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Distributed hybrid consensus of second-order dynamics over proximity nets*

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Abstract— **In this paper, we consider the distributed consensus problem of the second-order multi-agent system where the agents are connected via distance-dependent networks. For the feasibility of information processing, we introduce dwell times which may be different at different sampling time instants. Each agent can only receive the information and update its control laws at discrete sampling time instants, which in combination with the continuous-time dynamics yields the hybrid closed-loop dynamics. By analyzing the discrete-time system at the sampling instants and the continuous-time system between the sampling instants, we establish the sufficient condition for consensus of the hybrid second-order multi-agents system, without relying on the properties of the dynamics of the neighbor graphs.**

I. INTRODUCTION

Coordination of multi-agent systems (MAS) has attracted much attention of researchers in control and robotics fields, and many efforts have been paid on the distributed control design of the agents and the analysis of the collective behavior of the whole system, see [1]-[16] and the references therein. Consensus, meaning the agreement of the states of all agents, is a basic task and is related to many phenomena in natural systems, such as flocking in biological systems, superconductivity in physical systems, and collective decision-making in social and economic systems.

For many MAS, the agents can only receive and exchange information from neighbors due to limited sensing and communication capability. In the consensus study, the nearest-neighbor rule is one of the widely used protocols. The relationship between agents can be described by graphs or networks. Therefore, the algebraic properties of graphs (cf., [17]-[19] as well as the corresponding matrix representatives (cf., [20]) are often used to analyze the MAS. For different network topology and different dynamics of the agents, different analysis methods will be used. The consensus results have been established for the continuous-time and discrete-time single integrator, double integrator and nonholonomic unicycle dynamics under undirected or directed network topology. For continuous-time dynamics, the Lyapunov stability theorem and LaSalle's invariance principle are often applied in the consensus analysis. For example, in [26], the first-order continuous-time dynamics is considered, and the consensus results are given under the assumption that the neighbor graphs are connected or jointly connected. In [7], the flocking problem of the second-order continuous-time dynamics is solved if the neighbor graphs are connected. While for discrete-time dynamics, the theorems related to the product of stochastic matrices are often used. For example, the first-order discrete-time dynamics was considered in [1] and [6] under the undirected and directed network topology, and consensus results are given under certain assumptions on dynamical neighbor graphs. Following these results, the systems with nonlinear dynamics, measurement noises and communication delays are also investigated, see [21]-[25] for references. However, in almost all results for the consensus of MAS, the dynamical neighbor graphs are required to satisfy certain connectivity assumptions, and how to verify these assumptions is an unresolved issue.

For different network topology, we may apply different methods to guarantee and preserve the connectivity of neighbor graphs. One of the natural and commonly used approach to construct the network topology is based on the distance between agents. The graphs formed in such a manner are also called proximity nets or geometric graphs. In [27] and [28], the necessary and sufficient condition was established for the connectivity of random geometric graphs. How to preserve the connectivity of dynamical neighbor graphs is a challenging issue for the consensus of MAS. For continuous-time dynamics with continuous-time control laws, the potential function approach is widely used, and the attractive forces in the potential function are designed large enough to maintain the connectivity of neighbor graphs, see [30]-[32]. But, this method may not work for the sampled-data case since the connectivity of neighbors graphs may be lost between sampling instants.

For the feasibility of information receiving and processing, sampled-data technique is widely used, which means that the agents can only receive the information from neighbors and design control laws at sampling instants. The combination of the continuous-time dynamics with the discrete-time control laws leads to the hybrid closed-loop dynamics. How to analyze the dynamical behavior of the hybrid systems without connectivity assumptions of dynamical neighbor graphs is a challenging issue. In this paper, we study the distributed consensus of second-order dynamics, where the dwell time is introduced. The distributed sampled-data

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control law is designed. By analyzing the continuous-time dynamics between sampling instants and the discrete-time dynamics at sampling instants, and relying on the spectral graph theory, we establish sufficient conditions depending on the neighborhood radius, the initial moving speed, and the difference of sampling periods, but without the connectivity of dynamical neighbor graphs.

The rest of this paper is organized as follows. In Section II, we present the problem formulation and provide the main result for consensus of the second-order dynamics under sampled-data control. In Section III, we provide some preliminary lemmas, and the proof of the main theorem is given in Section IV. Concluding remarks are presented in Section V.

II. Problem Statement

Consider a group of n agents moving in an $m-$dimensional Euclidean space ($m=2,3$), with motion equations described by the following second-order dynamics,

$$\begin{cases} \dot{q}_i(t)=p_i(t) \\ \dot{p}_i(t)=u_i(t) \end{cases}, \quad i=1,\cdots,n, \tag{1}$$

where $q_i(t)\in\mathbb{R}^m$ and $p_i(t)\in\mathbb{R}^m$, respectively, denote the position and moving velocity of the agent i at time t $(t\geq 0)$, and $u_i(t)\in\mathbb{R}^m$ is a local feedback control designed according to the information of agent i's neighbors. The pair of agents i and j is said to be neighbors if the agent j falls in the open ball centered at the agent i's position with a given radius $r_n>0$. Denote $\mathscr{N}_i(t)$ as the neighbor set of the agent i, *i.e.*,

$$\mathscr{N}_i(t)=\{j:\|q_j(t)-q_i(t)\|<r_n\}, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm. It is clear that for any agent i and t, we have $i\in\mathscr{N}_i(t)$.

The neighbor relations can be described by a time-varying graph sequence $\mathscr{G}_t=(\mathscr{V},\mathscr{E}_t)$, where the vertex set $\mathscr{V}=\{1,2,\cdots,n\}$ is composed of all agents, while the edge set $\mathscr{E}_t=\left\{(i,j)\in\mathscr{V}\times\mathscr{V}:\|q_i(t)-q_j(t)\|<r_n\right\}$ is defined via the Euclidean distance between agents, and dynamically changes over time. The graph $\mathscr{G}_t$ is also called geometric graph or proximity net. The adjacency matrix of $\mathscr{G}_t$ is denoted as $A(t)=(a_{ij}(t))_{n\times n}$ where $a_{ij}(t)=1$ if $(i,j)\in\mathscr{E}_t$, and $a_{ij}(t)=0$ otherwise. The degree of the agent i, denoted as $d_i(t)$, is defined as $d_i(t)=\sum_{j\in\mathscr{N}_i(t)}a_{ij}(t)$. The degree matrix $D(t)$ is defined as $D(t)=diag(d_1(t),d_2(t),\cdots d_n(t))$. The Laplacian and the normalized Laplacian of the graph $\mathscr{G}_t$ is, respectively, defined as $L(t)=D(t)-A(t)$ and $\mathscr{L}(t)=D^{-1/2}(t)L(t)D^{-1/2}(t)$. The neighbor graphs formed in the above manner are undirected, and thus the matrices $A(t)$ and $\mathscr{L}(t)$ are symmetric. It is clear that $\mathscr{L}(t)$ is non-negative definite and 0 is one of the eigenvalues of $\mathscr{L}(t)$. Arrange all the eigenvalues of $\mathscr{L}(t)$ according to a non-increasing manner as $0=\lambda_0(t)\leq\lambda_1(t)\leq\cdots\leq\lambda_{n-1}(t)$. The connectivity of the graph $\mathscr{G}_t$ is equivalent to that $\mathscr{L}(t)$ has only one eigenvalue 0.

We assume that each agent can only sense the relative velocities of its neighbors. For the feasibility of information receiving and processing, we introduce a dwell time when the agents receive the information and design the control laws. For the agent i, it can receive the information of its neighbors at discrete time instants $t_{i0},t_{i1},t_{i2},\cdots$. Thus, the control law of the agent can only be updated at time instants $t_{i0},t_{i1},t_{i2},\cdots$ and keep unchanged for $t\in[t_{ik},t_{i(k+1)})$ with $k=0,1,2,\cdots$. We introduce the dwell time τ_{ik} to characterize the frequency with which the agent i receives the information, i.e., $t_{i(k+1)}-t_{ik}=\tau_{ik}$. Thus, for the agent i, it can receive the following information at time t_{ik}

$$\{p_j(t_{ik})-p_i(t_{ik}),\ j\in\mathscr{N}_i(t_{ik})\}, \tag{3}$$

where $\mathscr{N}_i(t_k)=\{j:\|q_j(t_k)-q_i(t_k)\|<r_n\}$.

The objective of this paper is to design the distributed control law $u_i(t)$ $(i\in\mathscr{V})$ for the system (1) based on the sampled-data information (3) such that the second-order integrators (1) reach consensus in velocity. By consensus we mean that all agents move with the same velocity eventually, i.e., there exists a vector $p_{ss}\in\mathbb{R}^m$, such that for all $i\in\mathscr{V}$, we have $\lim_{t\to\infty}p_i(t)=p_{ss}$.

In order to simplify the analysis, we assume that all agents receive the information from their corresponding neighbors and update the distributed control law at the same time instant, i.e., for any i and j, we have $t_{ik}=t_{jk}\triangleq t_k$ with $t_0=0$, and we denote the dwell time as τ_k. The analysis for the case where different agents may have different update time instants is much more complicated, and falls into our future research.

In this paper, we adopt the widely-used nearest-neighbor rule for the control law of each agent, i.e., for $t\in[t_k,t_{k+1})$,

$$u_i(t)=\frac{1}{n_i(t_k)}\sum_{j\in\mathscr{N}_i(t_k)}(p_j(t_k)-p_i(t_k)). \tag{4}$$

Substituting (4) into (1), we obtain the following hybrid closed-loop system for $t\in[t_k,t_{k+1})$,

$$\begin{cases} \dot{q}_i(t)=p_i(t) \\ \dot{p}_i(t)=\frac{1}{n_i(t_k)}\sum_{j\in\mathscr{N}_i(t_k)}(p_j(t_k)-p_i(t_k)) \end{cases}. \tag{5}$$

It is clear that the closed-loop system (5) evolves according to continuous-time dynamics between sampling instants, but evolves according to discrete-time dynamics at sampling instants.

The positions and velocities of all agents at the initial time instant is important for the dynamical behavior of the closed-loop system (5). For example, if the agents are distributed in two disjoint clusters at the initial time and the distance between these two clusters is large, then there are no interactions between the agents in different clusters for small neighborhood radius, and the consensus of the system can not be achieved. In addition to this, if the initial velocities of the agents is large, then the neighbor graphs will change a lot and the connectivity of the neighbor graphs can not be guaranteed even if the initial neighbor graph is connected. In this paper, we proceed with our analysis under the following assumptions on the initial configuration of all agents.

Assumption 2.1: 1) All agents are uniformly and independently distributed in the unit square $[0,1]^2$;

2) The initial velocities of all agents satisfies $\max_{1\leq i\leq n}\|p_i(0)\|\leq v_n$.

Under the above assumptions, we establish sufficient conditions depending on the the neighborhood radius and the maximum initial speed to guarantee the consensus of the closed-loop system (5).

Theorem 2.1: Assume that the neighborhood radius and the initial moving speed satisfy

$$\sqrt[6]{\frac{\log n}{n}} \ll r_n \ll 1, v_n \leq \frac{\tau_0\pi^2 r_n^5}{288^2\cdot 324\log n},$$

and the dwell time satisfies $\max_k|\tau_k-\tau_0|\leq\frac{\pi r_n^2}{8\cdot 288\cdot 324}$, then under Assumption 2.1, the closed-loop multi-agent system (5) reaches consensus for large population size.

Remark 2.1: Throughout the sequel, the following standard notions will be used: for two positive sequences $\{a_n, n\geq 1\}$ and $\{b_n, n\geq 1\}$, $a_n=O(b_n)$ means that there exists a positive constant C independent of n, such that $a_n\leq Cb_n$ for any $n\geq 1$; $a_n=o(b_n)$ means that $\lim_{n\to\infty}\frac{a_n}{b_n}=0$.

III. Preliminary Lemmas

Denote $p(t)=[p_1'(t),p_2'(t),\cdots,p_n'(t)]'\in\mathbb{R}^{mn}$, and $Q(t)=D^{-1}(t)L(t)$. By the second equation of (5), we have for $t\in[t_k,t_{k+1})$

$$\dot{p}(t)=(-Q(t_k)\otimes I_m)p(t_k), \tag{6}$$

where $\otimes$ denotes the Kronecker product, and I_m is an identity matrix with order m. For $t\in[t_k,t_{k+1})$, the degree matrix and the Laplacian matrix will not change since the neighbor graphs keep unchanged. Thus, the solution of the equation (6) can be written as

$$p(t)=((I_n-Q(t_k)(t-t_k))\otimes I_m)p(t_k),\ t\in[t_k,t_{k+1}]. \tag{7}$$

For $t=t_{k+1}$, we have

$$p(t_{k+1})=((I_n-Q(t_k)\tau_k)\otimes I_m)p(t_k). \tag{8}$$

By the definition of $Q(t_k)$, we see that $((I-Q(t_k)\tau_k)\otimes I_m)\mathbf{1}_{mn}=\mathbf{1}_{mn}$, where $\mathbf{1}_{mn}$ denotes the $mn-$dimensional vector with all elements 1. Furthermore, if $\tau_k\leq 1$, then all elements of the matrices $((I-Q(t_k)\tau_k)\otimes I_m)$ are nonnegative, and thus the matrices $((I-Q(t_k)\tau_k)\otimes I_m)$ are stochastic. We proceed with the following analysis under assumption $\tau_k\leq 1$ without further explanations.

In the consensus analysis, we are concerned with the convergence behavior of the dissimilarity of the velocities $\delta p(t)$ which is defined as

$$\delta p(t)=\max_{1\leq i,j\leq n}\|p_i(t)-p_j(t)\|.$$

The dynamical behavior of $p(t)$ depends on the properties of the matrices $Q(t)$, which are determined by the neighbor graphs. We will first provide a preliminary lemma for the evolution of $p(t_k)$ for the case where the neighbor graphs and the dwell times keep unchanged.

Lemma 3.1: Assume that the neighbor graphs and the dwell times keep unchanged, and denoted by $\mathscr{G}$ and τ, respectively. The normalized Laplacian and the degree of $\mathscr{G}$ are denoted as $\mathscr{L}$ and D, and the eigenvalues of $\mathscr{L}$ are denoted in a non-decreasing manner as $\lambda_0\leq\lambda_1\leq\cdots\leq\lambda_{n-1}$. Then we have for $t\in[t_k,t_{k+1})$

$$\max_{1\leq i,j\leq n}\|p_i(t)-p_j(t)\|\leq\sqrt{2}\kappa\widetilde{\lambda}^{\lceil\frac{t}{\tau}\rceil}\|p(0)\|,$$

where $\lceil x\rceil$ denotes the largest natural number not more than x, $\kappa=\sqrt{\frac{d_{\max}}{d_{\min}}}$ with $d_{\max}=\max_{1\leq i\leq n}d_i$ and $d_{\min}=\min_{1\leq i\leq n}d_i$, and $\widetilde{\lambda}\triangleq\max\{|1-\lambda_1\tau|,|1-\lambda_{n-1}\tau|\}$.
The proof details are omitted due to space limitations.

By the assumption that $\tau\leq 1$ and the fact that the eigenvalues of $\mathscr{L}$ satisfy $0\leq\lambda_i\leq 2$, we see that $\widetilde{\lambda}\leq 1$. Furthermore, if the neighbor graphs keep unchanged and are connected, then we have $\widetilde{\lambda}<1$, and the consensus can be achieved with the convergence rate $\widetilde{\lambda}$. However, the neighbor graphs are determined via the positions of all agents, and will change over time. It is clear that if the positions of all agents change too much, then it is difficult to characterize the change of matrices $Q(t)$. In this paper, we consider the case where the matrices $Q(t)$ slowly change with time.

Lemma 3.2: Denote the neighbor graph at time interval $[t_k,t_{k+1})$ as $\mathscr{G}(t_k)$ and the corresponding degree, minimum degree, maximum degree, degree matrix and the normalized Laplacian are denoted as $d_i(t_k), d_{\min}(t_k), d_{\max}(t_k), D(t_k)$ and $\mathscr{L}(t_k)$, respectively. If $\|Q(t_k)\tau_k-Q(t_0)\tau_0\|\leq\varepsilon$, then we have

$$\max_{1\leq i,j\leq n}\|p_i(t_k)-p_j(t_k)\|\leq\sqrt{2}\kappa(\widetilde{\lambda}(0)+\kappa\varepsilon)^k\|p(0)\|,$$

where $\widetilde{\lambda}(0)=\max\{1-\lambda_1(0)\tau_0, 1-\lambda_{n-1}(0)\tau_0\}$.

The above lemma can be derived by using Lemma 3.1 and Lemma 2 in [33]. We omit the proof details to save space.

By Lemma 3.2, we see that in the consensus analysis of (7) and (8), we need to estimate the upper bound of $\|Q(t_k)\tau_k-Q(0)\tau_0\|$, as well as some characteristics concerning the initial states including the maximum and minimum degrees, the spectrum of the normalized Laplacian $\mathscr{L}$.

The quantity $\|Q(t_k)\tau_k-Q(0)\tau_0\|$ is determined by the positions of all agents, while the positions depend on the velocities, and the velocities are affected by $\|Q(t_k)\tau_k-Q(0)\tau_0\|$. Thus, $\|Q(t_k)\tau_k-Q(0)\tau_0\|$, positions and headings of all agents are coupled together. We deal with this coupled relationship in the next section. Here, we first provide a preliminary result for the estimation of $\|Q(t_k)\tau_k-Q(0)\tau_0\|$.

Lemma 3.3: Assume that in comparison with the initial neighbor graph $\widehat{G}(0)$, the number of agents changed in the neighborhood of the k-th $(1\leq k\leq n)$ node in the graph $\widehat{G}(t_k)$ satisfies $R_k\leq R_{\max}<d_{\min}$, then

$$\begin{aligned}\|Q(t_k)\tau_k-Q(0)\tau_0\|\leq&\frac{2(d_{\max}(0)+R_{\max})|\tau_k-\tau_0|}{d_{\min}}\\&+2\frac{R_{\max}(d_{\max}(0)+d_{\min}(0))\max_k\tau_k}{d_{\min}(0)(d_{\min}(0)-R_{\max})},\end{aligned}$$

where $Q(t_k)$ is defined as $Q(t_k)=D^{-1}(t_k)L(t_k)$.

The proof details are omitted due to space limitations.

We introduce the following set $\mathscr{R}_i$ to characterize the change of neighbors of the agent $i(1\leq i\leq n)$,

$$\mathscr{R}_i=\{j:(1-\eta_n)r_n\leq d_{ij}(0)\leq(1+\eta_n)r_n\}, \tag{9}$$

where $d_{ij}(0)=\|q_i(0)-q_j(0)\|$ is the distance between agents i and j at the initial time, and η_n can be taken as follows:

$$\eta_n = \frac{\pi r_n^2}{288\cdot 324}. \tag{10}$$

The cardinality of the set $\mathscr{R}_i$ is denoted as R_i, and $R_{max}=\max_{1\le i\le n} R_i$. Assume that for any time instants t_k, $\|d_{ij}(t_k)-d_{ij}(0)\|\le \eta_n r_n$. If $d_{ij}(0)<(1-\eta_n)r_n$, then $d_{ij}(t_k)<r_n$, and if $d_{ij}(0)\ge(1+\eta_n)r_n$, then $d_{ij}(t_k)\ge r_n$. Thus, the set (9) characterizes the change of neighbors of the agents i.

Under Assumption 2.1, we can obtain estimates for the characteristics concerning the initial states of the agents..

Lemma 3.4: [33] For the initial random geometric graph G_0, if the interaction radius satisfies $\sqrt[6]{\frac{\log n}{n}}\ll r_n\ll 1$, then the following results hold almost surely for all large n,

1) The maximum and minimum degrees satisfy

$$d_{\max}=n\pi r_n^2(1+o(1));\quad d_{min}(0)=\frac{n\pi r_n^2}{4}(1+o(1)).$$

2) The maximum number of agents in (9) satisfies

$$R_{max}\le 4n\pi\eta_n r_n^2(1+o(1)). \tag{11}$$

3) The second smallest eigenvalue of the normalized Laplacian $\mathscr{L}(0)$ of the graph $G(0)$ satisfy

$$\lambda_1(0)\ge\frac{\pi r_n^2}{144}(1+o(1));$$
$$\lambda_{n-1}(0)\le 2\left(1-\frac{1}{4(1+2\sqrt{3})^2}\right)(1+o(1)).$$

Remark 3.1: The constants κ and $\widetilde{\lambda}$ in Lemmas 3.1 and 3.2 can be taken as

$$\kappa=2(1+o(1));$$
$$\widetilde{\lambda}\le 1-\frac{\pi r_n^2\tau_0}{144}(1+o(1)).$$

Remark 3.2: By [34], we see that similar results as those of Lemma 3.4 hold for $m=3$.

IV. Proof of Theorem 2.1.

The analysis of the hybrid closed-loop system needs the combination of the discrete-time dynamics at sampling instants with the continuous-time dynamics between sampling instants. In addition, we need to deal with the entanglement relations of positions and headings. For the closed-loop second-order dynamics under consideration, the velocity of each agent is determined via the velocities of the neighbors, while neighbors are determined by the position of the agents, and the position of each agent depends on its velocities. We deal with this coupled relation in the following proposition.

Proposition 4.1: Assume the dwell time satisfies $\max_k|\tau_k-\tau_0|\le\frac{\eta_n}{8}$. If the moving speed and the neighborhood radius satisfy the following conditions:

$$\sqrt[6]{\frac{\log n}{n}}\ll r_n\ll 1,\quad v_n\le\frac{\tau_0\pi\eta_n r_n^3}{288\log n},$$

then we have

$$|d_{ij}(t_k)-d_{ij}(0)|\le\eta_n r_n; \tag{12}$$
$$\|Q(t_k)\tau_k-Q(t_0\tau_0)\|\le 162\eta_n, \tag{13}$$

where η_n is taken as in (10).

The proof details are omitted due to space limitations.

Proof of Theorem 2.1

By the translational velocity update equation (6), we know that $\max_{1\le i\le n}\|p_i(t_k)\|$ (resp. $\min_{1\le i\le n}\|p_i(t_k)\|$) is a non-increasing (resp. non-decreasing) sequence, so $\max_{1\le i\le n}\|p_i(t_k)\|$ and $\min_{1\le i\le n}\|p_i(t_k)\|$ have bounded limits as $k\to\infty$. On the other hand, by Proposition 4.1, we have for all $k\ge 0$

$$\begin{aligned}
&\max_{1\le i,j\le n}\|p_i(t_k)-p_j(t_k)\|\\
\le\ & 2\sqrt{2n}v_n\left(1-\frac{\pi r_n^2(1+o(1))}{288}\right)^k\\
\to\ & 0,\qquad\qquad \text{as } k\to\infty.
\end{aligned}$$

It is easy to see that the translational velocity tends to the same value. This completes the proof of the theorem. ■

V. Concluding Remarks

In this paper, we investigated the distributed sampled-data consensus of the second-order dynamics connected via the proximity networks, where the sampled periods may be different and independent of the network topology. The combination of the continuous-time dynamics with the discrete-time control law leads to the hybrid closed-loop dynamics. We provide sufficient conditions depending on the moving speed, neighborhood radius and the initial sampling period for consensus. The investigation for the case where different agents may have different sampling time instants is interesting and more complicated, and falls into our future research.

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