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Distributed Consensus Networks of Neutral Type*

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***Abstract*— We propose and study a network of autonomous agents which evolve their state under a distributed consensus algorithm of non-linear neutral type. We provide sufficient conditions for convergence to a common consensus point, by means of a stability in variation argument and fixed point theory. Our approach provides both an estimation on the rate of convergence and an implicit expression for the consensus point.**

***Index Terms*— Consensus systems, distributed delays, fixed point theory, stability and rate of convergence.**

I. INTRODUCTION

The consensus problem is known in the literature as the dynamic averaging of a state of interest among a finite number of autonomous agents. The mathematical representation of the algorithm involves a finite number $N < \infty$ of entities (agents, birds, terminals) each of which possesses a state of interest, usually a real valued function of time $x_i(t) \in \mathbb{R}$ and executes the following distributed algorithm

$$\dot{x}_i(t) = \sum_j a_{ij}(t)\big(x_j(t) - x_i(t)\big)$$

where $a_{ij}(t)$ are coupling functions of time that model the effect of agent j on i and they are assumed to be non-negative. Due to its applicability in diverse fields of the control and the applied science communities, this algorithm has been extensively studied under numerous significant variations. [8], [5], [7], [9], [2], [3], [18]. For a detailed review of recent related results the interested reader is referred to [17].

A. Motivation and Contribution

All the aforementioned works discuss consensus algorithms under the instrumental assumption that the rate of change of the state of an agent i strictly depends on the agents current state. This is an assumption that although mathematically convenient, it is over-simplistic for a number of reasons. In real-world scenarios the agents' ability to operate cannot exclusively depend on their current state. Robots have terminals that may take some time to keep processing data after a while due to the rate that such data were processed at some previous time. For example, the rate at which earlier data information was processed may cause excessive memory overflows or other buffering issues that diminish the current processing rate. On the other hand, birds may get tired after maneuvering beyond their physical abilities. Such a physical constraint is dictated by the way they adjusted their velocities in previous times. These phenomena are very important as they affect both the performance and the collective stability of the corresponding dynamic network algorithms. The mathematical equations now read as functional differential equations of neutral type. To the best of our knowledge there is no work towards this path in the theory of consensus systems and for good reason. For one, the classical ordinary differential equation theory is no longer applicable and one needs to switch to the theory of functional differential equations [4] and in particular to differential equations of neutral type. Although the mathematical theory has been fully developed, the stability tools are by no means as strong as the ones used in the ordinary case (or even the functional case) let alone when we are focused in the Lagrange type stability, the consensus systems enjoy (i.e. stability with respect to a subset of the state space).

In the present work, we consider a finite population of autonomous agents, connected over a linear time invariant (static) communication network. This network is sufficiently connected so that in the non-neutral (ordinary) case the agents can execute a dynamic consensus algorithm and converge to a common value according to the standard agreement protocol [8].

Based on this (nominal) system we consider its neutral variation. Our aim is to establish sufficient conditions for asymptotic convergence to a constant value via a stability in variation argument and application of fixed point theory. Stability by fixed point theory is an emerging field in the study of differential equations which was motivated by the seminal work of T.A. Burton [1]. As an alternative to the mainstream Lyapunov techniques one can discuss the stability problem of differential equations by applying fixed point theorems on linear spaces after representing the solutions of those equations in integral form. The authors have implemented such an approach in the study of delayed consensus systems both in linear [15], [16], [13] and nonlinear [14], [12] versions.

The contribution of fixed points in the present work is twofold: On the one hand, it ensures the existence of a (unique) solution for the non-linear algebraic equation that characterizes the consensus point and on the other hand it will be used to prove the existence of a fixed point on the integral operator that represents the solution of the neutral equation. This fixed point will lie in an specially designed compact subset of the Banach space of bounded functions so that the existence in the large, asymptotic stability and

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estimate of the rate of convergence are proved at the same time.

The paper is organized as follows: In Section II we provide the underlying notation and discuss the based theory that is to be used throughout this work. In Section III we formulate our problem in rigorous mathematical terms, state the assumptions, the main result and we conclude with a couple of comments on it. In Section IV we prove a number of preliminary results that are to be used in the section to follow. In Section V we prove the main result and in Section VI we conduct a thorough discussion that includes examples, a number of important remarks and it concludes with a number of open questions for future research.

II. NOTATIONS & DEFINITIONS

The dynamics evolve in the Euclidean space $\mathbb{R}^N$ where N is the number of agents. All the vectors are assumed to be column vectors. By $\mathbf{1}$ we understand the vector of ones. The space $\mathbb{R}^N$ is endowed with an appropriate vector norm $||\cdot||$ with $|\cdot|$ to denote the corresponding matrix norm. The last notation will also denote the absolute value of a real number. Each agent is defined through its state x_i which evolves under a dynamic algorithm to be defined below. The set of agents consists a population each of which may affect each other with coupling weights, assumed to be non-negative constants. In particular, agent j affects agent i if and only if $a_{ij} > 0$. The matrix $A = [a_{ij}]$ is the adjacency matrix and the matrix $D = \text{Diag}[d_i]$, $d_i = \sum_j a_{ij}$ is the degree matrix. We are interested in the communication graph to be sufficiently connected such that A corresponds to a routed-out branching graph. This means that there exists an agent i out of which a connectivity path is paved towards any other agent j of the population via a path of existing edges via strictly positive weights a_{ii_2}, $a_{i_2i_3}$,..., $a_{i_{l-1}j}$. This is the mildest type of static connectivity for ststic networks. Consequently $L = D - A$, called the Laplacian matrix, is characterized by the spectrum

$$0 = \lambda_1 < \Re\{\lambda_2\} \leq \cdots \leq \Re\{\lambda_N\}.$$

Under this connectivity condition, the right eigenvector $\mathbf{c}$ of L with respect to λ_1 is unique up to normalization. Henceforth $\mathbf{c} = (c_1, \ldots, c_N)^T$ is such that $\mathbf{c}^T L = 0$ and $c_i \geq 0$ with $\sum_i c_i = 1$, [8].

A. *Elements of Fixed Point Theory*

A pair $(\mathbb{S}, \rho)$ is a metric space if $\mathbb{S}$ is a set and $\rho : \mathbb{S}\times\mathbb{S} \to [0, \infty)$ a metric function. A metric space is complete if every Cauchy sequence in $(\mathbb{S}, \rho)$ has a limit in that space. A set L in a metric space $(\mathbb{S}, \rho)$ is compact if each sequence $\{x_n\} \subset L$ has a sub-sequence with limit in L. $(\mathbb{B}, |\cdot|)$ will constitute the Banach space of functions defined on $\mathbb{R}$ and take values in $\mathbb{R}^N$ such that for $\boldsymbol{\phi} \in \mathbb{B}$, $|\boldsymbol{\phi}| = \sup_{t\in\mathbb{R}} ||\boldsymbol{\phi}(t)|| < \infty$. Two important fixed point theorems in linear spaces are cited below. The first and most celebrated result of the whole theory is the result of Banach that dates back in 1932:

Theorem 2.1 (The Contraction Mapping Principle): Let $(\mathbb{S}, \rho)$ be a complete metric space and let $\mathcal{P} : \mathbb{S} \to \mathbb{S}$. If there is a constant $\alpha < 1$ such that for each $y_1, y_2 \in \mathbb{S}$, we have

$$\rho(\mathcal{P}y_1, \mathcal{P}y_2) \leq \alpha\rho(y_1, y_2)$$

then there exists a unique point $y^* \in \mathbb{S}$ with $\mathcal{P}y^* = y^*$.

Proof: See [11]. ■

The condition imposed above is the contraction condition and it will be repeated in the next result. In 1958, Krasnoselskii studied a paper of Schauder and obtained the following working hypothesis:

Theorem 2.2: Let $\mathbb{M}$ be a closed, convex, non-empty subset of a Banach space $(\mathbb{B}, |\cdot|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$, for any $x, y \in \mathbb{M}$,
(ii) $\mathcal{A}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact set,
(iii) $\mathcal{B}$ is a contraction with constant $\alpha < 1$.

Then there exists $y^* \in \mathbb{M}$ with $\mathcal{A}y^* + \mathcal{B}y^* = y^*$.

Proof: See [11]. ■

While Theorem 2.1 formulates the results in complete metric spaces and guarantees both existence and uniqueness, Krasnoselskii's result requires compactness and it is formulated in complete normed spaces. The proofs of these results can be found in [11]. See also [1] where the properties of Theorems 2.1 and 2.2 are discussed through numerous examples in a very similar context with the present work. In Section VI we will argue for our decision to use Theorem 2.2 as our central stability theorem. For the moment we recall that the standard way of proving compactness (a prerequisite for applying Theorem 2.2) of a map is through the Arzela-Ascoli theorem [10]. However this theorem applies only to compact intervals of time. The stability analysis requires unbounded t-intervals and thus we shall need the following generalization of the Arzela-Ascoli's result for the particular class of functions considered for the main stability problem of the paper.

Proposition 2.3: Let $\mathbb{R}_+$ and $h(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $h(t) \to 0$ as $t \to \infty$. If $\{\mathbf{g}_k(t)\}$ is an equicontinuous sequence of $\mathbb{R}^N$ valued functions on $[a, \infty)$ such that $||\mathbf{g}_k(t)|| \leq h(t)$ for $t \in \mathbb{R}_+$, then there is a subsequence that converges uniformly on $\mathbb{R}_+$ to a continuous function $\mathbf{g}^*(t)$ with $||\mathbf{g}^*(t)|| \leq h(t)$.

Proof: See [1]. ■

III. PROBLEM FORMULATION

A network of $N < \infty$ autonomous agents evolves its state $\mathbf{x} = (x_1, \ldots, x_N)^T$ according to the following algorithm

$$\frac{d}{dt}\left(x_i + \int_{-r}^{0} f_i\big(x_i(t+s)\big)p_i(s)\,ds\right) = \sum_j a_{ij}\big(x_j - x_i\big)$$

for $i = 1, \ldots, N$, $x_i = x_i(t)$ and $\int_{-r}^{0} p_i(s)\,ds \equiv 1$. The latter models a delay, the uncertainty of which imposes the integrable distribution function $p_i(s) : [-r, 0] \to \mathbb{R}$. Now we observe that we can rewrite the above equation as

$$\frac{d}{dt}\left(x_i + \int_{-r}^{0} \tilde{f}_i(x_i(t+s))p_i(s)\,ds\right) = \sum_j a_{ij}\big(x_j - x_i\big)$$

where $\tilde{f}_i(x_i(t+s)) = f_i(x_i(t+s)) - f_i(k)$ for some k to be determined in the following. We arrive at the following initial value problem in vector form

$$\begin{aligned} &\frac{d}{dt}\left(\mathbf{x} + \int_{t-r}^{t} \tilde{\mathbf{F}}\big(\mathbf{x}(q), \mathbf{p}(q-t)\big)\, dq\right) = -L\mathbf{x}, t \geq 0 \\ &\mathbf{x}(t) = \boldsymbol{\phi}(t), -r \leq t \leq 0. \end{aligned} \tag{1}$$

At the moment, we know very little about the fundamental properties of $\mathbf{x} = \mathbf{x}(t, 0, \boldsymbol{\phi}),\ t \geq 0$. We need assumptions that ensure the existence and perhaps uniqueness.

A. Assumptions

Let us now impose the set of Assumptions that will come at hand in our analysis.

Assumption 3.1: The communication graph is routed-out branching.

This is a necessary and sufficient condition for convergence of the ordinary model and all the properties of the Laplacian discussed in the previous section to hold. Next we need an assumption on the non-linear neutral terms f_i, the most reasonable of which is a global Lipschitz condition.

Assumption 3.2: For every $i = 1, \ldots, N$, $f_i : \mathbb{R} \to \mathbb{R}$ is integrable and there exists $K_i \in \mathbb{R}_+$ such that

$$|f_i(x) - f_i(y)| \leq K_i |x - y|$$

for any $x, y \in \mathbb{R}$.

This is a particularly convenient condition but at the same time it is very restricting. Possible extensions and relaxations are to be discussed in Section VI. For the moment we keep in mind that such a condition at least ensures uniqueness of a solution [4]. The existence (in the large) property of the solutions is yet to be established together with stability. For the moment we are ready to present the central result of this work:

B. Statement of the main result

Theorem 3.3: Let Assumptions 3.1 and 3.2 hold. If there exists

$$0 < \gamma < \Re\{\lambda_2\} \tag{2}$$

such that

$$\left(1 + \frac{W|L|}{\Re\{\lambda_2\} - \gamma}\right) \max_i K_i \int_{-r}^{0} e^{-\gamma q} |p_i(q)|\, dq < 1 \tag{3}$$

where W is defined as in (6), then the solution $\mathbf{x}(t) = \mathbf{x}(t, 0, \boldsymbol{\phi})$, $t \geq 0$ of (1) satisfies

$$||\mathbf{x}(t) - \mathbf{1}k|| \leq C e^{-\gamma t}$$

where k is the unique solution of (9) and some finite constant $C > ||\boldsymbol{\phi}(0) - \mathbf{1}k||$.

At this point we would like to make a comment on conditions (2) and (3). The first condition signifies the fact that (1) is a perturbation to (5), defined below. Thus the rate of convergence of the solutions of the perturbed system cannot be better than the nominal one. The second condition characterizes the existence and uniqueness of the consensus point. Indeed (2) implies

$$\max_i K_i < 1$$

so that Lemma 4.2 is applied to conclude the existence of a point k, that is the solution of (9). Now, taking a look in (9) we are tempted to consider for a moment a linear version of $f_i(x) = -|K_i|x$. Then the consensus point has the closed form solution

$$k = \frac{\sum_i c_i \phi_i(0) - \sum_i c_i |K_i| \int_{-r}^{0} \phi(s) p_i(s)\, ds}{1 - \sum_i c_i |K_i|} \tag{4}$$

and this could create instability $k = \infty$ at values of $|K_i|$ close to 1. For this reason we conclude that so long as we are searching for asymptotic consensus solutions $\max_i K_i < 1$ is not unnecessarily strict. However, condition (3) is undoubtedly a very hard one as it imposes severe smallness conditions on both K_i and r, which is the result of the stability in variation argument. It is shown however that such strict conditions occur very regularly in the literature and examples can be constructed that justify them for the sake of the stabilization of solutions (see also [4]).

IV. PRELIMINARY RESULTS

In this section, we review a collection of preliminary results. At first we observe that (1) is similar to

$$\frac{d}{dt}\mathbf{y}(t) = -L\mathbf{y}(t) \tag{5}$$

and the dynamics of (1) may be resembled under smallness conditions on f and r.

A. The simple delayed dynamics problem

The dynamic behavior of

$$\dot{y}_i(t) = \sum_j a_{ij}\big(y_j(t) - y_i(t)\big)$$

is very well understood in the literature. The solution $\mathbf{y}(t) = e^{-Lt}\mathbf{y}(0)$ satisfies under Assumption 3.1 $\mathbf{y}(t) \to \mathbf{1}\mathbf{c}^T\mathbf{y}(0)$ with $\mathbf{c}^T L = 0$ the (unique) right eigenvector of the Laplacian so that

$$||\mathbf{y}(t) - \mathbf{1}\mathbf{c}^T\mathbf{y}(0)|| \leq W e^{-\Re\{\lambda_2\} t} \tag{6}$$

for a constant $W > 0$ that depends on the norm and it is henceforth assumed to be known [8].

B. Deriving the solution operator

Using the simple variation of constants formula we see that the solution $\mathbf{x}$ of (1) satisfies

$$\begin{aligned} \mathbf{x}(t) &= \\ &= e^{-Lt}\tilde{\boldsymbol{\phi}} - \int_0^t e^{-L(t-s)} \frac{d}{ds} \int_{s-r}^{s} \tilde{\mathbf{F}}\big(\mathbf{x}(q), \mathbf{p}(q-s)\big)\, dq ds \\ &= e^{-Lt}\tilde{\boldsymbol{\phi}} - \int_{t-r}^{t} \tilde{\mathbf{F}}\big(\mathbf{x}(q), \mathbf{p}(q-t)\big)\, dq + \\ &\qquad + \int_0^t e^{-L(t-s)} L \int_{s-r}^{s} \tilde{\mathbf{F}}\big(\mathbf{x}(q), \mathbf{p}(q-s)\big)\, dq ds \end{aligned} \tag{7}$$

where $\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi}(0) + \int_{-r}^{0} \tilde{\mathbf{F}}\big(\boldsymbol{\phi}(q), \mathbf{p}(q)\big)\, dq$. This integral form of the solution will constitute the operator through which, stability results will be established.

C. The space of solutions

Let $C^0 = C\big([-r,\infty), \mathbb{R}^N\big)$ be a subspace of $(\mathbb{B}, |\cdot|)$ which constitutes the continuous bounded functions. For fixed $\boldsymbol{\phi} \in C^0([-r,0], \mathbb{R}^N)$ $k \in \mathbb{R}$, $C > 0$ and $\gamma > 0$ we define the following set

$$\mathbb{M} = \{\mathbf{z} \in C^0 : \mathbf{z} = \boldsymbol{\phi}, \sup_{t\geq 0} e^{\gamma t}\big|\big|\mathbf{z}(t) - \mathbf{1}k\big|\big| \leq C\} \qquad (8)$$

Lemma 4.1: The set $\mathbb{M}$ as defined in (8) is closed, convex and non-empty, if $C \geq ||\boldsymbol{\phi}(0) - \mathbf{1}k||$.

Proof: The set is obviously closed as, it is constructed to contain all of its limit points. It can be also easily shown that it is convex: For any pair $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{M}$, $\mathbf{z}_3 := \beta\mathbf{z}_1 + (1-\beta)\mathbf{z}_2$ is also a member of $\mathbb{M}$ for any $\beta \in [0,1]$. Indeed $\mathbf{z}_3(t) = \beta\boldsymbol{\phi}(t) + (1-\beta)\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t)$ for $-r \leq t \leq 0$ and for $\mathbf{z}_3(t) - \mathbf{1}k = \beta(\mathbf{z}_1(t) - \mathbf{1}k) + (1-\beta)(\mathbf{z}_2(t) - \mathbf{1}k)$ it holds that $e^{\gamma t}||\mathbf{z}_3(t) - \mathbf{1}k|| \leq \beta C + (1-\beta)C = C$. Finally under the imposed condition, the function

$$\mathbf{z}(t) = \begin{cases} \boldsymbol{\phi}(t), & -r \leq t \leq 0 \\ \mathbf{1}k + (\boldsymbol{\phi}(0) - \mathbf{1}k)e^{-\gamma t}, & t \geq 0 \end{cases}$$

is clearly a member of $\mathbb{M}$ so the set is not empty. ■

D. The consensus point

Lemma 4.2: Let Assumption 3.2 and Eq. (3) holds. Then there exists a unique $k \in \mathbb{R}$ such that

$$k = \sum_i c_i \bigg(\phi_i(0) + \int_{-r}^{0} \big(f_i(\phi_i(q)) - f_i(k)\big)p_i(q)\, dq\bigg) \quad (9)$$

Proof: Consider the complete metric space $(\mathbb{R}, \rho)$ where $\rho(x,y) = |x-y|$ is the standard distance between two points on the line. Then it is easily seen that for the operator $F(k) = \sum_i c_i \bigg(\phi_i(0) + \int_{-r}^{0} \big(f_i(\phi_i(-q)) - f_i(k)\big)p_i(q)\, dq\bigg)$ that maps $\mathbb{R}$ into itself

$$\rho(F(k_1), F(k_2)) \leq \big(\max_i K_i\big)\rho(k_1,k_2), \ \forall k_1, k_2 \in \mathbb{R}$$

and the result follows from Theorem 2.1. ■

Now that we have obtained these easy yet significant results we can proceed to the full proof of Theorem 3.3.

V. Proof of Theorem 3.3

The proof of our main result is based on Theorem 2.2. Having defined k and established a first estimate of C we are ready to further elaborate on our solution space and the solution operator.

The first step is to show that the function $\mathcal{P} : \mathbb{M} \to \mathbb{R}$

$$(\mathcal{P}\mathbf{z})(t) = \begin{cases} \boldsymbol{\phi}(t), & -r \leq t \leq 0 \\ \mathbf{z}_{(7)}(t), & t \geq 0 \end{cases} \qquad (10)$$

is under conditions an operator $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. The first step towards this is to examine $\lim_{t\to\infty}(\mathcal{P}\mathbf{z})(t)$. Indeed we see that

$$e^{-Lt}\tilde{\boldsymbol{\phi}} \to \mathbf{1}\mathbf{c}^T\tilde{\boldsymbol{\phi}} = \mathbf{1}\mathbf{c}^T\tilde{\boldsymbol{\phi}}(k)$$

but

$$\int_{t-r}^{t} \tilde{\mathbf{F}}\big(\mathbf{z}(q), \mathbf{p}(q-t)\big)\, dq \to 0 \qquad (11)$$

and

$$\int_0^t e^{-L(t-s)} L \int_{s-r}^{s} \tilde{\mathbf{F}}\big(\mathbf{z}(q), \mathbf{p}(q-s)\big)\, dq ds \to 0 \qquad (12)$$

exactly because $e^{-L(t-s)}L = \big(e^{-L(t-s)} - \mathbf{1}\mathbf{c}^T\big)L$ and $|e^{-L(t-s)} - \mathbf{1}\mathbf{c}^T| \leq We^{-\Re\{\lambda_2\}(t-s)}$. Now from (11), (12) is justified as it is the convolution of an L^1 function with a function that goes to zero. Finally,

$$\lim_{t\to\infty}(\mathcal{P}\mathbf{z})(t) = \\ = \mathbf{1}\sum_i c_i \bigg(\phi_i(0) + \int_{-r}^{0} \big(f_i(\phi_i(-q)) - f_i(k)\big)\, dq\bigg)$$

so that if k is defined as in (9) we conclude that $(\mathcal{P}\mathbf{z})(t) \to \mathbf{1}k$ for $\mathbf{z}(t) \to \mathbf{1}k$. We define two operators $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{S}$ as follows:

$$(\mathcal{A}\mathbf{z})(t) = \int_0^t e^{-L(t-s)} L \int_{s-r}^{s} \tilde{\mathbf{F}}\big(\mathbf{z}(q), \mathbf{p}(q-s)\big)\, dq ds$$
$$(\mathcal{B}\mathbf{z})(t) = e^{-Lt}\tilde{\boldsymbol{\phi}} - \int_{t-r}^{t} \tilde{\mathbf{F}}\big(\mathbf{z}(q), \mathbf{p}(q-t)\big)\, dq \qquad (13)$$

We now proceed to check the conditions of Theorem 2.2 one by one:

a) Condition (i): Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{M}$. Then $(\mathcal{A}\mathbf{z}_1)(t) + (\mathcal{B}\mathbf{z}_2)(t)$ behaves in the limit exactly as $(\mathcal{P}\mathbf{z})(t)$, simple because it is only the time-varying (state-independent) part that contributes to the limit point. Hence, since $\mathbf{z}_1 \equiv \mathbf{z}_2$ in $[-r,0]$, it is only left to prove the convergence estimate: Note that under (2) simple calculations yield

$$\sup_t e^{\gamma t}||(\mathcal{A}\mathbf{z}_1)(t)|| \leq \frac{W|L|\max_i K_i \int_{-r}^{0} e^{-\gamma q}p_i(q)\, dq}{\Re\{\lambda_2\} - \gamma} C$$

and

$$\sup_t e^{\gamma t}||(\mathcal{B}\mathbf{z}_2)(t)|| \leq ||\tilde{\boldsymbol{\phi}}|| + \max_i K_i \int_{-r}^{0} e^{-\gamma q}p_i(q)\, dq C$$

so that (3) implies that the first condition of Krasnoselskii's Theorem is satisfied as this way it is always possible to pick a finite C large enough so that $\sup_t e^{\gamma t}||(\mathcal{A}\mathbf{z}_1)(t)|| + \sup_t e^{\gamma t}||(\mathcal{B}\mathbf{z}_2)(t)|| \leq C$. In fact it suffices to pick $C > \max\{||\boldsymbol{\phi}(0) - \mathbf{1}k||, D\}$ where

$$D = \frac{||\tilde{\boldsymbol{\phi}}||}{1 - \big(1 + \frac{W|L|}{\Re\{\lambda_2\} - \gamma}\big)\max_i K_i \int_{-r}^{0} e^{-\gamma q}|p_i(q)|\, dq}$$

b) Condition (ii): We note that $\mathcal{A}\mathbb{M}$ is a subset of the $\mathbb{B}$ as it maps $\mathbb{M}$ to a subset of functions which vanish to zero as fast as $e^{-\gamma t}$, in view of (2). It suffices to show that $\mathcal{A}\mathbb{M}$ is equicontinuous as then it follows that it is continuous with respect to the supremum norm in $\mathbb{B}$. The former can be shown by differentiating $(\mathcal{A}\mathbf{z})(t)$ with respect to t:

$$\frac{d}{dt}(\mathcal{A}\mathbf{z})(t) = \int_{-r}^{0} \tilde{\mathbf{F}}\big(\mathbf{z}(t+q), \mathbf{p}(q)\big)\, dq - \\ - \int_{0}^{t} e^{-L(t-s)} L^2 \int_{-r}^{0} \tilde{\mathbf{F}}\big(\mathbf{z}(s+q), \mathbf{p}(q)\big)\, dq ds$$

and it is only a tedious algebraic exercise to show that for $\mathbf{z} \in \mathbb{M}$, $\sup_t ||\frac{d}{dt}(\mathcal{A}\mathbf{z})(t)|| < \infty$, actually bounded by a constant that is independent of the element $\mathbf{z}$ and depends only on $\mathbb{M}$. This uniform condition implies equi-continuity and hence Proposition 2.3 applies to show that $\mathcal{A}$ is a compact map that is also continuous.

c) Condition (iii): Now, since $\mathbb{M}$ is a closed subset of $\mathbb{B}$ it also constitutes a (complete) metric space under the weighted metric $\rho(\mathbf{z}_1, \mathbf{z}_1) = \sup_t e^{\gamma t}||\mathbf{z}_1(t) - \mathbf{z}_2(t)||$. Then

$$\rho\big(\mathcal{B}\mathbf{z}_1, \mathcal{B}\mathbf{z}_2\big) \leq \left[\max_i K_i \int_{-r}^{0} e^{-\gamma q} |p_i(q)|\, dq\right] \rho(\mathbf{z}_1, \mathbf{z}_1)$$

which is automatically a contraction in view of (3).

Then we see that all condition of Theorem 2.2 are satisfied, hence $\mathcal{P} = \mathcal{A} + \mathcal{B}$ has as fixed point in $\mathbb{M}$.

VI. DISCUSSION

Assuming that the dynamics of a living organism or a modern computing machine should evolve in time as a function of the current or previous state only is a rather simplistic hypothesis. In real world problems the rate of change of a state also depends on the rate of change of the same state at a previous time. Any athlete knows the effect of the way they change their speed at present time, on the way they change their speed in future times. In Nature the acceleration of a flying bird at a particular moment cannot but be also a function of its acceleration at a previous time. These sorts of correlations are consistently ignored when designing mathematical models, exactly because their analysis is particularly difficult.

In this paper we introduced and developed a theoretical framework on such distributed systems of autonomous agents that execute a simple consensus algorithm with an additional non-linear neutral term. This addition turns the ordinary differential equation problem into a neutral functional differential equation one. We developed a novel fixed point theory argument based on a combination of the contraction mapping principle and Krasnoselskii's result on perturbed operators. The imposed conditions are effectively based on the smallness of the delays and/or the Lipschitz constants. The reason for using Theorem 2.2 is because f may not be obey a Lipshcitz condition. In such case the necessary modifications would involve to prove that (9) attains a solution via a different argument, and f to be bounded in by an appropriate function so that in the proof of Theorem 3.3, $\mathcal{A}$ can be proved to be a compact map. The linearity of the problem, in connection with the (strong) Lipschitz properties on f_i produced very elegant results that characterize both the convergence, the rate as well as the consensus point.

Our primary goal was to initiate the subject of neutral distributed consensus networks. We claim that proving simple convergence to a common constant is the first step yet less exciting phenomenon a research might come across. Theorem 3.3 is a combination of conservative imposed conditions and the over-simplistic static communication network. We conjecture that the methods developed in this work can be adapted to the study of networks with more realistic and more interesting neutral components or networks with time varying or non-linear couplings or even delayed state arguments. Then one could investigate the existence (and perhaps stability) of more interesting asymptotic phenomena such as periodic or chaotic solutions.

Towards this path we need to make a couple of comments. In the study of time-varying dynamics it is hopeless for the reader to make any comment on the consensus point, hence it is desirable to study the dynamics of the spread of the state $\mathbf{x}$, i.e. $\max_i x_i - \min_i x_i$ or to study the behavior of $\dot{\mathbf{x}}$, which vanishes so long as $\mathbf{x}$ converges to a constant.

Another serious difficulty is the derivation of the solution operator. Although the method of variation of parameters in dynamical systems is very popular and well-studied over the years [6], experience has proved that regardless if we are to follow a Lyapunov or a Fixed Point Method, stability problems are to be studied on a case by case basis. In our elementary model, the nominal system, exhibits satisfactory robust stability, but the derivation of the solution operator to a useful form required the integration by parts step. It is not clear how one could proceed for example if the nominal system, involved propagation delays or if the nominal system was nonlinear. In the latter case we would be forced to use a non-linear variation of parameters formula along the lines of the corresponding method [6]. This, however, would result in excessive technical difficulties.

A future step would be to impose monotonic conditions on the non-linear neutral terms (for instance, $f_i' > 0$), so that small Lipschitz constants would be dropped in the proof of Lemma 4.2. Indeed, for $r = 0$, Eq. (1) reads in the i^{th} component

$$\dot{x}_i = \sum_j \frac{a_{ij}}{1 + f'(x_i)}\big(x_j - x_i\big)$$

i.e. a stable (non-linear) network. The challenge that arises in this case is how the monotonic condition in the presence of delays would favor a milder (3), most probably with the method of combining solution forms proposed in [16].

Nevertheless, the field of neutral distributed consensus dynamics is very new, compared to the conventional consensus networks. It has a realistic application justification and promises a handful of new and most exciting global phenomena which under the fundamental nature of these decentralized cooperative algorithms, always occur out of local interactions.

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