Bipartite Consensus for Global Trust in Social Network Services

Peixin Gao
Electrical & Computer Eng and
Institute for Systems Research
University of Maryland
College Park, Maryland 20742
Email: gaopei@umd.edu

Zhixin Liu
Key Laboratory of Systems & Control
Academy of Mathematics & Systems Science
Chinese Academy of Sciences
Beijing, P. R. China
Email: lzx@amss.ac.cn

John S. Baras
Electrical & Computer Eng and
Institute for Systems Research
University of Maryland
College Park, Maryland 20742
Email: baras@umd.edu

***Abstract*—As social network services (SNS) are gaining popularity, establishing trust relationships among users in SNS is necessary for user interaction and decision making. However, the evaluation of global trust values (i.e. reputation) of users suffers from the curse of opinion divergence in the network. In this paper, we consider the scenario of opinion divergence within social networks, and propose a method to reach different global trust values between groups of users with controversial opinions in the network. We model the global trust opinion formation in discrete-time dynamics and introduce bipartite consensus as the approach to establish global trust in such circumstances. Such approach works upon the property of structural balance, and can be extended to more general situations where eventual positivity applies. Via our approach, non-trivial global trust can be reached within the network, which can guide users' social behaviors, and support many SNS-based applications.**

I. INTRODUCTION

Along with the fast development of the Internet and IT technologies, social network services (SNS), like Facebook and Twitter, are growing rapidly with tremendous popularity [1]. There are a large amount of interactions among users in SNS, and the social features are playing an increasingly important role in the interactions. Among all social features, trust relationship among users is seen as a critical component, as it provides a guideline for users to interact and make decisions [2]. Trust is fundamental in many SNS-based applications, e.g. social recommendation and personalized services [3]–[5].

The concept of trust has been widely applied in many different domains [5]–[8]. In SNS scenario, trust relationships are usually directed between users, and can be interpreted as a compound of integrity, preference/taste similarity and social closeness. Based on the interactions within SNS, there is an associated network of trust, where users form trust opinions about others based on direct interactions. Within SNS, the concept of trust has two levels, namely local and global level. Local trust refers to the subject trust opinions of each user about her neighbors in the network, while global trust is a combination of local trust opinions gathered from users within the network, and is also called as reputation in many cases. The value of global trust is considered as a measure of the credibility and homophily within the population. As a more objective measure compared to local trust, global trust is very important in network security, decision making processes as well as improving the quality of SNS-based applications.

There has been significant work on trust metrics and inference for local trust evaluation, for example [5], [8]–[12]. However, the research on evaluation of global trust is limited and is still at its early stage [13]. A classic approach is via aggregation of local trust opinions of users in the network, e.g. the ratings of users on Ebay. This scheme works when users in the network are homophily and malicious users do not exist [14]. When considering a more general situation where users may have distrust relationships due to controversial opinions or existence of malicious users, the aggregation of users' local trust opinions for global trust would suffer from the curse of opinion divergence in the network. Simple average over population with controversial opinions can hardly offer meaningful insights about true reputation. The case of two-party political system would be a good example. It's a community with two sub-communities that have similar opinions within the group but controversial ones between groups. The true reputation (or global trust) of each person cannot be reached by simple combination of opinions from both groups.

Recent research shows the limitation of global trust evaluation when considering distrust (i.e. negative trust). Kamvar et al. proposed to use EigenTrust [7], a PageRank-like algorithm to evaluate global trust of peers in P2P network. The issue of malicious nodes is addressed by computing trust value on other peers and majority voting. Li and Wang [13] proposed a subjective probability based approach for global trust evaluation in service-oriented computing (SOC). However, distrust is not well-modeled in both approaches, and the clustering of controversial opinions within the network cannot be captured. This renders both approaches ineffective for global trust evaluation in SNS scenarios. In [14], DeFigueiredo et al. discussed trust between users in online interactions, and they concluded that applying single universal trust ratings (i.e. global trust) is vulnerable to manipulation by malicious users.

The problem of consensus in signed network has attracted increasing interest in research community [15]–[18]. Shi et al. [15] studied opinion dynamics over signed social graphs, where phase transition phenomena is discussed. In [16]–[18], the authors discussed bipartite consensus in networks with

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antagonistic interactions (signed networks), where consensus can be reached separately in each antagonistic group.

In this work, we propose to solve the problem of global trust evaluation in social network services (SNS). Specifically, we consider a general case where both trust and and distrust relationships exist within the social network, and a trust network can be established on the trust and distrust relationships.

The trust network is modeled as a directed signed graph where both trust and distrust relationships are considered. The direction of an edge in the graph represents a directed trust relationship, with the signed weight of the edge implying the level of trust (distrust) associated with the edge. we formulate the problem of evaluation of global trust in SNS as an bipartite consensus problem within the corresponding trust network. First we consider bipartite consensus for global trust in a structurally balanced trust network; we further extend the results to a more general case with the notion of eventual positivity. User reputation and grouping information based on global trust opinions is available for decision making in social network environment and can be used to boost social-aware applications, for example trust-aware social recommendations.

Our contributions in this paper are three-fold. (1) In evaluation of global trust in SNS, we consider a more general scenario with the existence of distrust relationships, and interpret the differences of global trust (reputation) within two groups as the result of controversial trust opinions between the two. (2) To the best of our knowledge, we are the first to introduce the approach of bipartite consensus in global trust evaluation under SNS settings. (3) We consider applying the bipartite global trust results reached via our approach into SNS-based applications, in order to improve the quality of the services.

The rest of this paper is arranged as follows. In Sec. II, we discuss some preliminaries and propose our formulation of global trust evaluation in SNS settings as a bipartite consensus problem in a discrete-time system, considering the existence of distrust relationships in SNS (Sec. II). In Sec. III, we consider reaching bipartite consensus for global trust in SNS under structural balance, and extend the results to a more general case. We discuss the integration of global trust in SNS-based applications in Sec. IV. We conclude our work in Sec. V and highlight the future research directions.

II. PRELIMINARIES AND FORMULATION

A. Notations and Terminologies

All vectors are column vectors and denoted by lower case letters, and matrices are denoted by upper case letters. For a square matrix M, M^T denotes its transpose and M^k denotes its k-th power, and the ij-entry of M is denoted as M_{ij}.

1) Graph theory: In this work, we utilize the concept of weighted signed graphs.

Definition 1. A *weighted signed graph* $\mathcal{G}$ can be denoted by a triple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V}$ is a finite set of vertices (nodes), $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} \in \mathbb{R}^{n\times n}$ is the adjacency matrix of the signed weights of the edges in graph $\mathcal{G}$. For $\mathcal{A}$: $\mathcal{A}_{ij} \neq 0 \Leftrightarrow (v_i, v_j) \in \mathcal{E}$.

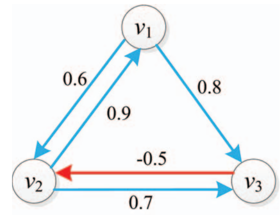

Fig. 1. An example of weighted signed digraph

For simplicity, we use $\mathcal{G}(\mathcal{A})$ to represent a weighted signed graph with adjacency matrix $\mathcal{A}$. We call a graph undirected if the order of the nodes is irrelevant in representing edges, and the matrix is symmetric $\mathcal{A} = \mathcal{A}^T$. For a directed graph (i.e. digraph), the edge $(v_i, v_j) \in \mathcal{E}$ is directed, where v_i is the tail and v_j is the head of the edge. In a digraph, the pair of edges between the same nodes is called a digon. A digraph is digon sign-symmetric if $A_{ij}A_{ji} \geq 0$ for all $i, j \in \{1, \cdots, n\}, i \neq j$.

Fig. 1 is a weighted signed digraph with $\mathcal{V} = \{v_1, v_2, v_3\}$, $\mathcal{E} = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3), (v_3, v_2)\}$, and

$$\mathcal{A} = \begin{bmatrix} 0 & 0.6 & 0.8 \\ 0.9 & 0 & 0.7 \\ 0 & -0.5 & 0 \end{bmatrix}$$

A (directed) path of length k is a sequence of distinct nodes, $\{v_1, v_2 \cdots v_{k+1}\}$, such that $\{v_m, v_{m+1}\} \in \mathcal{E}, m \in \{1, \cdots, k\}$. A (directed) cycle is a path beginning and ending with the same node. We say that the graph is strongly connected if for any $v_i, v_j \in \mathcal{V}$, there exists a path $\{v_i, \cdots, v_j\}$ in $\mathcal{G}$.

We say matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is *nonnegative* ($\mathcal{A} \geq 0$) by meaning $\mathcal{A}_{ij} \geq 0$ for $\forall i, j \in \{1, \cdots, n\}$, and $\mathcal{A} \neq 0$; we say $\mathcal{A}$ is positive ($\mathcal{A} > 0$) when $\mathcal{A}_{ij} > 0$ for $\forall\ i, j \in \{1, \cdots, n\}$.

$sp(\mathcal{A}) = \{\lambda_1(\mathcal{A}), \cdots, \lambda_n(\mathcal{A})\}$ denotes the spectrum of $\mathcal{A}$, where $\lambda_i(\mathcal{A}), i \in \{1, \cdots, n\}$ are the eigenvalues of matrix $\mathcal{A}$. The spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is the smallest real positive number such that $\rho(\mathcal{A}) \geq |\lambda_i(\mathcal{A})|, \forall\ i \in \{1, \cdots, n\}$.

2) Structural balance: Based on [19], structural balance of a signed network can be defined as follows.

Definition 2. A signed network is *structurally balanced* if it admits a bipartition of $\mathcal{V}$ into $\mathcal{V}_1, \mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, and $\mathcal{V}_1 \cap \mathcal{V}_2 = 0$, such that $\mathcal{A}_{ij} \geq 0$, for $v_i, v_j \in \mathcal{V}_m$, and $\mathcal{A}_{ij} \leq 0$, for $v_i \in \mathcal{V}_m$ and $v_j \in \mathcal{V}_n$, $m \neq n$, $m, n \in \{1, 2\}$.

The sign of a cycle is the product of all edges' signs; a (directed) cycle is positive if it contains even number of negative edges, and is negative otherwise. It can be shown that a signed graph is structurally balanced iff all the cycles in the graph are positive [16], [19]. As an extension, a signed digraph is structurally balanced iff all directed cycles are positive [16].

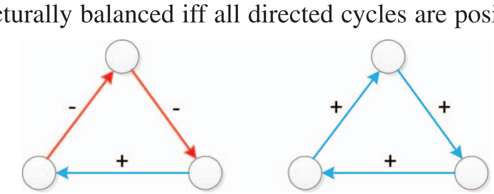

Fig. 2. Structural balance

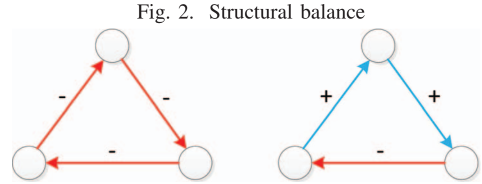

Fig. 3. Structural unbalance

3) Perron-Frobenius property and eventual positivity: Following [20], we give a definition of Perron-Frobenius property:

Definition 3. A real square matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is said to have *Perron-Frobenius property* if $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ (i.e. Perron-Frobenius eigenvalue), with corresponding nonnegative left and right eigenvectors. When $\mathcal{A}$ nonnegative and regular,

- $\rho(\mathcal{A})$ is a simple positive eigenvalue.
- The eigenvector corresponding to $\rho(\mathcal{A})$ can be chosen to be positive (called a Perron vector).
- No other eigenvalue has the same modulus, i.e. for any other λ an eigenvector of $\mathcal{A}$, $\lambda < \rho(\mathcal{A})$.

We use PF_n to denote the set of matrices in $\mathbb{R}^{n\times n}$ satisfying the Perron-Frobenius property.

Definition 4. A real square matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is *eventually positive* if $\exists\ k_0 \in \mathbb{N}_+$ such that $\mathcal{A}^k > 0$, $\forall\ k \geq k_0,\ k \in \mathbb{N}_+$.

The smallest k_0 in Definition 4 is called the power index of $\mathcal{A}$. We follow the notation in [17], [21] and denote eventually positive matrices as $\mathcal{A} \overset{\vee}{>} 0$.

Although Perron-Frobenius property is defined on nonnegative square matrices, it is show in [17], [22] that eventually positive matrices, as well as their transpose, also have the Perron-Frobenius property. As discussed in [17], the Perron-Frobenius property can be used for adjacency matrices with eventual positivity to evaluate the formation of unanimous opinions, by introducing the concept of holdability [23]:

Definition 5. For a discrete-time dynamical system with $x_0 \in \mathbb{R}^n$ as the initial state, a set $S \subset \mathbb{R}^n$ is *holdable* if for $\forall\ x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} dist(x(k), S) = 0$, where $dist(x(k), S) = \inf_{y\in S} \|x(t) - y\|$ with a certain norm in $\mathbb{R}^n$, and $\exists k_o \in \mathbb{N}_+$ such that $x(k) \in S$, for $\forall k \geq k_0$.

B. Problem Statement

In SNS scenarios, trust relationships among users reflect preference (fondness), social closeness (subjective similarity) and integrity of users in the network. Based on the trust relationships between users in the social network, there is an associated trust network. Referring to previous work [5], [10], we define trust network in SNS setting as follows:

Definition 6. A *trust network* in SNS is a directed weighted signed graph $\mathcal{G}_T(\mathcal{V}, \mathcal{E}, \mathcal{A})$ established via social interaction, where $\mathcal{V}$ is the set of nodes (i.e. users), $\mathcal{E}$ is the set of directed edges (i.e. trust links), and $\mathcal{A}$ the matrix of signed edge weight. $e_{ij} = (v_i, v_j) \in \mathcal{E}, v_i, v_j \in \mathcal{V}$, is a directed trust link from node v_i to v_j, and its value (weight) is an entry $\mathcal{A}_{ij}$ of $\mathcal{A}$. $N_i = \{v_j | e_{ij} \in \mathcal{E}\}$ is the neighbor set of node v_i.

Remark: Here $\mathcal{A}_{ij} \in [-1, 1]$ indicates the extent of trust that v_i has on v_j. $\mathcal{A}_{ij} = 1$ means v_i "totally agrees with"or "likes"v_j, while $\mathcal{A}_{ij} = -1$ means v_i "totally disagrees with"or "dislikes"v_j. Weights are not necessarily symmetric.

Global trust evaluation in SNS is typically very challenging. Due to lack of a central authority, different agents may provide different or even contradictory information in the evaluation process, and a unanimous global trust opinion is difficult to compute. In order to tackle the challenge and obtain meaningful results for global trust in SNS settings, we consider reaching opposite global trust opinions between groups of controversial opinions. We introduce the idea of bipartite consensus in signed networks and formulate our problem of global trust evaluation in SNS as a bipartite consensus problem over the signed trust network. In this formulation, opinion divergence between two antagonistic groups is well-handled.

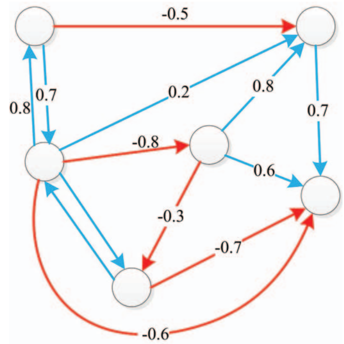

Fig. 4. An example of trust network

Consider a strongly connected trust network $\mathcal{G}_T(\mathcal{V}, \mathcal{E}, \mathcal{A})$. We assume that the bipartite consensus for global trust is evaluated via distributed opinion update scheme on $\mathcal{G}_T$ through a discrete-time linear dynamic system:

$$x(k+1) = W(k)x(k) \tag{1}$$

where vector $x(k) = (x_1(k), \cdots, x_n(k))$ is the temporary trust opinion held by nodes in $\mathcal{V}$ at time k. The update matrix $W(k)$ can be decomposed into 2 parts:

$$W(k) = \Sigma(k) + F(k) \tag{2}$$

where matrix $F(k)$ at time k is the off-diagonal matrix used in integrating opinions from neighbors, and $\Sigma(k)$ is a diagonal one describing the influence that users put on themselves. A starting point for $F(k)$ would be a static matrix in accordance with $\mathcal{A}$ of the trust graph $\mathcal{G}_T(\mathcal{A})$, and $\Sigma(k)$ can be arranged correspondingly such that users put a trust value of 1 on themselves. Note that in order to reach consensus on global trust, $W(k) = W$ is normalized in the system (1) as follows:

$$w_{ij} = \frac{\mathcal{A}_{ij}}{1 + \sum_{j\in adj(i)} |\mathcal{A}_{ij}|} \tag{3}$$

such that $\sum_{j\in\mathcal{V}} |w_{ij}| = 1$ for any $i \in \{1, \cdots, n\}$.

The monotonicity of (1) can be defined on a partial orthant order [24]. A partial orthant order in $\mathbb{R}^n$ is a vector:

$$\sigma = [\sigma_1, \cdots \sigma_n]^T, \qquad \sigma_i \in \{1, -1\}, \forall i \in \{1, \cdots, n\}$$

where $\sigma_i = 1$ denotes the natural order, and $\sigma_i = -1$ denotes the opposite. Corresponding to σ, we define the matrix $D_\sigma = D_\sigma^T = diag(\sigma) \in \mathbb{R}^{n\times n}$ as the gauge transformation matrix [16], [18]. D_σ can be used to define an orthant $K_\sigma = \{x \in \mathbb{R}^n | D_\sigma x \geq 0\}$. The partial order σ can be indicated by $\leq_\sigma$:

$$x_1 \leq_\sigma x_2 \Leftrightarrow x_1 - x_2 \in K_\sigma$$

The discrete-time system (1) is called monotone w.r.t σ if for any initial conditions $x_1(0), x_2(0)$ s.t. $x_1(0) \leq_\sigma x_2(0)$,

$$x_1(t) \leq_\sigma x_2(t) \qquad \forall t > 0 \tag{4}$$

The monotonicity of the system (1) can be verified using the off-diagonal matrix F; the system is monotone w.r.t σ iff:

$$\sigma_i\sigma_j F_{ij} \geq 0 \qquad \forall i, j \in 1, \cdots, n, i \neq j \tag{5}$$

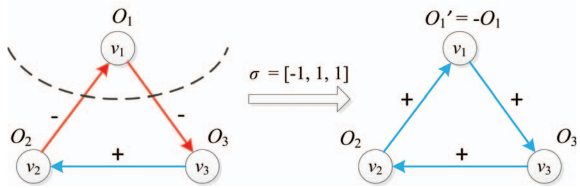

Fig. 5. Gauge transformation

A gauge transformation matrix can be applied to the adjacency matrix of a trust graph to change the sign of the edges as well as opinions held by nodes in the graph. As an example, Fig. 5 shows applying gauge transformation to a signed trust digraph. For the trust network on the left, its adjacency matrix has negative weighted entries $\mathcal{A}_{21}$ and $\mathcal{A}_{13}$. After a gauge transformation with $D = diag(-1, 1, 1)$, all the entries in the gauge transformed matrix $\mathcal{A}' = D\mathcal{A}D$ are positive. The system is monotone w.r.t $\sigma = [-1, 1, 1]$. Accordingly, the opinion held by v_1 changes from O_1 to $O_1' = -O_1$.

III. MAIN RESULTS

In order to evaluate the formation of global trust opinions within a trust network of controversial opinions, we formulate it as a bipartite consensus problem with the goal of reaching global trust. We start from the case where the trust network is of structural balance, and extend the result to a trust network of approximate structural balance in the next section.

Remark: By bipartite consensus for global trust, we mean that the global trust of users have the following convergence result:

$$\lim_{k\to\infty} |x_i(k)| = c > 0 \qquad \forall i \in \{1, \cdots, n\} \tag{6}$$

Additionally, if the final state is

$$c = \frac{1}{n}|\omega^T x(0)| \tag{7}$$

for some constant weight vector ω, then we say that all the agents in the network reach bipartite consensus.

A. Global Trust Evaluation in Structurally Balanced Networks

A structurally balanced signed social network, as discussed in Sec. II, can be partitioned into two disjoint antagonistic groups, where each group contains only friends, while any two individuals from different groups are adversaries. The dynamics of opinion forming in structurally balanced communities obeys monotone dynamics.

Given the trust network $\mathcal{G}_T(\mathcal{A})$, which is a digraph, We have the following lemma for structural balance.

Lemma III.1. For the trust network $\mathcal{G}_T(\mathcal{A})$ that is strongly connected and digon sign-symmetric, it is structurally balanced if and only if either of the following holds:

1) All directed cycles of $\mathcal{G}_T(\mathcal{A})$ are positive;
2) $\exists$ a gauge transformation matrix $D \in \mathcal{D}$, such that the adjacency matrix $\mathcal{A}' = D\mathcal{A}D$ is nonnegative;

Proof: 1) This comes from the definition of structurally balanced network as discussed in Sec. II.

2) From Definition 2, for the node set $\mathcal{V}$ of graph $\mathcal{G}_T$ which is structurally balanced, it can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ such that all and only the negative edges connect nodes in $\mathcal{V}_1$ and $\mathcal{V}_2$. If we choose a partial orthant order $\sigma = [\sigma_1, \cdots, \sigma_n]$, where $\sigma_i = 1$ when $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ if $v_i \in \mathcal{V}_2$, then through the gauge transform matrix $D_\sigma = diag(\sigma)$, the adjacency matrix $\mathcal{A}$ would satisfy that $\mathcal{A}' = D_\sigma \mathcal{A} D_\sigma$ is nonnegative. For the sufficient condition, it can be proved via contradiction. Suppose there doesn't exist such orthant order σ and corresponding matrix $D_\sigma = diag(\sigma)$ such that $D\mathcal{A}D$ is nonnegative, then for any bipartition of $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, from the proof of the necessary condition, $\mathcal{V}_1$ and $\mathcal{V}_2$ can not satisfy the condition that all and only negative edges connect the two set, which is equivalent to structural unbalance. ■

Based on Lemma III.1, we have the following theorem:

Theorem III.2. For the strongly connected trust network $\mathcal{G}_T(\mathcal{A})$ that is directed and digon sign-symmetric, the discrete-time system in (1) can reach a bipartite consensus on global trust, if $\mathcal{G}_T(\mathcal{A})$ is structurally balanced.

Proof: From Lemma III.1, we know that for $\mathcal{G}_T(\mathcal{A})$ that is strongly connected and digon sign-symmetric, there exists a gauge transformation matrix $D = diag(\sigma)$, such that $\mathcal{A}' = D\mathcal{A}D \geq 0$. If in system (1), we choose the off-diagonal matrix $F(k) = F = C\mathcal{A}$, and the diagonal matrix $\Sigma(k) = \Sigma = C\mathbb{I} = C$, where $C = diag(c_1, \cdots, c_n)$ is the normalization matrix with $c_i = 1/(1 + \sum_{j\in adj(i)} |\mathcal{A}_{ij}|) \geq 0, \forall i \in \{1, \cdots, n\}$, then for the same D, we have:

$$W' = DWD = D(C + C\mathcal{A})D = C + C\mathcal{A}' \geq 0 \tag{8}$$

The solution $y^* = D_t x^*$ would be the result of a usual consensus problem over a strongly connected unsigned graph, where the unsigned graph is the trust network $\mathcal{G}_T(\mathcal{A})$ after the gauge transformation on $\mathcal{A}$. After reaching the consensus $y^* = \lim_{k\to\infty} y(k) = \mathbf{c} = [c, \cdots, c] \in \mathbb{R}^n$, the bipartite consensus on the original graph $\mathcal{G}_T(\mathcal{A})$ can be evaluated using

$$x^* = D^{-1}y^* = Dy^* \tag{9}$$

which is the bipartite global trust reached by the users in the trust network with controversial opinions.

Based on the bipartite consensus result,the node set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1 = \{v_i \in \mathcal{V} | x_i^* = c, i \in \{1, \cdots, n\}\}$ and $\mathcal{V}_2 = \{v_j \in \mathcal{V} | x_j^* = -c, j \in \{1, \cdots, n\}\}$, which is the corresponding clustering due to structural balance. ■

Remark: As mentioned above, the bipartite consensus solution is in the form of Eq. (6).

Corollary 1. *As the consensus result,* $\lim_{k\to\infty} x(k) = \nu^T Dx(0)D\mathbf{1}$, *where D is the gauge transformation s.t. $D\mathcal{A}D$ nonnegative, and ν is the normalized nonzero left eigenvector of $W = DWD$ s.t. $\nu^T\mathbf{1} = 1$.*

Proof: From Theorem III.2, y^*, a standard consensus can be reached on the gauge transformed graph $\mathcal{G}(D\mathcal{A}D)$. According to [25], $y^* = \lim_{k\to\infty} y(k) = \nu^T y(0)\mathbf{1}$, ν is the normalized nonzero left eigenvector of DLD s.t. $\nu^T\mathbf{1} = 1$. Thus $x^* = D^{-1}y^* = Dy^* = (\nu^T Dx(0))D\mathbf{1}$. ■

Similarly, when $\mathcal{G}_T(\mathcal{A})$ is weight balanced, the consensus result would be $\lim_{k\to\infty} x(k) = (1/n)\mathbf{1}^T Dx(0)D\mathbf{1}$.

B. Global Trust Evaluation in Networks of Eventual Positivity

In real cases, structural balance can rarely be satisfied. By combining eventual positivity with the gauge transformation used in Sec. III-A, the approach can be extended to cases where structural balance property is not satisfied.

Definition 7. Matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ has the *signed Perron-Frobenius property* [17] if the following are satisfied:

1) Spectral radius $\rho(\mathcal{A})$ is a real positive eigenvalue of $\mathcal{A}$
2) $\rho(\mathcal{A}) > \lambda,\ \forall\lambda \in sp(\mathcal{A}), \lambda \neq \rho(\mathcal{A})$
3) All elements in v_r, the right eigenvector corresponding to $\rho(\mathcal{A})$, are nonzero, i.e. $v_{r,i} \neq 0, \forall i \in \{1, \cdots, n\}$

We denote the set of matrices of signed Perron-Frobenius property as SPF_n, and have the following proposition:

Proposition 1. *For the matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ of a directed trust graph $\mathcal{G}_T(\mathcal{A})$, $\exists$ an orthant order σ and $D = diag(\sigma)$ s.t. $D\mathcal{A}D \overset{\vee}{>} 0$, iff $\mathcal{A} \in SPF_n$, $\mathcal{A}^T \in SPF_n$, and the left eigenvector v_l and the right eigenvector v_r of $\mathcal{A}$ satisfies:*

1) $v_{l,i}v_{r,i} > 0,\ \forall i \in \{1, \cdots, n\}$*, or*
2) $v_{l,i}v_{r,i} < 0,\ \forall i \in \{1, \cdots, n\}$

Proof: If there exists a gauge transformation such that $\mathcal{A}' = D\mathcal{A}D \overset{\vee}{>} 0$, then from Sec. II-A3 and [17], [22], we know that $\mathcal{A} \in PF_n, \mathcal{A}^T \in PF_n$, and therefore $\rho(\mathcal{A}') > 0$ and strict larger than all other eigenvalues. As $sp(\mathcal{A}) = sp(\mathcal{A}'), sp(\mathcal{A}^T) = sp(\mathcal{A}')$ according to [16], thus $\rho(\mathcal{A}) = \rho(\mathcal{A}')$ is positive and such that $\rho(\mathcal{A}) > \lambda,\ \forall\lambda \in sp(\mathcal{A}), \lambda \neq \rho(\mathcal{A})$. From Definition 3, we can find positive v'_r and v'_l the right and left eigenvectors (Perron vectors), such that:

$$D\mathcal{A}Dv'_r = \mathcal{A}'v'_r = \rho(\mathcal{A}')(v'_r)$$
$$(v'_l)^T D\mathcal{A}D = (v'_l)^T\mathcal{A}' = \rho(\mathcal{A}')(v'_l)^T$$

as $D^2 = \mathbb{I}$ and $\rho(A) = \rho(\mathcal{A}')$,

$$\mathcal{A}v_r = \rho(\mathcal{A})v_r, \quad v_l^T\mathcal{A} = \rho(\mathcal{A})v_l^T$$

where $v_l = Dv'_l$, and $v_r = Dv'_r$.

Obviously both v_l and v_r have no elements of 0. Thus from the Definition 7 about signed Perron-Frobenius property, we know that $\mathcal{A} \in SPF_n,\ \mathcal{A}^T \in SPF_n$. Note that $v_{l,i}v_{r,i} = (D_{ii})^2 v'_{l,i}v'_{r,i} = v'_{l,i}v'_{r,i}$. If we choose both v'_l and v'_r to be positive, the condition (1) is satisfied. Similarly, if choose one of the two vectors to be negative (multiply it by -1), and the other to be positive, then the second condition can be satisfied.

The sufficient condition can be proved in a similar way. ■

Lemma III.3. Consider the system (1), where the normalized weight matrix $W = \Sigma + F$ with diagonal matrix $\Sigma = diag(c_1, \cdots, c_n)$ and off-diagonal matrix F. If $\exists\ d \geq 0$ s.t. $F+D \overset{\vee}{>} 0$ with $D = \Sigma - d\mathbb{I}$, then system (1) holds to $\mathbb{R}_{\{-,+\}}$.

Proof: Let $B = F + D$, then W can be written as:

$$W = \Sigma + F = d\mathbb{I} + D + F = d\mathbb{I} + B$$

Since $B = F + D \overset{\vee}{>} 0$, then from Proposition 1 and [22], $B, B^T \in SPF_n$, and $\rho(B)$ is a positive real eigenvalue of B that strictly larger than other eigenvalues. Let v_l and v_r be left and right eigenvectors of B, then $d > 0$ implies that W must have $d + \rho(B) \in \mathbb{R}_+$ as the largest eigenvalue, and v_l and v_r as left and right eigenvectors, as

$$v_l^T W = v_l^T d + v_l^T B = (d + \rho(B))v_l^T$$
$$Wv_r = dv_r + Bv_r = (d + \rho(B))v_r$$

Therefore, $W, W^T \in PF_n$ and $W \overset{\vee}{>} 0$, and we have:

$$x^* = \lim_{k\to\infty} x(k) = \frac{v_l^T x_0 v_r}{v_l^T v_r} \tag{10}$$

If $v_l^T x_0 > 0$ then $x^* \in int(\mathbb{R}^n_+) \cup \emptyset$, similarly $v_l^T x_0 < 0$ leads to $x^* \in int(\mathbb{R}^n_-) \cup \emptyset$. From Definition 5 about holdability, it can be shown that the system (1) holds to $\mathbb{R}^n_{\{-,+\}}$. ■

Based on Proposition 1 and Lemma III.3, we have the following theorem for a relaxed bipartite consensus on global trust within a trust network of eventual positivity:

Theorem III.4. For the strongly connected trust network $\mathcal{G}_T(\mathcal{A})$ that is described by system (1), if there exists $d \geq 0$ such that proposition 1 holds for $F + D$, where $D = \Sigma - d\mathbb{I}$, then the system (1) holds to the orthant pair $\mathbb{R}^n_{\{-\sigma,+\sigma\}}$.

Proof: For the system (1) that describes the trust dynamics of $\mathcal{G}_T(\mathcal{A})$, we have the normalized weight matrix $W = \Sigma + F$ with diagonal matrix $\Sigma = C\mathbb{I} = C$ and off-diagonal $F = C\mathcal{A}$, where $C = diag(c_1, \cdots, c_n)$ is the normalization matrix with $c_i = 1/(1 + \sum_{j\in adj(i)} |A_{ij}|) \geq 0, \forall i \in \{1, \cdots, n\}$.

Let $B = F + D$, then $W = d\mathbb{I} + B$, $B, B^T \in SPF_n$, with v_l and v_r the left and right eigenvectors of identical signs. From Proposition 1, we know that there exists an orthant order σ and a corresponding gauge transformation $D = diag(\sigma) \in \mathcal{D}$ s.t. $B' = D\mathcal{B}D \overset{\vee}{>} 0$. Let $y = Dx$, then the system w.r.t. y is:

$$y(k+1) = D(d\mathbb{I} + B)Dy(k) = (d\mathbb{I} + B')y(k) \tag{11}$$

Thus from Theorem III.3, the system (11) holds to orthant pair $\mathbb{R}^n_{\{-,+\}}$. Since $x = D^{-1}y = Dy$, we know that the system (1) holds to $\mathbb{R}^n_{\{-\sigma,+\sigma\}}$. ■

As shown above, via relaxing bipartite consensus for global trust in SNS as bipartite opinions holdable in two opposite orthants, structural balance is no longer required, instead it only requires eventual positivity after gauge transformation. Such extension makes the framework of global trust evaluation available to more general case.

IV. APPLICATION OF GLOBAL TRUST IN SNS

A. Clustering Effect for System Security

The bipartite consensus of global trust comes naturally with clusters of controversial opinions within the trust network. When the opinion differences come from different tastes, the clusters represents two communities of opposite preferences. However, it is also possible that one of the clusters is formed due to the identity of adversary, in which case the global trust evaluation process plays the role of clustering adversaries within social network based on the distrust relationship between users. The level of system security can be improved by implementing global trust evaluation in the network.

B. Distrust Filtering in Recommender System

Along with the popularity of recommender systems, there are various types of attacks towards the system, e.g. random attack and bandwagon attack [26].

In order to enforce the integrity of the recommendation, we propose to integrate the global trust information and filter out users of low reputation [4] . If the user's global trust value is lower than the threshold (e.g. negative), her rating will not be considered in rating prediction. For example, When applying global trust into recommender system, the classic user-based collaborative filtering (CF) [27] can be modified as:

$$\tilde{r_{ik}} = b_{ik} + \frac{\sum_{u_j \in S(k;i), w_j \geq \eta} s_{ij} \cdot (r_{jk} - b_{jk})}{\sum_{u_j \in S(k;i)} s_{ij}} \quad (12)$$

where $S(k;i)$ is the neighbor set of u_i about item o_k, with s_{ij} the similarity between u_i and u_j. w_j is the reputation of u_j and η is the threshold. b_{ik} and b_{jk} are the baseline estimates for r_{ik} and r_{jk} respectively, and r_{jk} is the rating of u_j about o_k. This means that only people of global trust values above threshold (e.g. positive) are considered as a source of reference for item recommendation. Note that here because of bipartite global trust, users of the same cluster in global trust evaluation will mark each other positive global trust, and the opposite if the users are from two groups of controversial opinions.

Apart from those mentioned above, there are more scenarios in SNS that can apply users' global trust to improve service quality, such as SNS advertisement allocation [28].

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we investigate the problem of global trust evaluation in SNS with controversial opinions. We consider both trust and distrust relationships in the associated trust network, and propose to reach different global trust between antagonistic groups. We introduce the approach of bipartite consensus in signed graphs and formulate the problem of global trust evaluation in SNS as bipartite consensus for global trust with controversial opinions. We use a discrete-time dynamic system to describe the distributed evaluation process. Under the condition of structural balance, we prove that the dynamic system considered in our formulation can reach a bipartite consensus for global trust. In order to further extend the result of bipartite consensus for global trust to a more general case, the concept of eventual positivity is introduced and the definition of bipartite consensus is accordingly adjusted to be holdable cones. Finally we discuss the impact of global trust reached via our approach on security of SNS and its application in SNS-based recommender system design which relies on global trust for recommendation of integrity.

In the future we will consider time-varying adjacency matrix in the dynamical system and explore its influence on reaching bipartite consensus for global trust. We will also discuss the robustness of the system. Meanwhile, we are interested in structural balance approximation, which would connect the ideal case of structural balance and general cases in solving the problem of global trust evaluation in SNS. We will apply global trust in more SNS-based scenarios.

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