS OF STATEMENTS

In this section, we establish the proofs of the various statements presented in the previous section.

A. Supporting Lemmas

The following is a martingale convergence lemma (see e.g. [33]).

Lemma 1: Let $\{v_t\}_{t \geq 0}$ be a sequence of non-negative random variables with $\mathbb{E}\{v_0\} < \infty$. Assume that for any $t \geq 0$,

$$\mathbb{E}\{v_{t+1} | v_0, \dots, v_t\} \leq (1 - \xi_t)v_t + \theta_t,$$

where $\{\xi_t\}_{t \geq 0}$ and $\{\theta_t\}_{t \geq 0}$ are two (deterministic) sequences of non-negative numbers satisfying $\forall t \geq 0, 0 \leq \xi_t \leq 1, \sum_{t=0}^{\infty} \xi_t = \infty, \sum_{t=0}^{\infty} \theta_t < \infty$, and $\lim_{t \rightarrow \infty} \theta_t/\xi_t = 0$. Then $\lim_{t \rightarrow \infty} v_t = 0$ a.s..

We define $h(t) := \min_{i \in \mathcal{V}} s_i(t)$, $H(t) := \max_{i \in \mathcal{V}} s_i(t)$, and $\mathcal{H}(t) := H(t) - h(t)$, which will be used throughout the rest of the paper. The following lemma holds.

Lemma 2: Assume that $\sum_{t=0}^{\infty} d_t < \infty$. Then under the relative-state-flipping model, for all initial states, each of $h(t)$, $H(t)$, $\mathcal{H}(t)$ converges almost surely, respectively, to some random variables that take finite values with probability one.

Proof. If $D_t = 0$, then there hold

$$\begin{aligned} H(t+1) &\leq H(t); \\ h(t+1) &\geq h(t); \\ \mathcal{H}(t+1) &\leq \mathcal{H}(t); \end{aligned} \tag{3}$$

in view of the condition $\alpha \in [0, (n-1)^{-1}]$.

Based on the first Borel-Cantelli Lemma (cf. Theorem 2.3.1, [32]), $\sum_t d_t < \infty$ ensures that

$$\mathbb{P}(D_t = 1 \text{ for finitely many } t) = 1.$$

In other words, we can define¹

$$\mathcal{T}_D := \sup_t \{D_t = 1\},$$

¹From its definition it is clear that \mathcal{T}_D is not a stopping time. Our argument however relies only on the fact that $\mathbb{P}(\mathcal{T}_D < \infty) = 1$.

which is finite with probability one. This is to say, the inequalities in (5) hold almost surely for $t \geq \mathcal{T}_D$.

As a result, we conclude that

$$h(\mathcal{T}_D) \leq h(t) \leq h(t+1) \leq H(t+1) \leq H(t) \leq H(\mathcal{T}_D) \quad (4)$$

for all $t \geq \mathcal{T}_D$. It is obvious that both $h(\mathcal{T}_D)$ and $H(\mathcal{T}_D)$ are random variables taking finite values with probability one. The almost sure convergence of $h(t)$ and $H(t)$ therefore follows directly from (4), which in turn implies the almost sure convergence of $\mathcal{H}(t)$.

We have now completed the proof. \blacksquare

Lemma 3: Assume that $\sum_{t=0}^{\infty} d_t < \infty$. Assume that for any $t \geq 0$, $\mathcal{G}_t \equiv \mathcal{G}$ for some digraph \mathcal{G} . Let \mathcal{V}^\dagger be a positive cluster of \mathcal{G} . Then $H^\dagger(t) := \max_{i \in \mathcal{V}^\dagger} s_i(t)$ and $h^\dagger := \min_{i \in \mathcal{V}^\dagger} s_i(t)$ converge almost surely, respectively, to some random variables that take finite value with probability one.

Proof. Similarly, there hold

$$\begin{aligned} H^\dagger(t+1) &\leq H^\dagger(t); \\ h^\dagger(t+1) &\geq h^\dagger(t); \\ \mathcal{H}^\dagger(t+1) &\leq \mathcal{H}^\dagger(t); \end{aligned} \quad (5)$$

if $D_t = 0$. The desired conclusion follows from the same argument as the proof of Lemma 2. \blacksquare

Lemma 4: Assume that $D_t = 0$ for $t = 0, \dots, 2(n-2)K-1$. Take $i \in \mathcal{V}$. Then for any $t = 0, \dots, 2(n-2)K-1$, there hold

- (i) If $s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0)$ for some $\zeta_0 \in (0, 1)$, then $s_i(t+1) \leq \lambda_* \zeta_0 h(0) + (1 - \lambda_* \zeta_0)H(0)$, where $\lambda_* = 1 - \alpha(n-1)$;
- (ii) If $s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0)$ for some $\zeta_0 \in (0, 1)$, $B_t = 1$, and $(i, j) \in G_t$, then $s_j(t+1) \leq \alpha \zeta_0 h(0) + (1 - \alpha \zeta_0)H(0)$.

Proof. Note that the conditions that $D_t = 0$ for $t = 0, \dots, 2(n-2)K-1$ and $\alpha \in (0, (n-1)^{-1})$ yield $H(t+1) \leq H(t)$ and $h(t+1) \geq h(t)$ for all $t = 0, \dots, 2(n-2)K-1$.

- (i). If $D_t = 0$ and $s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0)$ for some $\zeta_0 \in (0, 1)$, then

$$\begin{aligned} s_i(t+1) &= s_i(t) + \alpha B_t h_i^+(t) \\ &\leq s_i(t) - \alpha \sum_{j \in N_i^+(t)} (s_i(t) - s_j(t)) \\ &\leq (1 - \alpha |N_i^+(t)|) s_i(t) + \alpha |N_i^+(t)| H(t) \\ &\leq (1 - \alpha |N_i^+(t)|) (\zeta_0 h(0) + (1 - \zeta_0)H(0)) \\ &\quad + \alpha |N_i^+(t)| H(0) \\ &\leq \lambda_* \zeta_0 h(0) + (1 - \lambda_* \zeta_0)H(0) \end{aligned} \quad (6)$$

in light of the fact that $\alpha \in (0, (n-1)^{-1})$, where $\lambda_* = 1 - \alpha(n-1)$.

- (ii) If $s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0)$ for some $\zeta_0 \in (0, 1)$, $B_t = 1$, and $(i, j) \in G_t$, there holds that

$$\begin{aligned} s_j(t+1) &= s_j(t) - \alpha \sum_{k \in N_j^+(t)} (s_j(t) - s_k(t)) \\ &= (1 - \alpha |N_j^+(t)|) s_j(t) + \alpha s_i(t) \\ &\quad + \alpha \sum_{k \in N_j^+(t) \setminus \{i\}} s_k(t) \\ &\leq (1 - \alpha)H(t) + \alpha (\zeta_0 h(0) + (1 - \zeta_0)H(0)) \\ &\leq \alpha \zeta_0 h(0) + (1 - \alpha \zeta_0)H(0). \end{aligned} \quad (7)$$

This proves the desired lemma. \blacksquare

B. Proof of Theorem 1

- (i). Let $\sum_{t=0}^{\infty} d_t < \infty$. Then as long as $\sum_{t=0}^{\infty} b_t < \infty$, the first Borel-Cantelli Lemma guarantees that almost surely, each node revises its state for only a finite number of slots, which yields the desired claim follows straightforwardly. In the following, we prove the desired conclusion based on the assumption that $\sum_{t=0}^{\infty} b_t = \infty$.

With $\sum_{t=0}^{\infty} d_t < \infty$, from the first Borel-Cantelli Lemma,

$$K_* := \inf\{k \geq 0 : D_t = 0, \forall t \geq k\}$$

is a finite number almost surely. We note that K_* is not a stopping time for $\{D_t\}_{t \geq 0}$, but a stopping time for $\{B_t\}_{t \geq 0}$ by the independence of $\{B_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$. Hence, we can recursively define

$$K_{m+1} := \inf\{t > K_m : B_t = 1\}, \quad m = 0, 1, \dots$$

with $K_0 := \inf\{t \geq K_* : B_t = 1\}$, which are stopping times for $\{B_t\}_{t \geq 0}$. Now in view of the independence of $\{G_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$, we know that $\{G_{K_m}\}_{m \geq 0}$ is an independent process and each G_{K_m} satisfies $\mathbb{P}((i, j) \in E_{K_m}) \geq p_*$ for all $(i, j) \in \mathcal{G}$ under Assumption A1.

Let \mathcal{V}^\dagger be a positive cluster of \mathcal{G} . By assumption, $\mathcal{G}|\mathcal{V}^\dagger$ has a spanning tree. Since $\alpha < 1/(n-1)$, the above discussion shows that at times $K_m, m = 0, 1, \dots$, the considered relative-state-flipping model defines a standard consensus dynamics on independent random graphs where each arc exists with probability at least p_* for any fixed time slot. Therefore, applying Theorem 3.4 in [39] on randomized consensus dynamics with arc-independent graphs, we conclude that the connectivity of \mathcal{V}^\dagger ensures that

$$\mathbb{P}\left(\lim_{m \rightarrow \infty} \mathcal{H}^\dagger(K_m) = 0\right) = 1,$$

where $\mathcal{H}^\dagger(t) = \max_{i \in \mathcal{V}^\dagger} s_i(t) - \min_{i \in \mathcal{V}^\dagger} s_i(t)$. This immediately gives us

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} \mathcal{H}^\dagger(t) = 0\right) = 1$$

by the definition of the K_m .

Finally, according to Lemma 3, both $\max_{i \in \mathcal{V}^\dagger} s_i(t)$ and $\min_{i \in \mathcal{V}^\dagger} s_i(t)$ converge almost surely. Define their limits as, respectively, H_*^\dagger and h_*^\dagger , which are finite with probability one. The fact that $\mathbb{P}(\lim_{t \rightarrow \infty} \mathcal{H}^\dagger(t) = 0) = 1$ immediately leads to $H_*^\dagger = h_*^\dagger$ almost surely. As a result, we conclude that

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} s_i(t) = H_*^\dagger = h_*^\dagger\right) = 1$$

for all $i \in \mathcal{V}^\dagger$.

Since \mathcal{V}^\dagger is chosen arbitrarily, we further know $\mathbb{P}(\lim_{t \rightarrow \infty} s_i(t) \text{ exists}) = 1$ for all node $i \in \mathcal{V}$ and all initial states $s(0)$. This proves the desired statement.

(ii) Let \mathcal{V}_1 and \mathcal{V}_2 be the two positive-clusters of \mathcal{G} . Let $s_i(0) = 0, i \in \mathcal{V}_1$ and $s_i(0) = C_0, i \in \mathcal{V}_2$ for some $C_0 > 0$. We define

$$f_1(t) := \max_{i \in \mathcal{V}_1} s_i(t); \quad f_2(t) := \min_{i \in \mathcal{V}_2} s_i(t).$$

Since the either of the positive cluster contains positive links only and $\alpha \in (0, (n-1)^{-1})$, there always holds that

$$f_1(t+1) \leq f_1(t); \quad f_2(t+1) \geq f_2(t).$$

Now that there is a negative arc between any two nodes from different clusters, it is straightforward to see that

$$\begin{aligned} f_2(t+1) - f_1(t+1) &\geq (1+\beta)(f_2(t) - f_1(t)) \\ &\geq f_2(t) - f_1(t) + C_0 \end{aligned} \quad (8)$$

whenever $D_t = 1$ and either $(i_*, j_*) \in E_t$ or $(j_*, i_*) \in E_t$ with $i_* = \arg \max_{i \in \mathcal{V}_1} s_i(t)$ and $j_* = \arg \min_{i \in \mathcal{V}_2} s_i(t)$. In light of Assumption A1, the second Borel-Cantelli Lemma (cf., Theorem 2.3.6, [32]) leads to that the event defined in (8) happens infinitely often with probability one when $\sum_{t=0}^{\infty} d_t = \infty$. The desired conclusion follows immediately.

The proof is now complete. \blacksquare

C. Proof of Theorem 2

The proof relies on Lemma 1, cf., [29] for the analysis of randomized consensus. We proceed in steps.

Step 1. Consider $2n-3$ intervals $[mK, (m+1)K-1], m = 0, \dots, 2(n-2)$. In this step, we select a few key intervals from them.

With Assumption A4, there is a center node $v_m \in \mathcal{V}$ in each of $\mathcal{G}([mK, (m+1)K-1])$. Therefore, we count $2n-3$ center nodes (repetitions are allowed, of course) in the $2n-3$ intervals. There must hold that either $s_{v_m}(0) \leq (h(0)+H(0))/2$ or $s_{v_m}(0) > (h(0)+H(0))/2$ for each $m = 1, \dots, 2n-3$. This is to say, we can find $n-1$ center nodes (again, repetitions are allowed) out of the v_m 's, denoted $v_{m_1}, \dots, v_{m_{n-1}}$, such that either

$$s_{v_{m_j}}(0) \leq \frac{h(0)}{2} + \frac{H(0)}{2}, \quad j = 1, \dots, n-1 \quad (9)$$

or

$$s_{v_{m_j}}(0) > \frac{h(0)}{2} + \frac{H(0)}{2}, \quad j = 1, \dots, n-1 \quad (10)$$

holds.

Step 2. In this step, we establish a probabilistic bound for $\mathcal{H}(K_0)$. To this end, we first assume that (9) holds.

Let $D_t = 0$ for $t = 0, \dots, 2(n-2)K-1$. We carry out the following recursive argument:

- 1) By our selection v_{m_1} is a center node of the graph $\mathcal{G}([\tau_1 K, (\tau_1+1)K-1])$ for some $\tau_1 = 0, \dots, 2(n-2)$ with $s_{v_{m_1}}(0) \leq (h(0)+H(0))/2$. Applying Lemma 4.(i) we conclude that

$$s_{v_{m_1}}(K_0) \leq \frac{\rho_*^{K_0}}{2} h(0) + \left(1 - \frac{\rho_*^{K_0}}{2}\right) H(0),$$

where K_0 and ρ_* are defined in the statement of Theorem 2.

- 2) Since v_{m_1} is a center, there exist $t_1 \in [\tau_1 K, (\tau_1+1)K-1]$ and $j_* \neq v_{m_1} \in \mathcal{V}$ such that $(v_{m_1}, j_*) \in E_{t_1}$ with probability at least p_* . If $B_{t_1} = 1$ and $(v_{m_1}, j_*) \in E_{t_1}$, then we can apply Lemma 4 and then conclude

$$s_{j_*}(K_0) \leq \frac{\rho_*^{K_0}}{2} h(0) + \left(1 - \frac{\rho_*^{K_0}}{2}\right) H(0).$$

For convenience we re-denote v_{m_1} and j_* as u_1 and u_2 , respectively.

- 3) We proceed for v_{m_2} . If $v_{m_2} \notin \{u_1, u_2\}$, applying Lemma 4.(i) again and we can obtain the same bound for $s_{j_*}(K_0)$. Otherwise either $v_{m_2} = u_1$ or $v_{m_2} = u_2$ allows us to find another node u_3 with the bound for $s_{u_3}(K_0)$ obtained as step 2).

From the selection of $v_{m_1}, \dots, v_{m_{n-1}}$, the above procedure eventually gives us the same bound for nodes u_1, \dots, u_n , and calculating the probability of the required events in the above argument we obtain

$$\begin{aligned} \mathbb{P}\left(s_i(K_0) \leq \frac{\rho_*^{K_0}}{2} h(0) + \left(1 - \frac{\rho_*^{K_0}}{2}\right) H(0), i \in \mathcal{V}\right) \\ \geq p_*^{n-1} \prod_{t=0}^{K_0-1} (b_t(1-d_t)). \end{aligned}$$

This implies

$$\mathbb{P}\left(\mathcal{H}(K_0) \leq \left(1 - \frac{\rho_*^{K_0}}{2}\right) \mathcal{H}(0)\right) \geq p_*^{n-1} \prod_{t=0}^{K_0-1} (b_t(1-d_t)). \quad (11)$$

Using a simple coordinate change by $y_i(t) = -s_i(t)$ and repeating the above analysis to the $y_i(t)$, we immediately know that (11) continues to hold for the case with (10).

Step 3. In this step, we conclude the proof making use of Lemma 1.

From the definition of the algorithm there always hold

$$\mathbb{P}(\mathcal{H}(t+1) \leq (1+2\beta(n-1))\mathcal{H}(t)) = 1 \quad (12)$$

and

$$\mathbb{P}(\mathcal{H}(K_0) > \mathcal{H}(0)) \leq 1 - \prod_{t=0}^{K_0-1} (1-d_t). \quad (13)$$

Since $\{B_t\}_{t \geq 0}$, $\{D_t\}_{t \geq 0}$, and $\{G_t\}_{t \geq 0}$ define independent processes, we conclude from (11), (12), and (13) that

$$\mathbb{E}\{\mathcal{H}((m+1)K_0) | \mathcal{H}(mK_0)\} \leq (1 - X_m + Y_m) \mathcal{H}(mK_0).$$

Then from Lemma 1 we immediately conclude the almost sure convergence of $\mathcal{H}(mK_0)$ as m tends to infinity. The desired almost sure convergence of $\mathcal{H}(t)$ holds by further noticing (12). This completes the proof. \blacksquare

D. Proof of Theorem 3

We first establish a technical lemma.

Lemma 5: Let $\alpha \in [0, (n-1)^{-1}/2]$. Then

$$(i) \mathbb{P}\left(\mathcal{H}(t+1) \geq (1-2(n-1)\alpha)\mathcal{H}(t)\right) = 1;$$

$$(ii) \mathbb{P}\left(\mathcal{H}(t+1) < \mathcal{H}(t)\right) \leq b.$$

Proof. (i) Take $i, j \in \mathcal{V}$ satisfying $s_i(t) = h(t)$ and $s_j(t) = H(t)$. Similarly as the proof of Lemma 4, we can establish that almost surely,

$$s_i(t+1) \leq \lambda_* h(t) + (1-\lambda_*)H(t) \quad (14)$$

and

$$s_j(t+1) \geq \lambda_* H(t) + (1-\lambda_*)h(t) \quad (15)$$

hold, where $\lambda_* = 1 - \alpha(n-1)$. Noting that (14) and (15) yield

$$\begin{aligned} \mathcal{H}(t+1) &\geq |s_j(t+1) - s_i(t+1)| \\ &\geq |2\lambda_* - 1|\mathcal{H}(t) \\ &= (1-2(n-1)\alpha)\mathcal{H}(t). \end{aligned} \quad (16)$$

Claim A is proved.

(ii) If $b_t = 0$, only negative recommendations can be effective in the node state update. The desired conclusion is obvious. ■

Now we define

$$L_0 := \inf\{t \in \mathbb{Z} : (1+\beta)^t \geq 2(n-1)\}$$

and

$$K_{L_0} = K((n^2 - n)(L_0 - 1) + 1).$$

Consider the $K_{L_0}/K = (n^2 - n)(L_0 - 1) + 1$ time intervals

$$[mK, (m+1)K - 1]$$

for $m = 0, 1, \dots, (n^2 - n)(L_0 - 1)$. In view of Assumption A5, each of the graph $\mathcal{G}^-([mK, (m+1)K - 1])$ is weakly connected.

As a result, there must be a pair of nodes, saying (u_m, v_m) , such that $(u_m, v_m) \in \mathcal{G}^-([mK, (m+1)K - 1])$ and $|s_{u_m}(mK) - s_{v_m}(mK)| \geq \mathcal{H}(mK)/(n-1)$. Note that $(u_m, v_m) \in \mathcal{G}^-([mK, (m+1)K - 1])$ means that there is $t_m \in [mK, (m+1)K - 1]$ such that $(u_m, v_m) \in \mathcal{G}_{t_m}^-$. If $t_m > mK$, we now construct the following event²

$$\mathcal{E}_m := \left\{ B_t = D_t = 0, t \in [mK, t_m - 1] \right\}.$$

Conditioned on \mathcal{E}_m , obviously there holds $|s_{u_m}(t_m) - s_{v_m}(t_m)| \geq \mathcal{H}(t_m)/(n-1)$ since $\mathcal{H}(t_m) = \mathcal{H}(mK)$.

There are however at most $n(n-1)$ different arcs. Consequently, there are two nodes $i_*, j_* \in \mathcal{V}$ and L_0 intervals

$$[m_1K, (m_1+1)K - 1], \dots, [m_{L_0}K, (m_{L_0}+1)K - 1],$$

such that $(i_*, j_*) \in \mathcal{G}^-([m_zK, (m_z+1)K - 1])$ for $z = 1, \dots, L_0$ with

$$|s_{i_*}(m_1K) - s_{j_*}(m_1K)| \geq \mathcal{H}(m_1K)/(n-1). \quad (17)$$

²Note that, u_m, v_m, t_m are all deterministic.

There are therefore L_0 instants $\tau_1 < \tau_2 < \dots < \tau_{L_0}$ with $\tau_z \in [m_zK, (m_z+1)K - 1]$ such that $(i_*, j_*) \in \mathcal{G}_{\tau_z}^-$ for all $z = 1, \dots, L_0$.

Consider the following event:

$$\begin{aligned} \mathcal{E}_* &:= \left\{ D_{\tau_k} = 1, i_* = N_{j_*}^-(\tau_k), k = 1, \dots, L_0; \right. \\ &\left. B_t = 0 \text{ for all } t \in [0, K_{L_0} - 1] \right\}. \end{aligned} \quad (18)$$

From the above argument we know that the event \mathcal{E}_* implies³ $\mathcal{H}(K_{L_0}) \geq |s_{i_*}(K_{L_0}) - s_{j_*}(K_{L_0})| \geq \mathcal{H}(0)(1+\beta)^{L_0} \cdot (n-1)^{-1}$.

As a result, we can bound the probability of \mathcal{E}_* and conclude

$$\begin{aligned} \mathbb{P}\left(\mathcal{H}(K_{L_0}) \geq \mathcal{H}(0)(1+\beta)^{L_0} \cdot (n-1)^{-1}\right) \\ \geq (dp_*(1-p^*)^{n-2})^{L_0} (1-b)^{K_{L_0}}. \end{aligned} \quad (19)$$

We can now apply the same argument as the proof of Proposition 1 in [1]. With (19) and Lemma 5 there holds

$$\begin{aligned} \mathbb{E}\left\{ \log \mathcal{H}(K_{L_0}) - \log \mathcal{H}(0) \right\} \\ \geq (dp_*(1-p^*)^{n-2})^{L_0} (1-b)^{K_{L_0}} \\ \cdot \log\left((1+\beta)^{L_0} \cdot (n-1)^{-1}\right) + b \log(1-2(n-1)\alpha) \\ \geq (dp_*(1-p^*)^{n-2})^{L_0} (1-b)^{K_{L_0}} \log 2 \\ + b \log(1-2(n-1)\alpha) \\ > 0 \end{aligned} \quad (20)$$

when $b < b_*$ for some sufficiently small $b_* > 0$. We can proceed to define $U(m) = \log \mathcal{H}(mK_{L_0})$ for $m = 0, 1, \dots$. Recursively applying the above arguments to the process $\{U_m\}$ we obtain that $U(m)$ has a strictly positive drift when $b < b_*$, which implies that $\liminf_{m \rightarrow \infty} U(m) = \infty$ holds almost surely.

This completes the proof. ■

E. Proof of Theorem 4

Let us focus on a given positive cluster \mathcal{V}^\dagger of \mathcal{G} . We use the following notations

$$\Psi(t) = \max_{i \in \mathcal{V}^\dagger} s_i(t), \psi(t) = \min_{i \in \mathcal{V}^\dagger} s_i(t), \Theta(t) = \Psi(t) - \psi(t).$$

Applying Lemma 2 on the positive cluster \mathcal{V}^\dagger , we conclude that each of $\Theta(t)$, $\Psi(t)$, and $\psi(t)$ converge to a finite limit almost surely if $\sum_{t=0}^{\infty} d_t < \infty$.

In light of Assumption A6, applying the same argument we used in order to establish (11) of Theorem 2 on the cluster \mathcal{V}^\dagger , we similarly have

$$\begin{aligned} \mathbb{P}\left(\Theta((m+1)K_0) \leq \left(1 - \frac{\rho_*^{K_0}}{2}\right)\Theta(mK_0)\right) \\ \geq p_*^{n-1} \prod_{t=mK_0}^{(m+1)K_0-1} (b_t(1-d_t)). \end{aligned} \quad (21)$$

³Here it is crucial to notice that $\mathcal{H}(t+1) \geq \mathcal{H}(t)$ always holds if $B_t = 0$.

Moreover, from the effect of the negative recommendations on nodes in \mathcal{V}^\dagger , we can easily modify (12) and (13) to that for all t ,

$$\mathbb{P}(\Theta(t+1) > \Theta(t)) \leq d_t \quad (22)$$

and

$$\mathbb{P}(\Theta(t+1) \leq (1 + 2\beta(n-1))\mathcal{H}(t)) = 1. \quad (23)$$

With (21), (22) and (23), we arrive at

$$\begin{aligned} & \mathbb{E}\{\Theta((m+1)K_0) | \Theta(mK_0)\} \\ & \leq (1 - X_m)\Theta(mK_0) \\ & \quad + (1 + 2\beta(n-1)) \sum_{t=mK_0}^{(m+1)K_0-1} d_t \mathcal{H}(t), \end{aligned} \quad (24)$$

where X_m is defined in Theorem 2.

On the other hand, it holds from the structure of the algorithm that $\mathcal{H}(t+1) \leq \mathcal{H}(t)$ if $D_t = 0$, and that $\mathcal{H}(t+1) \leq (1 + 2\beta(n-1))\mathcal{H}(t)$ if $D_t = 1$. We therefore deduce that:

$$\mathbb{E}\{\mathcal{H}(t+1) | \mathcal{H}(t)\} \leq (1 + 2\beta(n-1)d_t)\mathcal{H}(t). \quad (25)$$

From (25) we know

$$\mathbb{E}(\mathcal{H}(t)) \leq \mathcal{H}_0 \prod_{t=0}^{\infty} (1 + 2\beta(n-1)d_t)$$

for all $t \geq 0$. Taking the expectation from the both sides of (24), we obtain:

$$\begin{aligned} & \mathbb{E}\{\Theta((m+1)K_0)\} \leq (1 - X_m)\mathbb{E}\{\Theta(mK_0)\} \\ & \quad + \left[(1 + 2\beta(n-1))\mathcal{H}_0 \prod_{t=0}^{\infty} (1 + 2\beta(n-1)d_t) \right] W(m), \end{aligned} \quad (26)$$

where $W(m) = \sum_{t=mK_0}^{(m+1)K_0-1} d_t$.

Note that it is well known that $\sum_{t=0}^{\infty} d_t < \infty$ implies $\prod_{t=0}^{\infty} (1 - d_t) > 0$ and $\prod_{t=0}^{\infty} (1 + 2\beta(n-1)d_t) < \infty$. Consequently, $\sum_{m=0}^{\infty} J(m) = \infty$ implies $\sum_{m=0}^{\infty} X_m = \infty$. In view of Lemma 1, we have

$$\lim_{m \rightarrow \infty} \mathbb{E}\{\Theta((m+1)K_0)\} = 0 \quad (27)$$

if $\lim_{m \rightarrow \infty} W(m)/J(m) = 0$. Invoking Fatou's lemma (e.g., Theorem 1.6.5, [32]), we further conclude that

$$\mathbb{E}\left\{ \liminf_{m \rightarrow \infty} \Theta((m+1)K_0) \right\} \leq \lim_{t \rightarrow \infty} \mathbb{E}\{\Theta((m+1)K_0)\} = 0, \quad (28)$$

which actually implies

$$\mathbb{E}\left\{ \lim_{m \rightarrow \infty} \Theta((m+1)K_0) \right\} = 0 \quad (29)$$

since $\Theta((m+1)K_0)$ converges almost surely. Therefore, we have reached

$$\mathbb{P}\left(\lim_{m \rightarrow \infty} \Theta((m+1)K_0) = 0 \right) = 1, \quad (30)$$

which in turn leads to

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} \Theta(t) = 0 \right) = 1, \quad (31)$$

again, from the fact that $\Theta(t)$ converges almost surely.

Finally, $\Theta(t)$ converging almost surely to zero means that $\Psi(t)$ and $\psi(t)$ must converge to the same limit (their convergence is established in the beginning of the proof), which must be the limit of the each node state in \mathcal{V}^\dagger . We have now completed the proof. ■

V. CONCLUSIONS

This paper continued the study of [35], [36] investigating a relative-state-flipping model for consensus dynamics over signed random networks. A sequence of deterministic signed graphs define potential node interactions that happen independently but not necessarily i.i.d. The positive recommendations are consistent with the standard consensus algorithm; negative recommendations are defined by relative-state flipping from its negative neighbors. Each node puts a (deterministic) weight to each recommendation, and then encodes these weighted recommendations in its state update through stochastic attentions defined by two Bernoulli random variables. We have established several fundamental conditions regarding almost sure convergence and divergence of the network states. A condition for almost sure state clustering was also proposed for weakly balanced graphs, with the help of martingale convergence lemmas. Some fundamental differences were also highlighted between the current relative-state-flipping model and the state-flipping model considered in [1], [34].

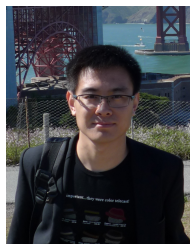
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