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Delay-Independent Stability of Consensus Networks with Application to Flocking ⋆

Christoforos Somarakis * **John S. Baras** **

* *Applied Mathematics Dept., The Institute for Systems Research, Univ. Of Md., College Park, MD, 20740, USA (e-mail: csomarak@math.umd.edu).*
** *Electrical & Computer Eng. Dept., The Institute for Systems Research, Univ. Of Md., College Park, MD, 20740, USA (e-mail: baras@umd.edu)*

Abstract: This work studies a class of linear first order and non-linear second order static distributed consensus networks with time-varying multiple propagation delays, in continuous time. We provide conditions for convergence of to a common constant value, under an increased connectivity condition. The results are delay-independent in the sense that they hold for arbitrary bounded delays. Our approach makes use of fundamental concepts from the Non-Negative Matrix Theory in a fairly elementary way.

Keywords: consensus, flocking, non-linear networks, time-varying delays, delay-independent convergence

1. INTRODUCTION

The field of network dynamics has a long history in both the Engineering and in the Applied Mathematics communities. Many diverse scientific fields have been, in one way or another, always interested in interacting populations that exhibit global behavioral patterns as a result of local interactions. Such a decentralized algorithm for exchanging information among autonomous agents is this of distributed linear consensus, the dynamics of which have been vastly explored over the last decade. A consensus system, concerns a finite population of N agents each of which $i = 1, \dots, N$ possesses a value of interest, say $z_i \in \mathbb{R}$ that dynamically changes according to the scheme

$$\dot{z}_i(t) = \sum_j a_{ij}(t)\big(z_j(t) - z_i(t)\big) \tag{1}$$

for some coupling weights $a_{ij}(t) \geq 0$ that characterize the effect of agent j to i. This is a distributed convex averaging scheme, so that under specific connectivity conditions, the values $z_i(t)$, $i = 1, \dots, N$ converge, asymptotically, to a common constant.

Consensus dynamics are by no means new. They have been reported in the literature quite early, from independent points of research (Tsitsiklis et al. (1986); Smith (1995)). In fact, the underlying mathematical tool which is to be used in the present paper as well, appears in Markov (1906), i.e. at the dawn of the previous century!

The interest in these systems had been reheated when Jadbabaie et al. (2003) provided a rigorous proof of a model proposed by Vicsek et al. (1995) for asymptotic flocking of a population of birds, z_i is considered to be the velocity value of the bird i. Asymptotic convergence of z_i means then velocity alignment of the whole flock. For a recent review on consensus algorithms on old and new results the reader is referred to Somarakis and Baras (2014). Nowadays, the linear models have been substantially studied and their dynamics are very well-understood. Research have been naturally elevated to non-linear variations of the consensus algorithm. Models of the type

$$\begin{aligned} \dot{z}_i(t) &= \sum_j f_{ij}(t, x_j - x_i) \\ \dot{z}_i(t) &= \sum_j g_{ij}(t, x_j) - g_{ij}(t, x_i) \end{aligned} \tag{2}$$

have been proposed by Papachristodoulou et al. (2010); Qing and Haddad (2008) as well as more complex models, closer to flocking dynamics of the type

$$\begin{cases} \dot{x}_i(t) = u_i(t) \\ \dot{u}_i(t) = \sum_j a_{ij}(t, \mathbf{x})\big(u_j(t) - u_i(t)\big) \end{cases} \tag{3}$$

have been proposed by Cucker and Smale (2007).

The non-linearities in these networks are typically detected in the coupling function scheme that controls the communication between two arbitrary agents of the network. In a series of papers, Somarakis and Baras (2012, 2014, 2015) the authors highlighted the fact that not only the vast majority of these types of algorithms share the same cooperative nature, but also that under appropriate conditions on the coupling function, they are mathematically handled with fairly similar techniques.

Interestingly enough, the same approach applies to delayed versions of consensus algorithms which are of great importance to the control community, as delays are unavoidable in real-world networked systems as they can slow the performance or even destabilize the network. Consensus networks with delays have been previously studied in the liter-

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ature (see for example Moreau (2004); Papachristodoulou et al. (2010); Olfati-Saber and Murray (2004)). The results in this case are naturally considerably weaker.

1.1 Motivation & Contribution

In this work, we study two types of continuous time consensus algorithms with multiple time-varying propagation delays. The first is the classical linear consensus system, (1) among individual agents with multiple propagation delays (to be defined below). Under an increased connectivity condition assumption, we will provide sufficient conditions for convergence to a commons state, emphasizing on the rate of convergence. The latter is to be explicitly stated as a function of the system's coupling strength and the imposed delays. We will apply these results to the non-linear flocking model of Cucker-Smale type, presented in (3) and we will provide sufficient conditions for asymptotic flocking (speed alignment). Both algorithms evolve on a finite population of interconnected autonomous agents which update their states after interacting with the delayed states of other agents.

This work can be considered as the delayed counterpart of Somarakis and Baras (2015). The main advantage here is that, just like the aforementioned ordinary case, the novel use of old mathematical tools, allows for explicit contraction rate estimates with the coupling weights allowed to be asymmetric and the imposed delays allowed to be multiple, time-varying, arbitrary, yet bounded. The main simplification condition is this of increased connectivity among the agents. Such an assumption was imposed due to space limitation. In the discussion section we mention milder connectivity regimes, analyzed for the ordinary case in Somarakis and Baras (2015).

1.2 Organization

This paper is structured as follows: In §2 the basic theoretical framework is developed. In §3 we present and briefly discuss the models that will concern us, as well as we state fundamental hypotheses for each model separately. In §4 we state a number of preliminary results for both models. Their proofs, if not stated, can be found in Somarakis and Baras (2014). We state and prove the central results of this work in §5. A thorough discussion and concluding remarks is put in §6.

2. NOTATIONS & DEFINITIONS

$N < \infty$ is a natural number equal to the cardinality of the set $V = \{1, \dots, N\}$ of the agents. Each agent $i \in V$ has a state of interest $z_i \in \mathbb{R}$ (or $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$) that evolves according to either of the two dynamical schemes defined in §3. The state space is, therefore, $\mathbb{R}^N$ (or $\mathbb{R}^N \times \mathbb{R}^N$) and a subset of interest is the so called *agreement subspace* $\Delta = \{\mathbf{z} \in \mathbb{R}^N : z_1 = \dots = z_N\}$ and for $\mathbf{u} \in \mathbb{R}^N$ accordingly. Considering Δ, a quantity of interest is the *spread* of a vector $\mathbf{z} \in \mathbb{R}^N$,

$$S(\mathbf{z}) = \max_i z_i - \min_i z_i.$$

The spread is a pseudo-norm because it vanishes in Δ. For the sake of clarification, given an appropriately defined function $\mathbf{z}(t) \in \mathbb{R}^N$, by $\mathbf{z}(t) \to \Delta$ as $t \to \infty$, we understand that there is an fixed vector in Δ to where all $z_i(t)$ converge asymptotically. Finally, by $\mathbf{1}_A^{(t)}$ we understand the set function and by $\delta(\cdot)$ the delta function.

The initial time is $t_0 \in \mathbb{R}$ is arbitrary but fixed and all the delays, denoted by $\tau_{ij}(t)$, are smooth functions of time defined in $[t_0, \infty)$. Additionally, we will use the notation $\lambda_{ij}(t) = t - \tau_{ij}(t)$. For any $t \geq t_0$, $\tau(t) := \max_{i,j} \tau_{ij}(t)$ and consequently $\lambda(t) := t - \tau(t)$. Also, $I_t := [\lambda(t), t]$. Although we could work in locally integrable function spaces, we will use continuous functions. In particular, $C^p(I_t, \mathbb{R}^N)$ is the space of functions defined in I_t take values in $\mathbb{R}^N$ and have $p \geq 1$ continuous derivatives. For $\boldsymbol{\phi} \in C^0(I_t, \mathbb{R}^N)$ we consider the set

$$W_t^{\boldsymbol{\phi}} = [\min_{s \in I_t, i \in V} \phi_i(s), \max_{s \in I_t, i \in V} \phi_i(s)]$$

together with its length $|W_t^{\boldsymbol{\phi}}|$. Observe that $|W_t^{\boldsymbol{\phi}}|$ is the functional counterparts of the spread S.

Elaborating more on functional spaces and rigorous definitions from the theory of functional differential equations is beyond the scopes of this paper. The interested reader is referred to Hale and Verduyn Lunel (1993).

The communication network is modeled through a directed graph $G = \{V, E\}$ with the nodes modeling the agents, the set E describes the connections between agents so that an element of E is the pair $(i, j) \in V \times V$ for which we say that j "affects" i where we take by default $(i, i) \notin E$. The number of agents j that affect i constitute the "neighborhood" of i, denoted as N_i. The sum of the communication effets on i is $d_i = \sum_j a_{ij}$, also known as *valency* of a node i. A *scrambling* graph is this for which there exists $i^* \in V$ such that $\forall\, j \neq i^*$, $(i^*, j) \in E$. In other words, there exists at least one agent that affects the rest of the population. Examples of these communication schemes are of increased connectivity, e.g. complete graphs, star shaped graphs, etc. Throughout this work we shall only consider static communication networks in the sense that E is a time invariant set.

The 1^{st} model, named as "consensus network", attains a state variable that will be denoted as the $\mathbb{R}^N$-valued function $\mathbf{z}(t) = \big(z_1(t), \dots, z_N(t)\big)^T$. The 2^{nd} non-linear model, named as "flocking network", attains the couple of state variables $\big(\mathbf{x}(t), \mathbf{u}(t)\big) \in \mathbb{R}^N \times \mathbb{R}^N$.

3. PROBLEMS SETUP

In this section, we state the equations under study, the accompanying set of assumptions and the convergence definitions.

3.1 Consensus network

The solution $\mathbf{z}(t), t \geq t_0$ of this population satisfies:

$$\begin{cases} \dot{z}_i(t) = \sum_{j \in N_i} a_{ij}(t)\big(z_j(\lambda_{ij}(t)) - z_i(t)\big), & t \geq t_0 \\ z_i(t) = \phi_i(t), & t \in I_{t_0} \end{cases} \tag{4}$$

for $i \in V$ where $\boldsymbol{\phi} = \big(\phi_1, \dots, \phi_N\big)^T \in C^0\big(I_{t_0}, \mathbb{R}^N\big)$ is the given initial data so that typically, $\mathbf{z} = \mathbf{z}(t, t_0, \boldsymbol{\phi})$, $t \geq t_0$. Each agent i dynamically updates its state after comparing it with delayed versions of the states of agents in N_i.

Assumption 1. For all $i, j \in V$ it holds that $a_{ij} \in C^0\big([t_0,\infty),[0,M]\big)$ and $\forall\ j \in N_i$ we have $a_{ij}(t) > a > 0$.

The convergence definition of interest for this system is:
Definition 2. We say that the solution $\mathbf{z} = \mathbf{z}\big(t, t_0, \boldsymbol{\phi}\big),\ t \geq t_0$ of the initial value problem (4) exhibits *unconditional asymptotic consensus* if for any initial functions $\boldsymbol{\phi}$, the solution is bounded and $\mathbf{z}(t) \to \Delta$ as $t \to \infty$.

At first we need to consider a connectivity regime that enables the possibility of asymptotic the in the kind of Definition 2. A natural adaptation to the graph theoretic framework introduced in §2 is by considering a communication graph $G^t_{(4)} = \big\{V, E^t_{(4)}\big\}$ with $(i,j) \in E^t_{(4)}$ if any only if $a_{ij}(t) > 0$. For the sake of simplicity we discuss only static networks with increased connectivity:
Assumption 3. For all $t \geq t_0$ the set $E_{(4)}$ does not depend on time and $G^t_{(4)}$ is scrambling.

The above is a hard assumption that is taken for simplicity. It is well known that there are much milder sufficient communication condition Jadbabaie et al. (2003); Somarakis and Baras (2015). We will revisit this point in the discussion section. With a little abuse of notation, we consider the following condition for the delays:
Assumption 4. $\forall j \in N_i,\ \tau_{ij}(t) \in C^1\big([t_0,\infty),[0,\tau]\big)$ for $\tau := \sup_t \tau(t) < \infty$ such that $1 - \dot{\tau}_{ij}(t) > 0$.

As a first remark although a_{ij} are taken continuous $\tau_{ij}(t)$ are assumed smooth enough so that $\lambda_{ij}(t)$ are invertible. By $\kappa_{ij}(t)$ we denote the inverse. Also we observe that combining Assumptions 1 and 4 $\exists\ D \leq (N-1)M$ such that

$$\max_i \sup_{t \geq t_0} \int_{t-\tau}^{t} d_i(s)\, ds \leq D\tau. \tag{5}$$

3.2 Flocking network

The convergence results that are to be obtained in §5 will be applied in the next, second order consensus, protocol. This algorithm is proposed for speed alignment of birds. Each bird is a node $i \in V$ and it is defined through its position and speed (x_i, u_i) so that the overall state vector is $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^N \times \mathbb{R}^N$. The evolution algorithm is:

$$\begin{cases} \dot{x}_i(t) = u_i(t) \\ \dot{u}_i(t) = \displaystyle\sum_{j \in N_i} a_{ij}\big(\mathbf{x}(t)\big)\big(u_j(\lambda_{ij}(t)) - u_i(t)\big), & t \geq t_0 \\ x_i(t_0 - \tau(t_0)) = x_i^0, u_i(t) = \phi_i(t), & t \in I_{t_0} \end{cases} \tag{6}$$

where $\mathbf{x}^0$ and $\mathbf{u} = \boldsymbol{\phi}$ are sufficient given initial data for the problem to be well-posed as $\mathbf{x}(t) = \mathbf{x}^0 + \int_{\lambda(t_0)}^{t} \mathbf{u}(s)ds$ for $t \geq \lambda(t_0)$. The definition of interest for (6) is:
Definition 5. We say that the solution $\big(\mathbf{x}(t), \mathbf{u}(t)\big)$ of the initial value problem (6) exhibits asymptotic flocking if

$$\mathbf{u}(t) \to \Delta \text{ as } t \to \infty \quad \& \quad \sup_{t \geq t_0} S\big(\mathbf{x}(t)\big) < \infty$$

The coupling rates $a_{ij}\big(\mathbf{x}\big)$ model the communication effect from j to i as a function of their relative distance. In the first appearance of the model these weights had the closed form

$$a_{ij}(|x_i - x_j|) = \frac{\Gamma}{(\beta + |x_i(t) - x_j(t)|^{2\alpha})}$$

and an algebraic approach was followed (Cucker and Smale (2007)) to establish asymptotic convergence as function of the systems' parameters $\Gamma, \beta, \alpha \geq 0$ and the initial data $(\mathbf{x}^0, \mathbf{u}^0)$. This is to model the natural working hypothesis that the more distant two birds are ($|x_i - x_j|$ large), the weaker the effect of the one to the other is (a_{ij} small). In more recent works, non-symmetric weights were assumed, S. and E. (2014). In their utmost generality the coupling weights are endowed with the following condition:
Assumption 6. For any $j \in N_i$, the coupling weights a_{ij} posses the smoothness conditions of Assumption 1 with the following modification: It holds that

$$a_{ij}(\mathbf{x}) \neq 0 \Rightarrow a_{ij}(\mathbf{x}) \geq f\big(S(\mathbf{x})\big)$$

for $f \in C^0\big([0,\infty),[0,M]\big)$ monotonically decreasing with the property that $\lim_{y\to\infty} f(y) = 0$.

The structure of the model and nature of the assumptions is postponed for §6. At the moment we only mention that the vanishing property of the function f constitutes the crucial difficulty in these systems and the study of their stability with respect to Definition (5) require the combination of an explicit estimate on the rate of convergence of the speed alignments and an appropriate Lyapunov functional.

4. PRELIMINARIES

A first remark is the obvious connection between (4) and (6) that leads to the following result.
Lemma 7. Under Assumption 1 the solution $\mathbf{z}$ of (4) satisfies $z_i(t) \in W^{\boldsymbol{\phi}}_{t_0}$ for any $t \geq t_0$.

Proof. Let $t^* \geq t_0$ be the first time that z_i escapes $W^{\boldsymbol{\phi}}_{t_0}$ for some $i \in V$, say to the right. Then $z_i(t^*) = \max_{j \in V, s \leq \bar{t}} z_j(s)$ and $\dot{z}_i(t^*) > 0$. But from (4) and Assumption 1

$$\dot{z}_i(t^*) = \sum_j a_{ij}(t)\big(z_j(\lambda_{ij}(t^*)) - z_i(t^*)\big) \leq 0.$$

A similar argument can be made for the lower bound. Hence the result follows by contradiction.

Two elementary yet crucial remarks are to be made now: The first is that the solution $\mathbf{z}$, together with $(\mathbf{x}, \mathbf{u})$, exist in the large for arbitrary initial conditions. The second is that t_0 is arbitrary hence it holds that

$$|W^{\mathbf{z}}_{t_1}| \leq |W^{\mathbf{z}}_{t_2}|,\ \forall\ t_1 \geq t_2 \geq t_0. \tag{7}$$

The latter condition is instrumental for the next result:
Proposition 8. Let Assumptions 1,3 and 4 hold. The solution $\mathbf{z} = \mathbf{z}(t, t_0, \boldsymbol{\phi}),\ t \geq t_0$ of (4) satisfies:

$$S\big(\mathbf{z}(t)\big) \leq (1-\rho)|W^{\mathbf{z}}_{t-\tau-B}| \tag{8}$$

for $\rho := a\frac{1-e^{-mB}}{m} \in (0,1)$.

Proof. Fix $B > 0$, $m := \sup_{t \geq t_0} \max_i d_i(t) \in (0,\infty)$ that after inversion from $t - B$ to t it reads:

$$z_i(t) = \sum_{j=1}^{N} \int_{t-\tau-B}^{t} b_{ij}(t,s) z_j(s)\, ds, \qquad t \geq t_0 + \tau + B$$

where

$$b_{ii}(t,s) = \mathbf{1}^{(s)}_{[t-B,t]} e^{-m(t-s)}\big(\delta(s-(t-B)) + (m - d_i(s))\big),$$
$$b_{ij}(t,s) = \mathbf{1}^{(s)}_{[\lambda_{ij}(t)-B,\lambda_{ij}(t)]} e^{-m(t-\kappa_{ij}(s))} a_{ij}(\kappa_{ij}(s)).$$

In vector form:

$$\mathbf{z}(t) = \int_{t-\tau-B}^{t} B(t,s)\mathbf{z}(s)\,ds \tag{9}$$

with $B(t,s) := [b_{ij}(t,s)]$. It can be easily shown that $\int_{t-2\tau}^{t} B(t,s)ds$ is a stochastic matrix with the properties that $\forall t \geq t_0 + \tau + B$ and $\forall t_1, t_2 \in [t-\tau-B, t]$

(1) $\sum_j \int_{t_1}^{t_2} b_{ij}(t,s)ds \equiv \text{const.}$
(2) $\sum_j \int_{t-2\tau}^{t} b_{ij}(t,s)ds \equiv 1$.

We fix $h, h' \in V$ and from (9)

$$z_h(t) - z_{h'}(t) = \sum_j \int_{t-\tau-B}^{t} \big(b_{hj} - b_{h'j}\big)(t,s) z_j(s)\,ds. \tag{10}$$

Now, for fixed t we consider the partition $\bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_l$ with $\bar{t}_0 = t - \tau - B$ and $\bar{t}_l = t$ is defined so that for any interval $[\bar{t}_{k-1}, \bar{t}_k]$, $(b_{hj} - b_{h'j})(t,s)$ does not change sign. Within this interval we apply the mean value theorem for integrals, the right hand-side of (10) is

$$\int_{t_{k-1}}^{t_k} \big(b_{hj} - b_{h'j}\big)(t,s)\,ds z_j(s_j^*)$$

for some $s_j^* \in [t_{k-1}, t_k]$. Let j' denote the indeces for which $u_{j'}^k := \int_{t_{k-1}}^{t_k} \big(b_{hj'} - b_{h'j'}\big)(t,s)\,ds > 0$ and j'' : $u_{j''}^k := \int_{t_{k-1}}^{t_k} \big(b_{hj''} - b_{h'j''}\big)(t,s)\,ds < 0$. Noting, by Property (1), that $\sum_j u_j^k \equiv 0$ we set

$$0 < \theta_k := \sum_{j'} u_{j'}^k = \sum_{j'} |u_{j'}^k| = \\ = -\sum_{j''} u_{j''}^k = \sum_{j''} |u_{j''}^k| = \frac{1}{2}\sum_j |u_j^k|.$$

and we have

$$z_h(t) - z_{h'}(t) = \sum_{k\geq 1}\sum_j \int_{t_{k-1}}^{t_k} (b_{hj} - b_{h'j})(t,s) z_j(s)\,ds =$$
$$= \sum_{k\geq 1}\sum_j u_j^k z_j(s_j^*(k))$$
$$\leq \sum_{k\geq 1} \theta_k \left(\frac{\sum_{j'} |u_{j'}^k| z_{j'}(s_{j'}^k)}{\theta_k} - \frac{\sum_{j''} |u_{j''}^k| z_{j''}(s_{j''}^k)}{\theta_k} \right)$$
$$\leq \left\{ \sum_{k\geq 1} \theta_k \right\} \big(\max_{k,i} z_i(s_i(k)) - \min_{k,i} z_i(s_i(k)) \big)$$
$$\leq \frac{1}{2}\max_{h,h'} \sum_j \int_{t-2\tau}^{t} \big|\big(b_{hj} - b_{h'j}\big)(t,s)\big|\,ds |W_{t-\tau-B}^{\mathbf{z}}|.$$

The last step is justified because of Lemma 7. We take now $\beta_{hj}(t) := \int_{t-\tau-B}^{t} b_{hj}(t,s)ds$ and in view of the identity $|x-y| = x+y-2\min\{x,y\}$, Property (2) and the fact that $h, h' \in V$ are fixed but arbitrary we obtain the contraction estimate:

$$S\big(\mathbf{z}(t)\big) \leq \left(1 - \min_{h,h'} \sum_{j=1}^{N} \min\{\beta_{hj}(t), \beta_{h'j}(t)\}\right) |W_{t-\tau-B}^{\mathbf{z}}|$$

From Assumptions 1 and 3 and the form of $B(t,s)$ we note that

$$\inf_{t\geq t_0+2\tau} \min_{h,h'} \sum_{j=1}^{N} \min\{\beta_{hj}(t), \beta_{h'j}(t)\} > \frac{\delta}{m}(1 - e^{-mB})$$

and the proof is complete.

The above Proposition is a novel use of an old tool of the theory of non-negative matrix, known as the coefficient of ergodicity adapted for the case of continuous time dynamics. See also Hartfiel (1998). It illustrates the contractive nature of stochastic matrices when they operate on vectors with respect to Δ. Here due to the presence of delays the contraction takes place over $W_t^{\mathbf{z}}$ and the analysis will conclude when an upper bound of $|W_t^{\mathbf{z}}|$ will be established. This is the topic of the next section.

5. MAIN RESULTS

The stability analysis of the present section relies on the rate at which $|W_t^{\mathbf{z}}|$ contracts. We begin with the linear consensus model and conclude with the nonlinear flocking model.

5.1 Convergence of (4).

Theorem 9. Let Assumptions 1,3 and 4 hold. Fix $B > 0$. The solution $\mathbf{z} = \mathbf{z}(t, t_0, \boldsymbol{\phi})$, $t \geq t_0$ of (4), satisfies

$$\max_i |z_i(t) - k| \leq \frac{|W_{t_0}^{\boldsymbol{\phi}}|}{1 - \rho e^{-D\tau}} e^{-\gamma t} \tag{11}$$

where $\gamma = -\frac{\ln(1-\rho e^{-D\tau})}{2\tau + B}$, $k \in W_{t_0}^{\boldsymbol{\phi}}$ and D as in (5).

Proof. Fix t and consider the set $W_t^{\mathbf{z}}$. Let i, j and t_1, t_2 be the indeces and the times on the solution segment such that $|W_t^{\mathbf{z}}| = z_i(t_1) - z_j(t_2)$. Assume without loss of generality that $t_1 \geq t_2$. Then, recalling Lemma 7, we work as follows:

$$|W_t^{\mathbf{z}}| =$$
$$= e^{-\int_{t_2}^{t_1} d_i(s)\,ds}\big(z_i(t_2) - z_j(t_2)\big) +$$
$$+ \int_{t_2}^{t_1} e^{-\int_s^{t_1} d_i(w)\,dw} \sum_l a_{il}(s)\big(z_i(\lambda_{il}(s)) - z_j(t_2)\big)\,ds$$
$$\leq e^{-\int_{t_2}^{t_1} d_i(s)\,ds} S(\mathbf{z}(t_2)) +$$
$$+ \int_{t_2}^{t_1} e^{-\int_s^{t_1} d_i(w)dw} d_i(s)ds |W_{t-2\tau}^{\mathbf{z}}|$$
$$\leq e^{-\int_{t_2}^{t_1} d_i(s)ds}(1-\rho)|W_{t_2-\tau-B}^{\mathbf{z}}| +$$
$$+ (1 - e^{-\int_{t_2}^{t_1} d_i(s)\,ds})|W_{t-2\tau}^{\mathbf{z}}|$$
$$\leq \left[e^{-\int_{t_2}^{t_1} d_i(s)\,ds}(1-\rho) + (1 - e^{-\int_{t_2}^{t_1} d_i(s)ds}) \right] |W_{t-2\tau-B}^{\mathbf{z}}|$$
$$\leq \big(1 - \rho e^{-\int_{t_2}^{t_1} d_i(s)ds}\big)|W_{t-2\tau-B}^{\mathbf{z}}|$$
$$\leq \big(1 - \rho e^{-D\tau}\big)|W_{t-2\tau-B}^{\mathbf{z}}|$$

Then for any $t \geq t_0 + B + 2\tau$, there exists $n \geq 1$ such that $t_0 + n(B+2\tau) \leq t \leq t_0 + (n+1)(B+2\tau)$ and so a recursive argument yields the estimate (11) which proves asymptotic exponential convergence.

Now, from Lemma 7 the solutions are bounded and the forward limit set exists with all the nice properties known from the theory of functional differential equations, (see Hale and Verduyn Lunel (1993)), among which the one of interest is that any solution that starts at t_0 from this set will have with initial data $\boldsymbol{\phi}$, that satisfy $|W_{t_0}^{\mathbf{z}}| = 0$ and hence any solution from the forward limit set must satisfy $\dot{z}_i \equiv 0$, equivalently $z_i(t) \equiv k$ for any $i \in V$ and the proof is complete.

5.2 Convergence of (6).

Proving convergence of the flocking network according to Def. (5) occurs if one can provide conditions so that $S\big(\mathbf{x}(t)\big)$ remains bounded. Then Theorem 9 takes over to prove flocking. Following Somarakis and Baras (2015) we implement a type of Lyapunov functionals firstly introduced in Ha and Liu (2009), appropriately modified for the delayed case. Indeed, from Assumption 6 $f(y)$ substitutes δ and consequently $\rho = \frac{1-e^{-mB}}{m} f(S(\mathbf{x}))$. The stability result is stated as follows:

Theorem 10. Let Assumptions 3, 4 and 6 hold. Consider the solution $(\mathbf{x}, \mathbf{u})$ of (6). Then asymptotic flocking occurs according to Definition 5, if the initial data satisfy

$$|W_{t_0}^{\boldsymbol{\phi}}| < \frac{1-e^{-mB}}{m(2\tau+B)} e^{-D\tau} \int_{P_{\mathbf{x}^0,\boldsymbol{\phi}}^{2\tau+B}}^{\infty} f(s)\, ds \qquad (12)$$

where $P_{\mathbf{x}^0,\boldsymbol{\phi}}^{2\tau+B} := \max\{S(\mathbf{x}^0), |S(\mathbf{x}^0) - |W_{t_0}^{\boldsymbol{\phi}}|(2\tau+B)|\}$ and D as defined in (5).

Proof. At first we note that $\frac{d}{dt} S\big(\mathbf{x}(t)\big) \le S\big(\mathbf{u}(t)\big) \le |W_t^{\mathbf{u}}|$ and we introduce the Lyapunov functional

$$V(\mathbf{x}, \mathbf{u}) = \int_{t-2\tau-B}^{t} |W_s^{\mathbf{u}}|\, ds + \\ + \frac{1-e^{-mB}}{m} e^{-D\tau} \int_0^{S(\mathbf{x})} f(s)\, ds.$$

We evaluate it along $\mathbf{x}, \mathbf{u}$ and take the time derivative to obtain

$$\frac{d}{dt} V(t) \le 0,$$

in view of the contraction estimate obtained in the proof of Theorem 9. Then we have that $V(t) \le V(2\tau + B)$ for $t \ge t_0 + 2\tau + B$

$$\int_{t-2\tau-B}^{t} |W_s^{\mathbf{u}}|\, ds + \frac{1-e^{-mB}}{m} e^{-D\tau} \int_0^{S(\mathbf{x}(t))} f(s)\, ds \le$$

$$\int_0^{2\tau+B} |W_s^{\mathbf{u}}|\, ds + \frac{1-e^{-mB}}{m} e^{-D\tau} \int_0^{S(\mathbf{x}(2\tau+B))} f(s)\, ds \qquad (13)$$

Assume the condition:

$$\int_0^{2\tau+B} |W_s^{\mathbf{u}}|\, ds < \frac{1-e^{-mB}}{m} e^{-D\tau} \int_{S(\mathbf{x}(2\tau+B))}^{\infty} f(s)\, ds \qquad (14)$$

Then we can pick q such that

$$\int_0^{2\tau+B} |W_s^{\mathbf{u}}|\, ds = \frac{1-e^{-mB}}{m} e^{-D\tau} \int_{S(\mathbf{x}(2\tau+B))}^{q} f(s)\, ds \qquad (15)$$

Substituting (15) into (13) we get

$$\int_0^{S(\mathbf{x}(t))} f(s)\, ds \le \int_{S(\mathbf{x}(2\tau+B))}^{q} f(s) ds + \int_0^{S(\mathbf{x}(2\tau+B))} f(s)\, ds$$

so that if $S(\mathbf{x}(t)) \ge S(\mathbf{x}(2\tau+B))$ then necessarily $S(\mathbf{x}(t)) \le q$. Hence $\sup_t S(\mathbf{x}(t)) < \infty$ throughout the solution and the exponentially fast alignment of the flock velocity is achieved. It is only left to show that the imposed condition (12) implies (14). Indeed, on the one hand the left part of the inequality in (14) is upper bounded by $|W_{t_0}^{\boldsymbol{\phi}}|(2\tau+B)$. On the other hand, unless $S(\mathbf{x}(t)) \le S(\mathbf{x}^0)$, the rate at which $S(\mathbf{x}(t))$ may shrink can be deduced from the extreme scenario $\mathbf{x}^0 = (x^0, 0, \dots, 0)$ with $x^0 \ne 0$ so that $S(\mathbf{x}^0) = 0$ and $\boldsymbol{\phi}(s) = (\phi_1(s), 0, \dots, 0)$, $s \in [-\tau, 0]$. Neglecting the averaging effect which will inevitably diminish $|W_t^{\mathbf{u}}|$, $x^0 < 0$ implies that the first agent will have approached or bypassed the rest of the group by $-|x^0| + |u^0|t$. Finally, at $t = 2\tau + B$ the spread of $\mathbf{x}(2\tau+B)$ is lower bounded by

$$S(\mathbf{x}(2\tau+B)) \ge \max\{S(\mathbf{x}^0), |S(\mathbf{x}^0) - |W_{t_0}^{\boldsymbol{\phi}}|(2\tau+B)|\} = \\ = P_{\mathbf{x}^0,\boldsymbol{\phi}}^{2\tau+B}$$

and the proof is concluded.

The following corollary is a straightforward application of Theorem 10:

Corollary 11. Consider the initial value problem (6) and its solution $(\mathbf{x}, \mathbf{u})$. Under Assumptions 3, 4 and 6, unconditional flocking occurs according to Definition (5) if:

$$\int^{\infty} f(s)\, ds = \infty$$

where $f(\cdot)$ is as in Assumption 6.

6. DISCUSSION

In this paper we addressed the problem of the effect of delays in distributed consensus systems. The delays considered are time-varying and of propagation type. It is assumed that the signal that agent i receives on the state of agent j arrives with some delay. Convergence results have already been established showing that the effect of propagation delays does not cause any problem in the asymptotic stability. The reader is referred to Moreau (2004); Papachristodoulou et al. (2010). However, those methods relied on Invariance Principles which provide no estimate on the rate of convergence. This defect made the study of delayed flocking networks of Cucker-Smale type (i.e. the initial value problem (6)) impossible. For the latter type of systems it is necessary for the researcher to establish an explicit estimate on the rate that is a mathematically tractable function of the systems coupling and delayed parameters, so that the Luapunov functional method can be implemented as this in the proof of Theorem 10.

In recent years the authors studied the effect of delays on the rate of convergence in consensus systems using Fixed Point Theory methods (e.g. Burton (2006); Somarakis et al. (2014b); Somarakis and Baras (2013b,c,a); Somarakis et al. (2014a)) but the majority of these results suffered from delayed dependent conditions. Here we improve these results using a different approach. The price to pay is that the estimate is much weaker and the interested reader is refereed to a relevant discussion for a qualitative similarly scalar case in Somarakis et al. (2014a).

If one would categorize our method, the fact that we examine the contraction on sets of functions puts our results into these of Lyapunov-Razumikhin type, Hale and Verduyn Lunel (1993).

Let us at this point to make a small remark on this Assumption 4 as its contribution is two-fold: The delays are assumed uniformly bounded, a feature that preserves the exponential speed of convergence. A relaxation to unbounded case may be achieved with considerable effort and on condition that $a_{ij}(t)$ vanish as t gets large. Then the speed is likely to be degraded to sub-exponential Somarakis et al. (2014a). Moreover, the smallness on the growth of $\dot{\tau}$ is a standard feature in the theory of delayed equations and, it allows $\lambda_{ij}(t)$ to be invertible and it is imposed mainly for consistency reasons Burton (2006).

For the flocking model we imposed delays on the observed velocity of the neighboring agents only and not on the observed position. We assumed that that each bird, although receives instant information on the position of its fellow neighbors, the information for the speed is delayed. This is not an unrealistic claim as, in general, it is much easier to estimate the position of a moving objects than its velocity. The latter quantity can be subjected to environmental factors which allow only a distorted or, in our case, delayed version of it.

All the above results can be generalized to simply connected populations of agents or even switching signals just as in Somarakis and Baras (2015). Such a setup is to be developed for the journal version of this work.

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