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# Time Varying Control Set Design for UAV Collision Avoidance Using Reachable Tubes

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*Abstract*— **In this paper, we propose a reachable set based collision avoidance algorithm for unmanned aerial vehicles (UAVs). UAVs have been deployed for agriculture research and management, surveillance and sensor coverage for threat detection and disaster search and rescue operations. It is essential for the aircraft to have on-board collision avoidance capability to guarantee safety. Instead of the traditional approach of collision avoidance between trajectories, we propose a collision avoidance scheme based on reachable sets and tubes. We then formulate the problem as a convex optimization problem seeking time varying control sets for the ego aircraft given the predicted intruder reachable tube. We have applied the approach on a case study of two quadrotors collision avoidance scenario.**

## I. INTRODUCTION

Autonomous aircraft have been deployed for agriculture research and management, surveillance and sensor coverage for threat detection, and disaster search and rescue operations. In most of these scenarios, multiple aircraft are desired to increase the efficiency and coverage of the task. Since in many of these scenarios they will be deployed in the shared commercial airspace as well, they are required to have sophisticated collision avoidance algorithms in order to fly together with other conventional aircraft, which are piloted by humans. As the number of these UAVs increases, a centralized ground control based model alone is not sufficient. Thus an autonomous on-board collision avoidance system is required to be implemented in a decentralized manner. This limits the computation time and complexity of the algorithm.

Many collision avoidance algorithms have been proposed in robotics areas. An artificial potential function was proposed in [1]–[3] to produce control policies for robots to navigate towards a goal and avoid each other and obstacles. In [4]–[6], the authors proposed a decentralized collision avoidance rule based on heading and collision cones. However, the research works mentioned above focus only on designing a single path or trajectory for individual aircraft, so that they are separated by at least the threshold distance. The important challenge remaining is to provide guarantees for some uncertain behavior of the intruder aircraft. In most of the collision avoidance problems, an estimated position of intruder is provided to the planner at every time step. The ego vehicle has to adapt accordingly to the changes in prediction. The predictions commonly have uncertainties, so we captured them in our problem formulation as time varying sets instead of single trajectory. Because the predictions are updating frequently, the designed controller has to be robust to changes in the predicted behavior of the intruder. Therefore, instead of a control policy, we propose to use the control tube, a time varying set, to describe the robustness of the control. The problem then becomes how to find a control set update rule for the ego vehicle to avoid the predicted intruder tube while maximizing the robustness of the control. This problem implies that the collision avoidance problem needs to be solved between time varying sets instead of trajectories. Different from our previous work [7], the objective here is not to avoid all possible executions of the intruder, but to avoid the predicted tube of the intruder. We will also emphasize in this work that the control set obtained should be time varying instead of a fixed control set constraint, as in our previous work.

The reachable set of a dynamical system is defined as the set of states reachable from a given bounded initial set, control set and disturbance set. The practical problem mentioned earlier is closely related to reachability analysis. The collision avoidance problem between reachable sets has been previously studied under the frame of reachability analysis of nonlinear dynamical games [8], [9]. The other agent is considered as adversary or disturbance to the collision avoidance problem. A controller is synthesized to allow the aircraft to avoid the reachable sets of others. Commonly, the prediction of the intruder can be provided through ground radars or existing onboard systems such as a traffic collision avoidance system (TCAS). Besides the above mentioned level set approach [8], [9] to obtain reachable sets for nonlinear dynamics, there are several fast linear algorithms based on convex analysis. These algorithms employ linearized dynamics with convex initial state set and control disturbance sets. They commonly approximate reachable sets using specific covering sets including ellipsoids [10], [11] and polytopes [12]–[15]. In all these cases, support functions are commonly used to analytically derive, or estimate the reachable sets. However, many algorithms [12]–[15] compute the approximated convex set iteratively, which makes the resulting tubes impossible to be represented in analytic forms. In [16], the authors propose to use the invariant set to capture the reachable set, however, the method requires finding a particular parametrization of the control policy. We will use the reachable set tool set from [17] based on the ellipsoid methods in [10], because its solutions can be expressed efficiently in analytical expressions.

The main contribution of our work is that we provide a new formulation of the collision avoidance problem using reachable tubes, and propose a time varying update rule of

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control sets based on the optimization problem. The rest of the paper is organized as follows. In section II we present the fundamentals of reachability sets. Then in section III we define the reachable set collision avoidance problem and formulate it into a convex optimization problem incorporating the reachable sets. Afterward, we demonstrate our approach in scenarios involving collision avoidance between two quadrotors.

## II. PRELIMINARIES

We consider collision avoidance navigation between aircraft whose dynamics are given by nonlinear models as (1).

$$\dot{x}(t) = f(t, x, u, v) \tag{1}$$

where $x(t) \in \mathcal{X}$, $x(0) \in \mathcal{X}_0 \subseteq \mathcal{X} \subset \mathbb{R}^n$, $u(t) \in \mathcal{U}(t) \subset \mathbb{R}^m$ for all $t$, $v(t) \in \mathcal{V} \subset \mathbb{R}^n$ for all $t$. $\mathcal{U}(t)$ is the control set and $\mathcal{V}$ is a bounded disturbance set.

### A. Reachable Set

The reachable set of (1) (or forward reachable set) $\mathcal{R}[\vartheta] = \mathcal{R}(\vartheta, X_0)$, is the set of states that are reachable at time $\vartheta$ from a set of initial states $X_0$ and all possible controls and disturbances. Formally it is defined by the following,

**Definition 2.1 (Reachable Set):** The reachable set $\mathcal{R}[\vartheta] = \mathcal{R}(\vartheta, t_0, X_0)$ of the system (1) at time $\vartheta$ from a set of initial positions $X_0$ and time $t_0$ is the set of all points $x$ for which there exists a trajectory $x(s, t_0, x_0), x_0 \in X_0$ that transfers the system from $(t_0, x_0)$ to $(\vartheta, x), x = x(\vartheta)$, while satisfying the associated constraints.

Similarly the reachable tube is the set of all reachable sets over a time interval.

**Definition 2.2 (Reachable Tube):** The reachable tube $\mathcal{R}[\Theta] = \{X(\vartheta) = \mathcal{R}(\vartheta, t_0, X_0), \vartheta \in \Theta\}$

Reachable set computation for nonlinear model exists, but either it is impractical in collision avoidance due to slow computation time [8], or it relies on numerical methods to approximate the nonlinear model with linear models [12]. So instead of looking at the full nonlinear model, we linearize the dynamics around an operating point, resulting in dynamics that are of the following form.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t) \tag{2}$$

Control design using linearization of the dynamics around nominal trajectories (commonly used for fixed-wing aircraft) is studied in [18]. Interested readers should refer to detailed studies within. In this paper, because of the special structure of the quadrotor dynamics, linearization can be performed around an operating point for a short predicting horizon as in [19], [20]. Exemplary trajectories from nonlinear dynamics will be included in the results to confirm that the computation for the linearized quadrotor system does apply for the nonlinear case.

To simplify the computation, the following assumptions are used by noting Lemma 2.3, where $\mathcal{U}(t), \mathcal{X}_0$ and $\mathcal{V}$ are all convex and compact sets [10].

**Lemma 2.3:** With $\mathcal{U}(t), \mathcal{X}_0$ and $\mathcal{V}$ being convex and compact, the reachable set $\mathcal{R}[\vartheta]$ is also convex and compact.

The problem of defining the reachable set of the system can be reformulated as an optimization problem. Consider the system (2). Since the reachable set would be convex and compact due to the assumption on the control set and the initial set, the reachable set can be captured using its support function. Let $\rho(l|X)$ be the support function of the set $X$, i.e. $\rho(l|X) = \max\{\langle l, x\rangle \,|x \in X\}$, and $\langle l, x\rangle$ represents the inner product between vector $l$ and $x$. Then the support function of the reachable set $\mathcal{R}[\vartheta]$ is given by the following,

$$\begin{aligned}
\rho(l|\mathcal{R}[\vartheta]) &= \max\{\langle l, x\rangle \,|x \in \mathcal{R}[\vartheta]\} \\
&= \max\{\langle l, x(\vartheta, t_0, x_0, u, v)\rangle \,|u(\cdot) \in U(t), x_0 \in X_0\} \\
&= \max\left\{ \int_{t_0}^{\vartheta} l'\Phi(\vartheta, s)B(s)u(s) + l'\Phi(\vartheta, s)v(s)ds \right. \\
&\qquad \left. + l'\Phi(\vartheta, t_0)x_0 \middle| u(s) \in \mathcal{U}(s), x_0 \in X_0, v(s) \in \mathcal{V} \right\} \\
&= \int_{t_0}^{\vartheta} \rho(B'(s)\Phi'(\vartheta, s)l|\mathcal{U}(s)) + \rho(\Phi'(\vartheta, s)l|\mathcal{V})ds \\
&\qquad + \rho(\Phi'(\vartheta, t_0)l|X_0)
\end{aligned} \tag{3}$$

where $\Phi(t, s)$ is the transition matrix of the system (2). i.e. it satisfies $\frac{\partial}{\partial t}\Phi(t, s) = A(t)\Phi(t, s)$ and $\Phi(s, s) = \mathbf{I}$.

### B. Ellipsoid Formulation

Assume further that all the sets are represented by ellipsoids. Let $c_X$ and $M_X$ denote the center and shape matrix of the set. The following holds, if $x \in X = E(c_X, M_X)$,

$$\left\langle x - c_X, M_X^{-1}(x - c_X)\right\rangle \leq 1.$$

In terms of the support function, it can be expressed by

$$\langle l, x\rangle \leq \langle l, c_X\rangle + \langle l, M_X l\rangle^{1/2}.$$

The support function of the reachable set $\mathcal{R}[\vartheta]$ could be expressed in terms of the centers and shape matrices of the initial set, control and disturbance sets Equation (4).

$$\begin{aligned}
&\rho(l|\mathcal{R}[\vartheta]) \\
&= \langle l, \Phi(\vartheta, 0)c_{X_0}\rangle + \left\langle l, \int_0^{\vartheta} \Phi(\vartheta, s)B(s)c_{\mathcal{U}}(s)ds \right\rangle \\
&\quad + \left\langle l, \Phi(\vartheta, 0)M_{X_0}\Phi^T(\vartheta, 0)l\right\rangle^{1/2} \\
&\quad + \left\langle l, \int_0^{\vartheta} \Phi(\vartheta, s)c_{\mathcal{V}}(s)ds \right\rangle \\
&\quad + \int_0^{\vartheta} \left\langle l, \Phi(\vartheta, s)B(s)M_{\mathcal{U}}(s)B^T(s)\Phi^T(\vartheta, s)l\right\rangle^{1/2} ds \\
&\quad + \int_0^{\vartheta} \left\langle l, \Phi(\vartheta, s)M_{\mathcal{V}}(s)\Phi^T(\vartheta, s)l\right\rangle^{1/2} ds
\end{aligned} \tag{4}$$

In the following section where an optimization problem about the control sets is formulated, the disturbance and initial set will be constant terms to the optimization problem, since they are independent of the control set parameters. We will assume in what follows that $v(t) = 0$. The disturbance term affects the size of the reachable set. Therefore, it can be treated as an additional separation required in the collision avoidance problem.

### C. Control Set Reparameterization

The control set under the ellipsoid formulation is described by $\mathcal{U}(t) = E(c_{\mathcal{U}}(t), M_{\mathcal{U}}(t))$. To simplify the parameterization and formulation of the optimization problem, we adopt the following from [16].

We define the control set by a scaling factor $\alpha_{\mathcal{U}}(t)$ of the original system control constraint and the nominal control $c_{\mathcal{U}}(t)$, i.e. $\mathcal{U}(t) = E(c_{\mathcal{U}}(t), \alpha(t) M_{\bar{\mathcal{U}}})$. $\bar{\mathcal{U}} = E(c_{\bar{\mathcal{U}}}, M_{\bar{\mathcal{U}}})$ is an ellipsoid inner-approximation of the physical control limitations. Naturally we have the following constraint on $c_{\mathcal{U}}(t)$ and $\alpha(t)$.

$$E(c_{\mathcal{U}}(t), \alpha(t) M_{\bar{\mathcal{U}}}) \subseteq \bar{\mathcal{U}} \quad \forall t$$

## III. Collision Avoidance Using Reachability Analysis

In a collision avoidance scenario, the predicted path of the intruder is commonly known apriori. Furthermore, the uncertainty of the intruder can be predicted through onboard sensors or ground radar. The main problem we want to address, is to generate an update policy for the control set, so that the reachable tube of the ego aircraft is collision free from the predicted tube of the intruder.

More specifically, this collision avoidance problem will be defined as the following problem involving reachable tubes.

**Problem 3.1 (Reachability Based Collision Avoidance):** Denote the estimated collision time as $T$. The collision avoidance using reachability can be formulated as a two steps optimization problem. In the first step, We seek a control set policy $\mathcal{U}(t)$ over the time interval, $[0, T]$, for the ego aircraft, such that the resulting reachable tube $\mathcal{R}_x^e([0, T])$ does not intersect with the intruder aircraft reachable tube $\mathcal{R}_x^i([0, T])$ counting the separation. These control sets should be chosen to maximize the flexibility of the subsequent Model Predictive Control (MPC) design, and therefore, the objective will be to maximize the size of the control sets over time $[0, T]$. In the second step, we seek controls within these control sets, so that the ego aircraft can safely reach their objectives in an optimal manner.

The following remark summarizes all the assumptions discussed in the previous section,

**Remark:**

1) The control sets associated to the control policy and initial set are represented by convex and compact ellipsoids. The disturbance set is assumed to be empty since it is a constant contribution to the optimization problems.
2) The time varying control sets are parametrized by time varying scalars that capture the ratios between the new control sets and the original control bounds.
3) The nonlinear dynamics are linearized around the operating point.
4) Note: There are no assumptions about the stabilizability or controllability of the dynamics, (although unstable systems may induce infeasible optimization problems.)

We will focus mainly on the first step of Problem 3.1 in this paper. As assumed in the previous section, let the initial state and control set of the ego aircraft be the ellipsoids $X_0^e = E(c_{X_0}^e, M_{X_0}^e)$ and $\bar{\mathcal{U}} = E(c_{\bar{\mathcal{U}}}, M_{\bar{\mathcal{U}}})$ respectively. Formally, the Problem 3.1 can be formulated as the following optimization problem,

**Problem 3.2 (Reachable Set Collision Avoidance):**

$$\begin{array}{cc} \max\limits_{q(t), \alpha(t)} \max\limits_{l(t)} & \int_{t=0}^{T} \mu(t) \alpha(t) dt \\ \text{subject to} & \forall t \in [0, T]: \\ & E(q(t), \alpha(t) M_{\bar{\mathcal{U}}}) \subseteq E(c_{\bar{\mathcal{U}}}, M_{\bar{\mathcal{U}}}) \\ & -\rho(-l(t) | \mathcal{R}_x^i(t)) - \rho(l(t) | \mathcal{R}_x^e(t)) > 0 \end{array}$$

The parameters of the optimization are $q(t)$ and $\alpha(t)$, both related to the updated control sets, $\tilde{\mathcal{U}}(t) = E(q(t), \alpha(t) M_{\bar{U}})$. The objective is to maximize the sizes of the control sets over time. $\mu(t)$ is a fixed scalar function which specifies the importance of control sets over time. If the weight is uniform over time, the optimal solution can have very flexible control set at the start, but very tight control near collision, which is not desired. The inner maximization is a feasibility problem, which is to find the direction for a series of separation hyperplanes $l(t)$ induced by the reachable set separation constraint. The first constraint is due to the fact that $\tilde{\mathcal{U}}(t) \subset \bar{\mathcal{U}}$, the last constraint is to keep reachable sets separated at every time step.

The first constraint is equivalent to the following constraints [21] on a new function $\lambda(t) > 0, \forall t \in [0, T]$, such that

$$\begin{bmatrix} 1 - \lambda(t) & 0 & (q(t) - c_{\bar{\mathcal{U}}})^T \\ 0 & \lambda(t) I & a(t)(M_{\bar{\mathcal{U}}})^{1/2} \\ q(t) - c_{\bar{\mathcal{U}}} & a(t)(M_{\bar{\mathcal{U}}})^{1/2} & M_{\bar{\mathcal{U}}} \end{bmatrix} \succeq 0$$

$$a(t)^2 = \alpha(t)$$

Let us assume the norm of the best separation hyperplane $l^*(t)$ can be estimated based on the initial predicted tubes of the intruder and ego aircraft. In other words, we assume the direction that minimizes distance between the reachable sets is not affected by changes in the control constraint. The intuition behind this assumption is that even if the direction is altered, the outer maximization is achieved at a similar constraint set. In real applications, the autonomous aircraft will be given such direction to avoid either based on the approaching angle autonomously or based on instructions from the other pilots. Let us define ellipsoids $E(c_X^i(t), M_X^i(t)) \supseteq \mathcal{R}_x^i(t), t \in [0, T]$, which tightly over-approximate the reachable tube of the intruder. The last constraints can be written as

$$
\begin{aligned}
&\left\langle l^*(t), c_X^i(t)\right\rangle - \left\langle l^*(t), M_X^i(t) l^*\right\rangle^{1/2} - \left\langle l^*(t), e^{At} c_{X_0}^e\right\rangle \\
&\underbrace{-\left\langle l^*(t), \int_0^t e^{A(t-s)} B q(s) ds\right\rangle}_{h(q,t)} - \left\langle l^*(t), e^{At} M_{X_0}^e e^{At} l^*(t)\right\rangle^{1/2} \\
&\underbrace{-\int_0^t \overbrace{\left\langle l^*(t), e^{A(t-s)} B M_{\bar{\mathcal{U}}} B^T (e^{A(t-s)})^T l^*(t)\right\rangle^{1/2}}^{r(s,t)} a(s) ds}_{g(a,t)} > 0.
\end{aligned} \tag{5}
$$

Furthermore, we assume the control set is only allowed to be updated every $\delta t$ and collision happened at $N$ steps in the future, i.e. $N\delta t = T$. We have

$$
\mathcal{U}(s) = E(q_d(k), \alpha_d(k) M_{\bar{\mathcal{U}}}) \quad \forall s \in [k\delta t, (k+1)\delta t)
$$

Therefore the optimization variables can be captured in terms of $\boldsymbol{a} \in \mathbb{R}^N, \boldsymbol{q} \in \mathbb{R}^{m \times N}$ such that

$$
\boldsymbol{a}_i = \left(\alpha_d(i-1)\right)^{1/2} = a_d(i-1)
$$

and the $i$th column of $\boldsymbol{q}$ is

$$
\boldsymbol{q}_i = q_d(i-1).
$$

It is also desired to keep the control set varying smoothly over time. Therefore, we also add a term in the objective to minimize variation in the nominal controller $q_d(k)$. Define constant matrix $H$ as

$$
H = \begin{bmatrix}
1 & -1 & 0 & 0 & \dots & 0 \\
0 & 1 & -1 & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & -1 & 1 & 0 \\
0 & \dots & 0 & 0 & 1 & -1
\end{bmatrix}
$$

The variation in nominal control can be captured using $\left\|H^T \boldsymbol{q}\right\|_F$

Let $\{P(i), i = 1, 2 \dots N\}$ be $N$ $n \times m$ constant matrices, such that the $j$th column of $P(i)$ is defined as

$$
P(i)_j = \int_{(i-1)\delta t}^{i\delta t} e^{A(k\delta t - s)} b_j ds,
$$

where $b_j$ is the $j$th column of the $B$ matrix.

Then $h(q,t)$ and $g(\alpha,t)$ in the separation constraints Equation (5) evaluated at $k\delta t$ become

$$
\begin{aligned}
h(\boldsymbol{q}, k\delta t) &= \left\langle l^*(k\delta t), \sum_{i=1}^{k} \int_{(i-1)\delta t}^{i\delta t} e^{A(k\delta t - s)} B \boldsymbol{q}_i ds \right\rangle \\
&= \left\langle l^*(k\delta t), \sum_{i=1}^{k} \sum_{j=1}^{m} \int_{(i-1)\delta t}^{i\delta t} e^{A(k\delta t - s)} b_j ds\, \boldsymbol{q}_{ji} \right\rangle \\
&= \sum_{i=1}^{k} l^{*T}(k\delta t)\, P(i) \boldsymbol{q}_i
\end{aligned} \tag{6}
$$

$$
g(\boldsymbol{a}, k\delta t) = \sum_{i=1}^{k} \boldsymbol{a}_i \int_{(i-1)\delta t}^{i\delta t} r(s, k\delta t) ds \tag{7}
$$

Let $\boldsymbol{\mu}_i = \mu(i\delta t)$.

Since $\boldsymbol{a}$ is positive the objective of maximizing $\mu(t)^T \alpha(t)$, is modified to maximize $\boldsymbol{\mu}^T \boldsymbol{a}$ instead, then the optimization problem 3.2 becomes the following,

**Problem 3.3 (Simplified Problem):**

$$
\begin{aligned}
\max_{\boldsymbol{a}, \boldsymbol{q}} \quad & \boldsymbol{\mu}^T \boldsymbol{a} - \left\|H^T \boldsymbol{q}\right\|_F \\
\text{subject to} \quad & \forall k \in 1, 2 \dots N: \\
& \lambda(k) > 0 \\
& \begin{bmatrix}
1 - \lambda(k) & 0 & (\boldsymbol{q}_k - c_{\bar{U}})^T \\
0 & \lambda(k) I & \boldsymbol{a}_k (M_{\bar{U}})^{1/2} \\
\boldsymbol{q}_k - c_{\bar{U}} & \boldsymbol{a}_k (M_{\bar{U}})^{1/2} & M_{\bar{U}}
\end{bmatrix} \succeq 0 \\
& \rho(-l(k\delta t) | \mathcal{R}_x^i(k\delta t)) - \rho(l(k\delta t) | \mathcal{R}_x^e(k\delta t)) > 0
\end{aligned}
$$

The last constraint will be affine on $\boldsymbol{q}, \boldsymbol{a}$ based on Equations (6) and (7). Therefore this problem can be solved using convex optimization.

## IV. Simulations and Results

The reachable set based method described above is demonstrated on the linearized quadrotor models described below.

### A. Quadrotor Model

To capture the dynamics of the quadrotor properly, we need two coordinate frames. One of them is a fixed frame and will be named as the earth frame, and the second one is the body frame which moves with the quadrotor. The transformation matrix from the body frame to the earth frame is $R(t)$. The quadrotor dynamics has twelve state variables $(x, y, z, v_x, v_y, v_z, \phi, \theta, \psi, p, q, r)$, where $\xi = [x, y, z]^T$ and $v = [v_x, v_y, v_z]^T$ represent the position and velocity of the quadrotor w.r.t the body frame. $(\phi, \theta, \psi)$ are the roll, pitch and yaw angles, and $\Omega = [p, q, r]^T$ are the rates of change of roll, pitch and yaw respectively.

The Newton-Euler formalism for the quadrotor rigid body dynamics in earth fixed frame is given by:

$$
\begin{aligned}
\dot{\xi} &= v \\
\dot{v} &= -g\mathbf{e_3} + \frac{F}{m} R\mathbf{e_3} \\
\dot{R} &= R\hat{\Omega} \\
\dot{\Omega} &= J^{-1}(-\Omega \times J\Omega + u)
\end{aligned} \tag{8}
$$

where $g$ is the acceleration due to gravity, $\mathbf{e_3} = [0, 0, 1]^T$, $F$ is the total lift force and $u = [u_1, u_2, u_3]^T$ are the torques applied. $F$ and $u$ are the control inputs. More details on the quadrotor dynamics can be found in [22], [23]. For this work, we linearize the dynamics (8) about the hover with yaw constraint to be zero, as it has been done in [19]. Since $\psi$ is constrained to be zero, we remove $\psi$ and $r$ from our system and make the system ten dimensional. Consequently, we only need three control inputs, $F, u_1$, and $u_2$ for the system. The linearized model is the same as what is done in [19], [20]. The system matrices for the linearized model are:

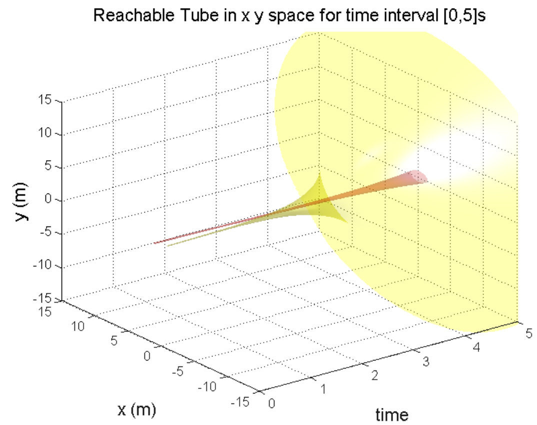


Fig. 1. The initial reachable tubes of of both aircraft projected to $x$ $y$. The reachable tubes are represented by external approximations computed by Ellipsoid Toolbox. The light yellow tube is the external approximation of the reachable tube of the ego aircraft, while the light red one is the external approximation of the reachable tube of the intruder. Clearly, the reachable tubes collide.

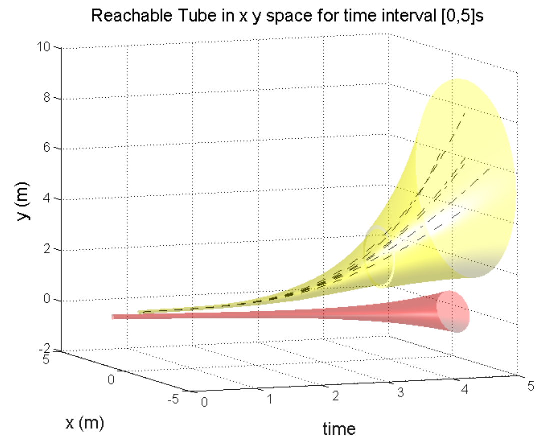


Fig. 3. The reachable tubes for two aircraft after control set design for the second scenario. Since the intruder is moving away, the reachable tube for the ego aircraft is larger in size comparing to the one in Fig. 2 . Clearly, there is no collision between the overapproximation of the reachable tubes. A closer examination also reveals that there is no violation of separation requirement over time.

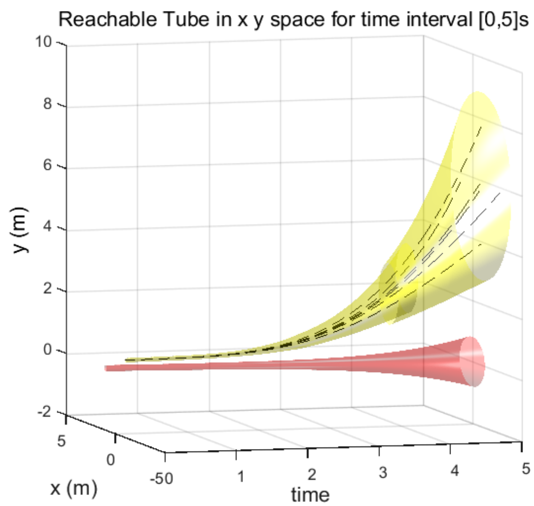


Fig. 2. The reachable tubes for two aircraft after control set design for the first scenario. Clearly, there is no collision between the overapproximation of the reachable tubes. A closer examination also reveals that there is no violation of separation requirement over time. The nonlinear trajectories are within the reachable tube bounds during the whole time horizon.

$$A = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} 0 & g \\ -g & 0 \\ 0 & 0 \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}; \ B = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \begin{bmatrix} 0 \\ 0 \\ 1/m \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2\times 3}J^{-1} \end{bmatrix}$$

$$I_{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

All zero and identity matrices in $A$ and $B$ are of proper dimensions.

### *B. Collision Avoidance Between Quadrotors*

We computed the reachable sets and performed convex optimization using a computer with a 3.4GHz processor and 8GB memory. The software we used include the Ellipsoid toolbox [17] and CVX [24], [25] for solving convex optimization problems.

The first scenario is the following: the ego aircraft starts at $(-1, 0.5, 0)$m with the initial speed of $(0.2, 0, 0)$m/s. The intruder aircraft is currently at $(1, 0, 0)$ approaching the ego aircraft with the initial speed of $(-0.2, 0, 0)$m/s. The separation requirement is 1m. The collision time $T = 5$s. The directions of the hyperplane $l^*(t)$ are estimated based on nominal trajectories. The initial reachable tubes of both aircraft projected to the $x$, $y$ and time axis are shown in Fig. 1. As can be seen, the reachable tubes clearly overlap with each other. By solving the optimization problem, we obtained the resulting reachable tubes for intruder and ego aircraft shown in Fig. 2. The newly computed control set is updated every 0.25s. The nonlinear example trajectories are obtained by forwardly integrating the nonlinear dynamics of Equation (8) with controls on the boundary of the resulting control tube for the ego aircraft. They are within the reachable tube computed. Clearly, the resulting reachable tubes avoid each other, and a closer examination shows that the separation requirement is satisfied.

There is miner a modification in the simulation scenario. The difference is that the intruder aircraft is moving slightly away in the $y$ direction from the ego aircraft with initial speed of $(-0.2, -0.05, 0)$m/s. All the other parameters are the same. The near miss time instant is close to the collision time in the first scenario. By solving the optimization problem, we obtained the resulting reachable tubes for intruder and ego aircraft shown in Fig. 3. The newly computed control set is updated every 0.25s. Since the intruder is moving away, the reachable tube for the ego aircraft is larger in size. The resulting tubes also met the same separation requirements we specified.

The computation time for the control set design through convex optimization for both cases is about 1.2s, which means the collision avoidance algorithm can be implemented online. The reachable set computation takes a lot more time. Since we would like to have good precision of the reachable set, 10 ellipsoids are used to obtain the external approximation. Given time varying control sets the reachable tube computation takes around 620s.

## V. Conclusion

In this paper, we have proposed a reachability based approach to collision avoidance algorithm for UAVs so that the resulting reachable tube is safe from the intruder. We transform the collision avoidance problem into an optimization problem. The reachable set can be captured equivalently by its support function. Thus we formulated the original problem as a continuous time optimal control problem using the support function of the reachable sets. To solve the problem using numerical techniques, the problem was discretized. It was shown that the discretized optimization problem has cost function which is convex in the optimization variables and the constraints are affine in the optimization variables. The optimization problem is solved numerically and the simulation results are presented. We want to emphasize that the present approach is robust, since the problem has been formulated using reachable tubes, instead of a single trajectory. Furthermore, as this approach gives limited constraints on the controller, standard optimization based or rule based controller design can be used after our method to obtain optimal and safe trajectories.

Although currently, the method is limited to linearized systems, reachability analysis for hybrid systems can extend our method to nonlinear and more general dynamics. We focused the analysis for two aircraft collision avoidance, but our method can be extended to the multiple aircraft case by adding constraints.

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## References

[1] N. E. Leonard and E. Fiorelli, "Virtual leaders, artificial potentials and coordinated control of groups," in *Proceedings of the 40th IEEE Conference on Decision and Control, 2001.*, vol. 3. IEEE, 2001, pp. 2968–2973.

[2] Y. Koren and J. Borenstein, "Potential field methods and their inherent limitations for mobile robot navigation," in *Proceedings of 1991 IEEE International Conference on Robotics and Automation, 1991.* IEEE, 1991, pp. 1398–1404.

[3] K. Sigurd and J. How, "UAV Trajectory Design Using Total Field Collision Avoidance," in *AIAA Guidance, Navigation, and Control Conference and Exhibit, American Institute of Aeronautics and Astronautics, Austin, TX, Aug.* American Institute of Aeronautics and Astronautics, Aug. 2003.

[4] C. Carbone, U. Ciniglio, F. Corraro, and S. Luongo, "A novel 3d geometric algorithm for aircraft autonomous collision avoidance," in *2006 45th IEEE Conference on Decision and Control,*. IEEE, 2006, pp. 1580–1585.

[5] E. Lalish, K. A. Morgansen, and T. Tsukamaki, "Decentralized reactive collision avoidance for multiple unicycle-type vehicles," in *American Control Conference, 2008.* IEEE, 2008, pp. 5055–5061.

[6] S.-C. Han and H. Bang, "Proportional navigation-based optimal collision avoidance for UAVs," in *2nd International Conference on Autonomous Robots and Agents*, 2004, pp. 13–15.

[7] Y. Zhou and J. S. Baras, "Reachable set approach to collision avoidance for UAVs," in *2015 54th IEEE Conference on Decision and Control (CDC)*, Dec 2015, pp. 5947–5952.

[8] I. Mitchell, A. Bayen, and C. Tomlin, "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947–957, Jul. 2005.

[9] M. P. Vitus and C. J. Tomlin, "Hierarchical, Hybrid Framework for Collision Avoidance Algorithms in the National Airspace," *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2008.

[10] A. B. Kurzhanski and P. Varaiya, "Ellipsoidal Techniques for Reachability Analysis," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, N. Lynch and B. H. Krogh, Eds. Springer Berlin Heidelberg, 2000, no. 1790, pp. 202–214.

[11] O. Botchkarev and S. Tripakis, "Verification of hybrid systems with linear differential inclusions using ellipsoidal approximations," in *Hybrid Systems: Computation and Control*. Springer, 2000, pp. 73–88.

[12] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, 2008, pp. 4042–4048.

[13] G. Frehse, C. Le Guernic, A. Donz, S. Cotton, R. Ray, O. Lebeltel, R. Ripado, A. Girard, T. Dang, and O. Maler, "SpaceEx: Scalable verification of hybrid systems," in *Computer Aided Verification*, 2011, pp. 379–395.

[14] A. Girard, C. Le Guernic, and O. Maler, "Efficient computation of reachable sets of linear time-invariant systems with inputs," in *Hybrid Systems: Computation and Control*. Springer, 2006, pp. 257–271.

[15] E. Asarin, O. Bournez, T. Dang, and O. Maler, "Approximate reachability analysis of piecewise-linear dynamical systems," in *Hybrid Systems: Computation and Control*. Springer, 2000, pp. 20–31.

[16] S. V. Rakovic, B. Kouvaritakis, R. Findeisen, and M. Cannon, "Homothetic tube model predictive control," *Automatica*, vol. 48, no. 8, pp. 1631 – 1638, 2012.

[17] A. A. Kurzhanskiy and P. Varaiya, "Ellipsoidal toolbox," EECS Department, University of California, Berkeley, Tech. Rep. UCB/EECS-2006-46, May 2006. [Online]. Available: http://code.google.com/p/ellipsoids

[18] Y. Zhou and J. S. Baras, "Reachable set approach to collision avoidance for UAVs," *submitted to CoRR*, 2016.

[19] E. M. Wolff, U. Topcu, and R. M. Murray, "Optimization-based trajectory generation with linear temporal logic specifications," in *Int. Conf. on Robotics and Automation*, 2014.

[20] D. J. Webb and J. van den Berg, "Kinodynamic rrt*: Asymptotically optimal motion planning for robots with linear dynamics," in *2013 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2013, pp. 5054–5061.

[21] A. A. Kurzhanskiy and P. Varaiya, "Ellipsoidal toolbox (et)," in *2006 45th IEEE Conference on Decision and Control*, Dec 2006, pp. 1498–1503.

[22] N. Michael, D. Mellinger, Q. Lindsey, and V. Kumar, "The GRASP multiple micro-UAV testbed," *IEEE Robotics & Automation Magazine*, vol. 17, no. 3, pp. 56–65, 2010.

[23] L. R. García Carrillo, A. E. Dzul López, R. Lozano, and C. Pégard, "Modeling the quadrotor mini-rotorcraft," *Quad Rotorcraft Control*, pp. 23–34, 2013.

[24] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, Mar. 2014.

[25] ——, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110, http://stanford.edu/~boyd/graph_dcp.html.