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The Effect of Delays in the Economic Dispatch Problem for Smart Grid Architectures

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Abstract— **We consider the Economic Dispatch Problem (EDP) in power systems for a smart-grid friendly environment. We develop a consensus based decentralized optimization algorithm that evolves in a time-varying communication network that suffers from multiple propagation time-dependent delays. This paper, being an improvisation of an earlier work of ours, addresses the effects of time dependent delays in a fully decentralized network. We show analytically and by simulation that propagation delays may not only affect the performance of the dynamic algorithm that solves the EDP but can also destabilize the system.**

I. INTRODUCTION

Energy supply systems are typically a structure of interconnected power generation plants that independently produce power to serve a load over a common distribution network [10], [11], [16]. Power units, however, produce energy at some cost. Hence, a fundamental optimization problem associated with power grids is the operation of the power units to serve a given load so that the cumulative cost is minimized. This is the well-known EDP.

Over the past years many optimization methods for the EDP have been proposed in the literature. The solution approaches vary from the lambda iteration or gradient based search algorithms [10], [16] to modern heuristic optimization techniques (see for example [1], [2], [3] and references therein). Although the performance and applicability of economic dispatch has been improved by these optimization techniques, they all require the maintenance of a central control center that can access the state of the entire system.

The deregulation of the electric utilities has led to research on a decentralized model of control where utilities, transmission system operators and independent power producers cooperate and compete using market and other mechanisms [16]. Therefore, centrally designed and controlled power networks may cause performance limitations due to the fundamental incompatibility with the modern power network design principles.

A. Smart-Grid Architecture

The new generation of power systems is expected to satisfy high standards of efficiency, resilience and reliability against cyber-attacks or natural disasters, improved integration of renewable energy resources and plug-in hybrid electrical vehicles. Such electrical grids, known as Smart-Grid (SG), are designed to monitor, predict and intelligently respond to the behavior of all electric power suppliers and consumers connected to it in order to deliver such standards [4], [7].

A necessary requirement is the development of advanced control and communication technology both in the physical and the algorithmic layer. In an SG environment, the communication and measurement requires a multi-agent systems (MAS) technology [7]. MAS are computation networks in which several agents co-operate to achieve a desired task. The development of monitoring and measurement in SG with the use of MAS technology involves a combination of several agents working without human intervention, collaboratively pursuing assigned tasks to achieve the overall goal of the system.

B. MAS architecture and consensus based algorithms

The central feature of MAS devices is their ability to operate in a decentralized and co-operative manner in order to achieve a global and common goal. Each agent has access to limited information of the network and typically supervises a part of it. Based on the available information, co-operative dynamic control laws are implemented so that the desired operational state is reached.

A central algorithm in networked systems is the agreement or consensus algorithm under which a collection of autonomous agents engage in a dynamic averaging of a state of interest so that in the long run, they all obtain the same value. The research in consensus systems is fairly vast as they are considered to be the underlying mathematical model for co-operative biological, social, robotic networks (see [6], [8], [15] and references therein).

C. Motivation & Contribution

Recent advances in consensus systems have reheated the subject of such algorithms executed over a finite number of processors in order to solve optimization problems in a decentralized way. The successful and reliable implementation of complex electrical networks such as the SG, requires advanced measurement and control methods that operate autonomously, while taking into account the mechanical and communication restrictions of such networks.

These limitations affect not only the performance but also the stability of the algorithms. In a number of papers Zhang et al. [17], [18], [19], solved the EDP using the consensus algorithm in a centralized version with constant weights and uniform delays. In [14] the authors extended

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the work of Zhang and Chow by proposing a generalized linear consensus based model for the decentralized solution of the EDP using the lambda iteration algorithm. In that paper, the main contribution is that in addition to time-varying communication parameters that would take into account realistic communication phenomena such as link failures, time-dependent delays were also considered in a way that did not affect the stability, but only had effects on the performance of the algorithm. For this strong result a number of unrealistic assumptions had to be considered.

The present work revisits the problem in the framework introduced in [14] and proposes a fundamentally decentralized scheme for the solution of the EDP. In particular we relax a number of simplifications taken in [14], such as the connectivity scheme and the mechanical limitations of the power generators. We demonstrate that the combination of the communication regime, the magnitude of communication delays and the mechanical limitations play a very important role in the stability of the proposed algorithm.

D. Organization Of The Paper

In §II we review the nominal mathematical model for the analysis that follows. In §III we review the classic EDP with operation constraints and we explain the adaptation to a decentralized environment. In §IV we continue the discussion of §III and introduce our main algorithm. The rigorous analysis is presented in §V where we demonstrate how the proposed algorithm can be restated as a perturbed leader follower consensus model with delays and we establish a sufficient stability in variation condition. In §VI a simulation example is presented to validate the theoretical results. A thorough discussion on the advantages and disadvantages on our approach together with prospects for future work is held in §VII.

II. Linear Consensus With Delays

A number of $N < \infty$ autonomous agents constitute the set $[N] = \{1, \ldots, N\}$. Each agent is defined through a state of interest x_i, $i \in [N]$ that is dynamically updated under the following scheme,

$$\begin{cases} \dot{x}_i(t) = \sum_j a_{ij}(t)\big(x_j(t-\tau_{ij}(t)) - x_i(t)\big), & t \geq t_0 \\ x_i(t) = \phi_i(t), & t \in I_{t_0} \end{cases} \tag{1}$$

where $a_{ij}, \tau_{ij} : [t_0, \infty) \to \mathbb{R}_+$ are the coupling weights and the propagation delays respectively, $I_{t_0} = [t_0 - r, t_0]$ is the time interval of the appropriate initial data $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$ all of which under standard conditions ensure the existence and uniqueness of a solution $\mathbf{x} = \mathbf{x}(t, t_0, \boldsymbol{\phi})$, $t \geq t_0$ [5]. The weights $a_{ij} \geq 0$ (with $a_{ii} \equiv 0$) constitute the communication graph that is represented by the adjacency matrix $A = [a_{ij}]$. In [8] it is shown that if there exists $B > 0$ such that $\int_t^{t+B} A(s)\,ds$ constitutes a sufficiently connected graph (routed-out branching), then $\{x_i(t)\}_{i \in [N]}$ converge exponentially fast to a common constant.

A leader in a consensus network, is an agent that affects the rest of the group without being affected by others. Non-negative Matrix Theory [12] ensures that for the same connectivity conditions, the existence of at most one leader in the network forces the state of the entire group to the match the leader's, provided the latter is (or becomes asymptotically) constant.

Theorem 2.1: [15] Consider (1) and its solution $\mathbf{x}$. If:

(i) τ_{ij} have continuous first derivative with $1 - \dot{\tau}_{ij} > 0$ and $\sup_{t \geq t_0} \tau_{ij}(t) = \tau < \infty$,

(ii) $\exists\ \epsilon > 0$ such that $a_{ij}(t) \neq 0$ implies that $a_{ij} > 0$ for at least an ϵ interval of time in which t is included,

(iii) $\exists\ B > 0$ such that the graph corresponding to $\int_t^{t+B} A(s)\,ds$, $t \geq t_0$ is routed-out branching,

there exists $x_\infty \in [\min_{s,i} \phi_i(s), \max_{s,i} \phi_i(s)]$ such that

$$\max_i |x_i(t) - x_\infty| \leq E e^{-\epsilon(t-t_0)}.$$

for some $E, \epsilon > 0$. In particular, if we have a leader-follower network, then x_∞ is the state of the leader.

In [15] the authors provide a unified framework for the stability of consensus networks with explicit estimates of the rate of convergence of (1), i.e. E, ϵ can be estimated by the initial data as well as $A = [a_{ij}]$, B and $\boldsymbol{\tau} = [\tau_{ij}]$.

III. The EDP with Operation Constraints

The EDP [16] consists of a system of N power generating units, connected to a single bus bar, serve a received electrical load P_L. The input to each unit, F_i, represents the cost rate of the unit. The output of each unit, P_i, is the electrical power generated by that particular unit. The total cost rate of this system is the sum of the costs of each of the individual units. We have two essential constraints:

- The sum of the output power P_i must equal the load demand P_L: $\sum_{i=1}^N P_i = P_L$.
- The units operate within bounds: $P_i \in [\underline{P}, \overline{P}]$, $i \in [N]$ for some $\underline{P}, \overline{P} > 0$.

This is a constrained optimization problem and it is attacked with standard calculus that involves the Lagrange function:

$$\mathcal{L} = F_T + \lambda \iota$$

where $F_T = \sum_{i=1}^N F_i(P_i)$ is the objective function and

$$\iota = 0 = P_L - \sum_{i=1}^N P_i$$

The necessary conditions for an extreme value of the objective function F_T result when we take the first derivative of $\mathcal{L}$ with respect to each independent variables and set the derivatives equal to 0:

$$\frac{\partial \mathcal{L}}{\partial P_i} = \frac{dF_i(P_i)}{dP_i} - \lambda = 0. \tag{2}$$

Following [16] we assume quadratic cost functions:

$$F_i(P_i) = \frac{1}{2}\chi_i P_i^2 + \psi_i P_i + \omega_i \tag{3}$$

for parameters $\chi_i, \psi_i, \omega_i > 0$ assumed to be known. Together with (2) we must add the two constraints mentioned above so that the optimal operation point must satisfy for $i \in [N]$:

$$\frac{dF_i(P_i)}{dP_i} = \lambda, \qquad \underline{P} \leq P_i \leq \overline{P}, \qquad \sum_{i=1}^N P_i = P_L. \tag{4}$$

A. decentralized approach for SG setting

We assume that the SG network is covered by a finite number N of stations each of which has full access to the parameters of their area of duty (load and generator). At the same time they have limited (local) and asynchronous information on the quantities of any other area of the network. Any information is propagated through a time-dependent communication network as well as it suffers from delays. In particular, given a network with weights defined through the adjacency matrix $A = [a_{ij}]$ (A-network), each controller $i \in [N]$ executes the following tasks:

(1). Attains instant information of the (constant) load in its section, $P_{load}^{(i)}$, (consequently $P_L = \sum_i P_{load}^{(i)}$). At this level, agent i acts as a leader of a consensus algorithm that transmits $P_{load}^{(i)}$ over the A-network and the rest of the agents learn this value. Thus we have the following consensus system for the i^{th} agent of the network:

$$\dot{p}_n^{(i)}(t) = \begin{cases} 0, & n = i \\ \sum_j a_{nj}(t)\big(p_j^{(i)}(t - \kappa_{nj}^{(i)}(t)) - p_n^{(i)}(t)\big), & n \neq i \end{cases}$$

In other words, $\mathbf{p}^{(i)}$ are dynamic vectors with elements that will approach asymptotically $P_{load}^{(i)}$. Thus through this process every agent will learn the load of the network by exchanging local information with their neighbors only. Now, the i^{th} agent's load observation is

$$P_{c_L}^{(i)}(t) = P_{load}^{(i)} + \sum_{j \neq i} p_i^{(j)}(t) \tag{5}$$

If the delays $\kappa_{jn}^{(i)}$ are smooth and uniformly upper bounded then by Theorem 2.1, there exist $L, \underline{l} > 0$ that depend on the network A and the delays $\kappa_{jn}^{(i)}$ such that:

$$|P_{c_L}^{(i)}(t) - P_L| \leq L e^{-\underline{l}(t - t_0)}, \qquad t \geq t_0. \tag{6}$$

(2). Instant information on the power the i^{th} generator produces, denoted by P_i, updated according to the differential equation

$$\frac{d}{dt} P_i(t) = -\gamma_i(t) P_i(t) + \gamma_i(t) \frac{\lambda_i(t) - \psi_i}{\chi_i}. \tag{7}$$

This equation models the mechanical constraints of the power generators that tend to approach the desired operation point $\frac{\lambda_i(t) - \psi_i}{\chi_i}$. These can be time-dependent factors (e.g. inertia, primary resources availability) that are model with the positive function γ_i such that

$$0 < \underline{\gamma} \leq \gamma_i(t) \leq \overline{\gamma} < \infty \tag{8}$$

so that convergence to $\frac{\lambda_i(t) - \psi_i}{\chi_i}$ occurs exponentially fast with rate at least $\underline{\gamma}$.

(3). Asynchronous information on the produced generator for the rest of the sensors. At this level, the observation each controller has on the cumulative generated power is

$$P_{c_G}^{(i)}(t) = \sum_{j=1}^{N} P_j\big(t - \sigma_{ij}(t)\big), \tag{9}$$

where σ_{ij} are smooth time-dependent and uniformly bounded delays with the convention that $\sigma_{ii} = 0$.

(4). The $\boldsymbol{\lambda}$ update. The lambda iteration algorithm is executed locally under the structure of the A network (see (10) below).

IV. The Model

As described in previous sections, $N < \infty$ generators attempt to serve a common load while at the same time operate at an optimal point. N agents control a part of the power generated and a part of the load. The lambda iteration algorithm is executed under a decentralized consensus based scheme that evolves in the A network and suffers from time dependent propagation delays. The initial value problem is written as follows:

$$\begin{cases} \dot{\lambda}_i(t) = \sum_j a_{ij}(t)\big(\lambda_j(t - \tau_{ij}(t)) - \lambda_i(t)\big) + \\ \qquad\qquad + w_i(t)\big(P_{c_L}^{(i)}(t) - P_{c_G}^{(i)}(t)\big), & t \geq t_0 \\ \lambda_i(t) = \phi_i(t), & t \in I_{t_0} \end{cases} \tag{10}$$

with $I_{t_0} = [t_0 - r, t_0]$, $P_{c_L}^{(i)}(t)$ as in (5), $P_{c_G}^{(i)}(t)$ as in (9) and w_i is a control coupling parameter. In the next two sections we will discuss (10) both in theory and in simulation.

V. Analysis

We conduct a theoretical analysis of (10) and derive sufficient conditions for asymptotic stability. Our strategy is to express (10) as a perturbation of a nominal consensus leader-follower network. Then a stability in variation expression of the solution and a fixed point theory argument will state a general asymptotic stability criterion.

A. Construction of the nominal system

By (7), the variation of parameters formula, yields:

$$P_i(t) = e^{-\int_{t_0}^t \gamma_i(s)\, ds} P_i(t_0) + \int_{t_0}^t e^{-\int_s^t \gamma_i(q)\, dq} \gamma_i(s) \frac{\lambda_i(s) - \psi_i}{\chi_i}\, ds$$

so that $P_{c_G}^{(i)}(\cdot)$ is, henceforth, considered in the central differential equation as a function of $\boldsymbol{\lambda}$ with the appropriate delays σ_{ij}. Now, from (4) P_i must operate within $[\underline{P}, \overline{P}]$. A sufficient condition for this is deduced from the above equation and it is

$$\underline{P} \leq P_i(t_0) \leq \overline{P} \quad \text{and} \quad \psi_i + \underline{P}\chi_i \leq \lambda_i(t) \leq \overline{P}\chi_i + \psi_i \tag{11}$$

Additionally, we will write $w_i(t)\big(P_{c_L}^{(i)}(t) - P_{c_G}^{(i)}(t)\big)$ as the sum of a term that makes λ_i tend to a constant value and another term that if λ_i tends to the aforementioned value, the latter term asymptotically vanishes. The "strength" of the latter term determines the stability of the overall algorithm. These remarks pave the way for the main result of this section.

Proposition 5.1: The system presented in (10) is equivalent to $\overline{\boldsymbol{\lambda}} = (\lambda_0, \ldots, \lambda_N)$ with

$$\lambda_0 = \frac{P_L + \sum_j (\psi_j / \chi_j)}{\sum_j \chi_j^{-1}}$$

and

$$\begin{cases}\dot\lambda_i(t) = \sum_j \overline{a}_{ij}(t)\big(\lambda_j(t-\tau_{ij}(t)) - \lambda_i(t)\big) + f_i(t,\boldsymbol{\lambda}_t), & t \geq t_0\\ \lambda_i(t) = \phi_i(t), & t \in I_{t_0}\end{cases}$$

for $i \in [N]$ where $\bar{A} = [\overline{a}_{ij}(t)]$ constitutes a leader-follower network with delays and $f_i(t,\boldsymbol{\lambda}_t)$ is a state-dependent perturbation.

Proof: By adding and subtracting appropriate terms we can write $w_i(t)\big(P^{(i)}_{c_L}(t) - P^{(i)}_{c_G}(t)\big)$

$$w_i(t)\big(P_L - P^{(i)}_{c_G}(t)\big) + w_i(t)\big(P^{(i)}_{c_L}(t) - P_L\big) = \\ = \overline{w}_i(t)\big(\lambda_0(t) - \lambda_i(t)\big) + \varepsilon_i(t) + \zeta_i(t,\boldsymbol{\lambda}_t)$$

with

$$\eta_{ij}(t) := 1 - e^{-\int_{t_0}^{t-\sigma_{ij}(t)}\gamma_j(s)\,ds},\ \overline{w}_i(t) := w_i(t)\sum_j \eta_{ij}(t)\chi_j^{-1}$$

$$\varepsilon_i(t) := -w_i(t)\sum_j \big(1-\eta_{ij}(t)\big)P_j(t_0) + w_i(t)\big(P^{(i)}_{c_L}(t) - P_L\big) + \\ + \overline{w}_i(t)\bigg(\frac{P_L + \sum_j(\eta_{ij}(t)\psi_j)/\chi_j}{\sum_j \eta_{ij}(t)/\chi_j} - \lambda_0\bigg)$$

$$\zeta_i(t,\boldsymbol{\lambda}_t) := \\ = -w_i(t)\sum_j\int_{t_0}^{t-\sigma_{ij}(t)} e^{-\int_s^{t-\sigma_{ij}(t)}\gamma_j(q)dq}\gamma_j(s)\frac{\lambda_j(s)-\lambda_i(t)}{\chi_i}\,ds$$

so that $f_i(t,\boldsymbol{\lambda}_t) = \varepsilon_i(t) + \zeta_i(t,\boldsymbol{\lambda}_t)$ and $\overline{A} = [\overline{a}_{ij}(t)]$ with

$$\overline{a}_{ij}(t) = \begin{cases}0, & i = 0\\ \overline{w}_{ij}(t), & i \neq 0, j = 0\\ a_{ij}(t), & i,j \neq 0\end{cases}$$

■

Combining the consensus point λ_0 with the feasibility condition (11) we see that it is necessary for $\underline{P}$ and $\overline{P}$ to be chosen appropriately. A relaxed sufficient condition is

$$\max_i \psi_i + \underline{P}\max_i \chi_i < \lambda_0 < \min_i \psi_i + \overline{P}\min_i \chi_i \qquad (12)$$

B. Stability in variation

We will construct a fixed point theory representation of the solution $\overline{\boldsymbol{\lambda}}$ of the system of Proposition 5.1 and prove that there is an appropriate functional space where the constructed solution representation, acting as an operator, is a contraction. Indeed, we will compare the aforementioned system with the unperturbed:

$$\begin{cases}\dot\nu_i(t) = \sum_j \overline{a}_{ij}(t)\big(\nu_j(t-\tau_{ij}(t)) - \nu_i(t)\big), & t \geq t_0\\ \nu_i(t) = \phi_i(t), & t \in I_{t_0}.\end{cases} \qquad (13)$$

For this we need to consider C as the Banach space of continuous function mapping the interval $[t_0 - r, t_0]$ into $\mathbb{R}^N$ with the topology of uniform convergence and the solution segment $\boldsymbol{\nu}(t,s,\boldsymbol{\nu}(s,t_0,\boldsymbol{\phi})) = \boldsymbol{\nu}_t(s,\boldsymbol{\nu}) \in C([t-r,t],\mathbb{R}^{N+1})$. By Proposition 5.1 and Theorem 2.1, $\boldsymbol{\nu}$ can be represented as

$$\boldsymbol{\nu}(t,t_0,\boldsymbol{\phi}) = T(t,t_0)\boldsymbol{\phi} \qquad (14)$$

where $T(t,s) : C \to C$ is a family of continuous linear operators. By virtue of Theorem 2.1, we set $\tilde{T}(t,s) := T(t,s) - \mathbf{1}(1,0,\ldots,0)$ that satisfies

$$\big|\tilde{T}(t,s)\boldsymbol{\phi}\big| \leq Ee^{-\underline{\epsilon}(t-s)} \qquad (15)$$

where $\mathbf{1} \in \mathbb{R}^{N+1}$ is the column vector of all ones. From [13] the solution $\overline{\boldsymbol{\lambda}} = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_N)$ can be expressed as

$$\overline{\boldsymbol{\lambda}}_t(\theta) = [T(t,t_0)\boldsymbol{\phi}](\theta) + \int_{t_0}^t [T(t,s)Y_0](\theta)\overline{\mathbf{f}}(s,\boldsymbol{\lambda}_s)\,ds \qquad (16)$$

for $\theta \in [-r,0]$ and $Y_0(\theta) = \mathbf{0}, \theta \in [-r,0)$, $Y_0(0) = I$ and $\overline{\mathbf{f}}(s,\boldsymbol{\lambda}_s) = \big(0, f_1(t,\boldsymbol{\lambda}_t), \cdots, f_N(t,\boldsymbol{\lambda}_t)\big)^T$.

Remark 5.2: For the sake of simplicity as well as the feasibility condition (12) we set the following initial data:

1) $t_0 = 0$
2) $\phi_i(t) \in \big[\max_i \psi_i + \underline{P}\max_i \chi_i, \min_i \psi_i + \overline{P}\min_i \chi_i\big]$ for $t \in I_0$ as initial data for the lambda iteration algorithm.
3) $P_i^0 = \underline{P}$ as initial data for the generated power at each unit.
4) $\mathbf{p}^{(i)} : p_i^{(i)} \equiv P^{(i)}_{load}$, $p_j^{(i)}(t) \equiv 0$ for $t \in I_0$ as initial data for the parallel learning of the network load P_L.
5) We set the mechanical reaction $\underline{\gamma} < \underline{\epsilon} < \overline{\gamma}$, for $\underline{\gamma}$ and $\overline{\gamma}$ as in (8).
6) we set the load learning rate $\underline{l} < \underline{\epsilon}$.

The last two assumptions are taken without loss of generality. Had either not being valid only would lead to a minor alternation of the following condition:

Assumption 5.3: It holds that

$$\Xi\bigg(1 - 2\overline{w}E\max\bigg\{\frac{1}{\underline{\epsilon}}, \frac{e^{\overline{\gamma}\overline{\sigma}} - e^{\underline{\epsilon}\,\overline{\sigma}}}{\overline{\gamma} - \underline{\epsilon}}\bigg\}\sum_j \chi_j^{-1}\bigg) > \\ \overline{w}E\bigg[L\frac{\big(\frac{\underline{l}}{\underline{\epsilon}}\big)^{\underline{\epsilon}} - \big(\frac{\underline{l}}{\underline{\epsilon}}\big)^{\underline{l}}}{\underline{l} - \underline{\epsilon}} + \frac{\big(\frac{\underline{\gamma}}{\underline{\epsilon}}\big)^{\underline{\epsilon}} - \big(\frac{\underline{\gamma}}{\underline{\epsilon}}\big)^{\underline{\gamma}}}{\underline{\gamma} - \underline{\epsilon}}e^{\underline{\gamma}\overline{\sigma}}J\bigg]$$

where

$$J := \bigg(N\underline{P} + P_L + 2\sum_j \psi_j\chi_j^{-1}\bigg)$$

$$\Xi := \min\big\{\lambda_0 - \max_i \chi_i\underline{P} - \max_i \psi_i, \min_i \chi_i\overline{P} + \min_i \psi_i - \lambda_0\big\}$$

and $E, \underline{\epsilon} > 0$ as in (15), $\overline{\gamma}, \underline{\gamma} > 0$ as in (8), $L, \underline{l} > 0$ as in (6) and $\overline{w} = \max_i \sup_t w_i(t)$, $\underline{\sigma} < \sigma_{ij} < \overline{\sigma}$.

We are ready now to state and prove the main result of this work:

Theorem 5.4: Assume the following:

1) The A-network and the delays $[\tau_{ij}], [\sigma_{ij}]$ and $[\kappa^{(i)}_{nj}]$ satisfy the conditions of Theorem 2.1.
2) Assumption 5.3 is true.

Then the EDP as it is stated in (10) has a unique solution and all units asymptotically tend to the optimal operating condition.

Proof: [Sketch] The proof is an application of the Contraction Mapping Principle [9]. We consider the space of functions.

$$\mathbb{M} = \big\{\mathbf{y} \in C^0\big([-r,\infty),\mathbb{R}^{N+1}\big) : y_i(t) = \phi_i(t)|_{t\in I_0}^{i\in[N]},$$
$$y_0 = \lambda_0, \max_i |y_i(t) - \lambda_0| \leq \Xi,\ t \geq 0,\ y_i(t) \to \lambda_0\big\}$$

that together with the metric $\rho(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}) = \sup_{t\geq 0}\max_i |y_i^{(1)}(t) - y_i^{(2)}(t)|$ constitute a complete metric space [9]. Observe that (16) can be rewritten as

$$\overline{\boldsymbol{\lambda}}_t(\theta) = [\tilde{T}(t,0)\boldsymbol{\phi}](\theta) + \mathbf{1}\lambda_0 + \int_0^t [\tilde{T}(t,s)Y_0](\theta)\overline{\mathbf{f}}(s,\boldsymbol{\lambda}_s)\,ds \tag{17}$$

Consequently we define the operator

$$(\mathcal{P}\mathbf{y})_t(\theta) = \begin{cases} \boldsymbol{\phi}(\theta), & t = 0 \\ \mathbf{y}_t(\theta)_{(17)}, & t \geq 0 \end{cases}$$

where $\mathbf{y}_t(\theta)_{(17)}$ stands for the right-hand side of (17). The first step is to show that for any $\mathbf{y} \in \mathbb{M}$ we have $\mathcal{P}\mathbf{y} \in \mathbb{M}$: Indeed, $\mathcal{P}\mathbf{y}$ coincides with $(\lambda_0, \boldsymbol{\phi})$ in I_0 and as $t \to \infty$

$$(\mathcal{P}\mathbf{y})_t(\theta) \to \mathbf{1}\lambda_0$$

in view of (15), the asymptotic behavior of $\varepsilon_i(t) \to 0$, $\zeta_i(t,\mathbf{y}) \to 0$ and consequently $\overline{\mathbf{f}}(t,\mathbf{y}_t) \to 0$, then the integral in (17) vanishes as the convolution of an L^1 function with a function that goes to zero. Finally, careful calculations of the estimates on ϵ_i, ζ_i provided in Proposition 5.1 with the convolution integral yield that condition in Assumption 5.3 is sufficient to prove that $\mathcal{P}$ maps $\mathbb{M}$ to itself and it is also a contraction under the metric ρ. Then the Contraction Mapping Principle ensure the existence of a unique fixed point in $\mathbb{M}$, i.e. a solution that solves the EDP with the operation constraints. ■

The purpose of the simulation example that follows is to illustrate Theorem 5.4 and to show that for large delays σ_{ij} our algorithm becomes unstable.

VI. A SIMULATION EXAMPLE

A network of $N = 5$ agents each of which controls a portion of the load and a portion of the generated power are interconnected and the objective is the decentralized solution of the EDP. The simulations were run with the `ddesd` routine of `MATLAB` and Euler approximations with $dt = 0.01$ sec.

a) The A-network and its delays.: The network is represented through the adjacency matrix

$$A = a(t)\begin{bmatrix} 0 & 1.5 & 0 & 0 & 0 \\ 1.3 & 0 & 0.9 & 0.8 & 0 \\ 0 & 1.3 & 0 & 1.6 & 0 \\ 0 & 0.7 & 0.1 & 0 & 0.8 \\ 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}$$

where $a(t) = 50\big(1 + \sin(t)\big)$. The delays with respect to λ and P_{gen} respectively:

$$T = 0.1\begin{bmatrix} 0&1&0&0&0\\1&0&1&1&0\\0&1&0&1&0\\0&1&1&0&1\\0&0&0&1&0 \end{bmatrix}, \Sigma = \sigma(t)\begin{bmatrix} 0&1&2&2&3\\1&0&1&1&2\\2&1&0&1&2\\2&1&1&0&1\\3&2&2&1&0 \end{bmatrix},$$

for some smooth function $\sigma(t) \in [\underline{\sigma}, \overline{\sigma}]$. The learning delays $\kappa^{(i)}$ are taken identical to these of Σ but with $\sigma(t) = 1 + 0.5\sin(t)$. The network load is $P_{load} = 2000MW$ and the cost functions are in the form (3).

Sensor	χ_i	ψ_i	ω_i	Load (MW)
1	0.003124	7.92	561	200
2	0.00388	7.85	310	500
3	0.00964	6.2	78	400
4	0.01752	3.2	150	300
5	0.0027	7.97	358	600

TABLE 1. The control loads and the properties of the cost functions F_i of the network in the simulation example. For instance sensor 1, controls 200 MW of the overall load and generates P_1 with fuel cost function $F_1(P_1) = 561 + 7.92P_1 + 0.001562(P_1)^2$.

Table 1 enlists all the load and cost data that each sensor of the network supervises. We assume that each generator works within $[\underline{P}, \overline{P}] = [10, 500]$ as (4) requires. The optimal operation point $\lambda = \lambda_\infty$ is explicitly calculated:

$$\lambda_\infty = \frac{2000 + \sum_{l=1}^{5}\frac{\psi_l}{\chi_l}}{\sum_{l=1}^{5}\frac{1}{\chi_l}} = 9.32\$/MWh$$

and the generators produce the optimal powers (in MW): $P_1^\infty = 448.1503$, $P_2^\infty = 378.8715$, $P_3^\infty = 323.6537$, $P_4^\infty = 349.3163$ and $P_5^\infty = 500.0080$.

b) Decentralized load learning : Figure 1 depicts the learning process of the overall network load, where each sensor acts both as a leader to the rest of the A-network and a follower.

Under Remark 5.2 we numerically calculate

$$|P_{C_L}^{(i)} - 2000| \leq 1800e^{-0.09t}$$

or $L = 1800$ and $\underline{l} = 0.09$ of (6).

c) Leader Follower Model: Given the cost parameters we calculate $\Xi = 1.1748$ and we appropriately chose initial values for ϕ_i so that $E = 0.001$ and $\underline{e} = 12$ of (15).

d) Power generators.: Under the Remark 5.2 we set $\underline{\gamma} = 3$ and $\overline{\gamma} = 13$. Consequently if $\overline{\sigma}$ is small enough, Assumption 5.3 is calculated to be correct. As $\overline{\sigma}$ increases, it is not guaranteed that the system will operate within the bounds and as $\underline{\sigma}$ increases even more it is not guaranteed that the system will be stable. Indeed in Figs. 2 (a) and (b) we have small $\sigma(t) = 0.2(1 + \sin(t))$ that satisfies (5.3) and solve the problem in terms of λ and P_i respectively. In Figs. 3 (a) and (b), we have taken $\sigma(t) = 1.5 - 0.5\sin(t)$ large enough that does not satisfy 5.3 and we see instability of the algorithm.

VII. DISCUSSION

We developed a rigorous framework for the solution of the EDP in a time varying network with multiple communication delays and a network scheme that is compatible with the modern SG architectures. The dynamics algorithm is motivated by a consensus model with delays and turns out to

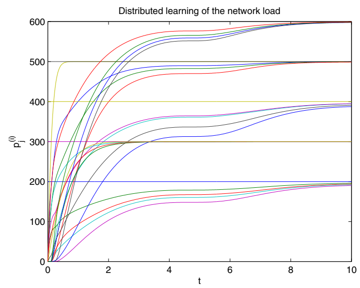

Fig. 1: Parallel learning of the example. Each sensor that controls part of P_L, acts as a leader in a leader-follower scheme under the A network and the prescribed delays. Adding each sensor's learning variable we obtain the quantity in (5).

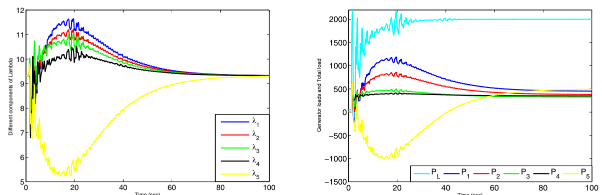

Fig. 2: Small σ delays that satisfy Assumption 5.3. The distributied solution of the EDP in terms of λ_i (a) and P_i (b).

a system of functional integro-differential equations. In this work, the delay on the generated power P_i is put under study. We demonstrated by theory and simulation that large delays may drive units out of bounds or even destabilize them.

There are a number of important parameters that are yet to be examined. We took into consideration multiple different types of delays but we suppressed most of them in order to study the effect of one of them (i.e. σ_{ij}). Simulations suggest that there are still parameter areas to be explored (such as w_i, κ_i, γ_j etc). Among the advantages of the followed method, it is the fact that it can handle general linear non-autonomous systems. The price to pay is conservative estimates (like Assumption 5.3). Moreover rate estimates from [15], although explicit, these are very conservative. Sharper rate estimates and stability conditions require frequency-based methods. In this case, however only networks with linear and constant parameters should be considered.

The main disadvantage of the modeling approach is that it is assumed that each sensor attains a delayed image of the generated power of every other sensor in the network. This is hardly a decentralized feature of the network dynamics. It is however critical for the performance of the algorithm. The sensor must somehow have information of the overall produced power so that it will know how much power to contribute from the unit it controls. The design of information networks that distribute critical global dynamically updated quantities is a challenging future research question.

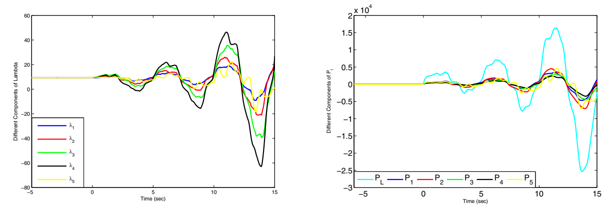

Fig. 3: Large σ delays that do not satisfy Assumption 5.3. The EDP in terms of λ_i (a) and P_i (b).

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