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Non-linear Flocking Networks with Collision Avoidance

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Abstract: We consider a second order non-linear consensus (flocking) network of a finite population of autonomous agents and prove that the long term behavior of its solution is towards a common value while the flock remains connected. We elevate the analysis to a collision avoidance type flocking after taking into consideration repelling forces between agents.

Key Words: Second order consensus, asymmetric coupling weights, flocking solutions, collision avoidance

1 Introduction

Dynamics of collectives is a perpetually interesting field of the applied science. Autonomous agents that form global patterns out of local interactions is a concept generally accepted to model living entities that cooperate towards a common goal [1, 15]. In mathematical modeling this problem has been considered with the introduction and study of distributed asymptotic consensus algorithms the literature of which is enormous. As only a brief introduction to the subject the interested reader is referred to [4, 5, 7, 9, 10, 12].

Recent results in the field of linear consensus dynamical systems [13, 14] shed light to some of the few remaining unexplored aspects of these algorithms and have motivated this technical paper in which we introduce and discuss the dynamics of two closely related non-linear collective algorithms regarding them as new members of the large family of flocking networks.

1.1 Related literature & contribution

Tne flocking networks are well-known in the literature [4, 5, 12, 15]. The standard framework consists of a finite number N of birds which exchange information according to the following scheme:

$$i \in \{1, \dots, N\} : \begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = \sum_j a_{ij}(\mathbf{x})\big(u_j - u_i\big) \end{cases} \tag{0}$$

The main objective is to derive initial conditions so that $\lim_{t\to\infty} |u_i(t) - k| = 0$, for some $k \in \mathbb{R}$, while at the same time $\sup_t |x_{ij}(t)| := |x_i(t) - x_j(t)| < \infty$. This is the very well-known asymptotic flocking definition introduced in [4, 5]. With the exception of [12], the rigorous investigation of the long-term behavior of the solutions of these models ask for symmetrical communication weights (i.e. $a_{ij} = a_{ji}$) so that tools from Algebraic Graph Theory apply. As symmetry is generally accepted to be an unrealistic mathematical abstraction [1], the problem remained open for networks of type (0) until a very interesting approach was discussed in [12]. The main defects of that work are that it required increased connectivity as well as smallness of the communication weights. In [13, 14] the authors followed a different implementation of fundamental concepts of Non-Negative matrix theory [6] and they managed to extend the convergence results so as to include asymmetric communications while at the same time they substantially relaxed the connectivity conditions to the mildest possible improving the results of [12]. In the case of collision avoidance flocking it is additionally asked that $|x_i(t)-x_j(t)| > d, t \geq t_0$ for some fixed $d > 0$. To the best of our knowledge, there is only one framework along this direction with dynamics of the type (0) and it is presented in [3]. The analysis relies in the symmetry of both the weights and the repelling forces that act on the dynamics preventing the agents from approaching one another.

In the spirit of [13, 14], the present technical paper continues the investigation by increasing the complexity of 0 by two levels. The first system we introduce is a second order consensus (flocking) algorithm with state-dependent coupling forces and non-linear observations of the agents' states. The second system is an elevation of the first and considers the case of collision avoidance. We provide conditions under which the agents achieve asymptotic speed alignments without collision between them whereas the communication is not vanished since the flock keeps communicating and exchanging information (i.e. it remains connected).

The core of the mathematical arguments deviate from the conventional methods used in the aforementioned literature. Initially we follow the spirit of [13] but in the second part we, employ a novel stability in variation argument and prove asymptotic convergence of the collision avoiding model by means of fixed point theory. We produce a number of new conditions between the initial values of the problem and its parameters which ensure exponential convergence to a common value.

1.2 Organization

This paper is structured as follows. In § 2 basic notations, definitions and related underlying theorems are stated. In § 3 we state the two models, we describe the sets of Assumptions that are to be used in the analysis to follow as well as the stability definitions of interest. In addition, we provide a series of instrumental preliminary results on which the proofs of the main two theorems rely. In § 4 we state and prove the two results of this work in the form of theorems whereas in § 5 we conduct a discussion about the implications of the derived results, the advantages and disadvantages of the followed methodology and we conclude with a number of generalizations for future work.

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2 Notations & Definitions

Henceforth $N < \infty$ denotes the number of autonomous agents and $[N] = \{1, \dots, N\}$. Each agent $i \in [N]$ is defined through the pair of position and velocity (x_i, u_i) or (z_i, v_i) for each model respectively so that both $(\mathbf{x}, \mathbf{u})$ and $(\mathbf{z}, \mathbf{v}) \in \mathbb{R}^N \times \mathbb{R}^N$. By $|\cdot|$ we denote any appropriate norm on $\mathbb{R}^N$ and for any $\mathbf{y} \in \mathbb{R}^N$ its *spread* is defined as

$$S(\mathbf{y}) = \max_i y_i - \min_i y_i.$$

We remark that S is always non-negative and satisfies the triangular inequality but it is only a pseudo-norm on $\mathbb{R}^N$ since $S(\mathbf{y}) = 0$ if and only if $y_1 = \cdots = y_N$. By $\mathbb{1}$ we understand the column vector of all ones in $\mathbb{R}^N$. Occasionally for $\mathbf{y} \in \mathbb{R}^N$ we will use the notation $y_{ij} = y_i - y_j$. As the analysis will include derivatives of $S(\mathbf{y})$ the non-smoothness of the spread suggests the use of generalized type of derivatives. Hence, all the time-derivatives $\frac{d}{dt}$ are assumed to be right-Dini derivatives. Furthermore the gradient $\nabla S(\mathbf{y})$ denotes a row vector with 1 at the place of the maximum element of $\mathbf{y}$, -1 at the place of the minimum element of $\mathbf{y}$ and zero elsewhere.

A square $N \times N$ matrix $P = [p_{ij}]$ is said to be m-stochastic, if $p_{ij} \geq 0$ for all i, j and $\sum_i p_{ij} \equiv m$.

By $C^r(I, B)$ we denote the space of functions defined on I and taking values on B with $r \geq 0$ continuous derivatives. By $\mathbb{B}$ we understand the space of continuous, bounded functions defined on $\mathbb{R}$ and taking values on $\mathbb{R}$. This space is endowed with the supremum norm so that for any $\phi \in \mathbb{B}$, $\sup_{t\in\mathbb{R}} |\phi(t)| < \infty$. It is well known that $(\mathbb{B}, |\cdot|)$ defines a metric space and any subset $\mathbb{M}$ of $\mathbb{B}$ is compact if each sequence $\{\phi_n\} \in \mathbb{M}$ has a sub-sequence with limit in $\mathbb{M}$. Also a set $\mathbb{M}$ is convex if for any $\phi_1, \phi_2 \in \mathbb{M}$ it follows that $\alpha\phi_1 + (1-\alpha)\phi_2 \in \mathbb{M}$ for any $\alpha \in [0, 1]$. The following result is known in the literature as Schauder's first fixed point theorem [2]:

Theorem 2.1 *Let $\mathbb{M}$ be a non-empty compact convex subset of a Banach space and let $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ be continuous. Then $\mathcal{P}$ has a fixed point in $\mathbb{M}$.*

The stability definition of interest is this of asymptotic flocking introduced in [5]. Let $(\mathbf{y}, \mathbf{w})$ be a solution of either of the flocking systems to be introduced in § 3 so that it exists in the large.

Definition 2.2 *We say that the solution $(\mathbf{y}, \mathbf{w})$ exhibits asymptotic flocking if*

$$S(\mathbf{w}(t)) \to 0 \text{ as } t \to \infty \qquad \& \qquad \sup_t S(\mathbf{y}(t)) < \infty$$

The objective of the agents is to align their speed fast enough so that their distances remain finite. It should be noted here that the nature of the schemes that are to be discussed favor a convergence to a common fixed value. This means that the first part of the Def. 2.2 translates to

$$\mathbf{w}(t) \to \mathbb{1}k$$

for some $k \in \mathbb{R}$. It will be remarked that the proposed algorithms sustain only such types of solutions .

3 The Models

In this section, we will state the evolution equations and the accompanying hypotheses together with a couple of introductory remarks. Additionally, we prove a number of preliminary results to be used in § 4.

3.1 The primary model

The first scheme to be studied consists of N autonomous agents each of which is defined through the pair (x_i, u_i). The velocity coordination is achieved through the set of equations

$$i \in [N] : \begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = \sum_j a_{ij}(\mathbf{x})\big(g_{ij}(u_j) - g_{ij}(u_i)\big) \end{cases} \tag{1}$$

with initial data $\mathbf{x}^0 = \big(x_1(t_0), \dots, x_N(t_0)\big)$, $\mathbf{u}^0 = \big(u_1(t_0), \dots, u_N(t_0)\big)$. In vector form we will use the representation

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}\mathbf{u}\\ \mathbf{C}(\mathbf{x}, \mathbf{u})\end{pmatrix} \tag{1}$$

The system (1) is a non-linear alternative of flocking networks introduced in the literature, [4, 12, 13]. Recent results in this type of models allow for further generalizations [13, 14]: In particular, (1) does not ask for a linear type of averaging of states between agents any more. Here each agent has a different, essentially non-linear, perception of the velocity it receives after interacting with the rest of the population. Similar first order consensus networks were studied in [11] under unreasonably strong assumptions. The conditions we impose on our model are:

Assumption 3.1 $\forall i \neq j \in [N]$, $a_{ij}(t, \mathbf{y}) : (\mathbb{R}^N \to [0, a]$ *integrable, for some $a < \infty$ such that*

$$a_{ij}(t, \mathbf{y}) \geq f\big(S(\mathbf{y})\big)$$

for some integrable, non-increasing function f with the property that $\lim_{z\to\infty} f(z) = 0$.

Assumption 3.2 $\forall i \neq j \in [N]$ *and fixed* $W \subset \mathbb{R}$ *there exist numbers* $0 < \underline{c} \leq \overline{c}$ *that depend on* W *such that:*

$$\underline{c} \leq \frac{g_{ij}(x) - g_{ij}(y)}{x - y} \leq \overline{c}, \qquad \forall x, y \in W$$

It is easy to understand that the larger the spread of the flock is, the weaker the coupling communication rate becomes. This is the actual difficulty in these models as the challenge is to derive sufficient conditions for flocking in the sense of Def. 2.2. We will see that this condition will be a formula that combines the initial data and the function f. Also, for the sake of simplicity, we effectively assumed the communication graph to be fully connected. This strong connectivity condition may be relaxed along the lines explained in [13, 14].

3.2 The collision free model

The flocking structure with colision avoidance is described via the following set of equations

$$i \in [N] : \begin{cases} \dot{z}_i = v_i \\ \dot{v}_i = \sum_j a_{ij}(t, \mathbf{z})\big(g_{ij}(v_j) - g_{ij}(v_i)\big) + \\ \quad +V(|z_{ij}|^2)J(z_{ij})\big|g_{ij}(v_j) - g_{ij}(v_i)\big|^{3/2} \end{cases} \tag{2}$$

with initial data $\mathbf{z}^0 = \big(z_1(t_0), \dots, z_N(t_0)\big)$, $\mathbf{v}^0 = \big(v_1(t_0), \dots, v_N(t_0)\big)$ and $J(z_{ij}) = \operatorname{sgn}(z_{ij})\sqrt{|z_{ij}|}$. In vector form

$$\frac{d}{dt}\begin{pmatrix}\mathbf{z}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \mathbf{C}(\mathbf{x},\mathbf{v}) + \mathbf{R}(\mathbf{x},\mathbf{v})\end{pmatrix} \tag{2}$$

with $\mathbf{R}$ standing for the repelling forces term. For the analysis of this system, together with Assumption 3.1 we shall need a slightly harder assumption on the g_{ij} functions.

Assumption 3.3 *$\forall i \neq j \in [N]$ it holds that $g_{ij} \in C^1(\mathbb{R},\mathbb{R})$ such that $g'_{ij} \in [\underline{c}, \overline{c}]$ for some $0 < \underline{c} \leq \overline{c} < \infty$.*

The extra term in (2) is the repelling force. Under the following condition the agents' positions are to remain away from each other by at least a prescribed distance:

Assumption 3.4 *There exists $d > 0$ such that the functions $V \in C^0((d,\infty],\mathbb{R}_+)$ and for all $d' > d$:*

$$\int_d^{d'} V^2(s)ds = \infty \qquad \& \qquad \int_{d'}^{\infty} V^2(s)ds < \infty \tag{3}$$

The purpose is for the agents not to approach each other a distance smaller than d. Examples of such repelling functions are provided in [3].

3.3 Preliminaries

The first technical proposition is an adaptation of a well known result on the effect of m-stochastic matrices [6].

Proposition 3.5 *Let $\underline{P} = [\underline{p}_{ij}]$ and $\overline{P} = [\overline{p}_{ij}]$ be two m-stochastic matrices and $\mathbf{z},\mathbf{y} \in \mathbb{R}^N$. If $\underline{P}\mathbf{y} \leq \mathbf{z} \leq \overline{P}\mathbf{y}$, then*

$$S(\mathbf{z}) \leq \big(m - \min_{h,h'\in[N]} \sum_j \min\{\underline{p}_{hj}, \overline{p}_{h'j}\}\big) S(\mathbf{y}).$$

Proof For fixed $h,h' \in [N]$, $z_h - z_{h'} \leq \sum_j p_j y_j$ with $p_j := \overline{p}_{hj} - \underline{p}_{h'j}$. Let j',j'' denote the indices in $[N]$ such that $p_{j'} > 0$ and $p_{j''} < 0$ and note that $\sum_j p_j \equiv 0$. Set $0 < \theta := \sum_{j'} p_{j'} = \sum_{j'} |p_{j'}| = -\sum_{j''} p_{j''} = \sum_{j''} |p_{j''}| = \frac{1}{2}\sum_j |p_j| = \frac{1}{2}\sum_j |p_j| = \frac{1}{2}\sum_j |\overline{p}_{h'j} - \underline{p}_{hj}|$ and see that for appropriate $h,h' \in [N]$

$$S(\mathbf{z}) = z_h - z_{h'} = \theta\bigg(\frac{\sum_{j'} p_{j'}y_{j'}}{\sum_{j'} p_{j'}} - \frac{\sum_{j''} p_{j''}y_{j''}}{\sum_{j''} p_{j''}}\bigg) \leq \theta S(\mathbf{y})$$

and the expression for $\theta = m - \min_{h,h'\in[N]} \sum_j \min\{\underline{p}_{hj}, \overline{p}_{h'j}\}$ can be obtained in view of the identity $|\alpha - \beta| = \alpha + \beta - 2\min\{\alpha,\beta\}$ and the fact that $\underline{P}$ and $\overline{P}$ are m-stochastic.

Henceforth, we set $\rho := \min_{h,h'\in[N]} \sum_j \min\{\underline{p}_{hj}, \overline{p}_{h'j}\}$.

3.3.1 Bounds on the solutions

We will briefly discuss two important remarks on the solutions $(\mathbf{x},\mathbf{u})$ and $(\mathbf{z},\mathbf{v})$ of (1) and (2) respectively. Based on Assumption 3.2 we have a valuable result regarding the bounds of $\mathbf{u}$:

Lemma 3.6 *Let the Assumptions 3.1 and 3.2 hold. Then the solution $(\mathbf{x},\mathbf{u})$ of (1) satisfies*

$$\min_{j\in[N]} u_j(t_0) \leq u_i(t) \leq \max_{j\in[N]} u_j(t_0), \qquad i \in [N]$$

Proof Let $t^* \geq t_0$ be the first time that for some $i \in [N]$, u_i is to escape the aforementioned interval, say to the right. Then it must hold $u_i(t^*) = \max_{j\in[N]} u_j(t_0)$ and $\dot{u}_i(t^*) > 0$. This set of conditions is incompatible with the dynamics of (1) in view of Assumption 3.2. Hence, the solution does not escape the interval to the right. A similar argument can be made for the lower bound $\min_{j\in[N]} u_j(t_0)$ and the proof is complete.

The above Lemma answers in the affirmative on the existence of $(\mathbf{x},\mathbf{u})$ in the large but it seizes to hold for (2). This is of course not the only issue regarding the collision free model. The imposed discontinuity with respect to $\mathbf{z}$ sets questions not only on the existence of a solution for all times (a prerequisite of stability) but also on its uniqueness. Assumption 3.4 reassures us that not only solutions are unique but, wherever they exist, appropriate agents' initial positions ensure collision-less evolution.

Lemma 3.7 *Let the Assumptions 3.1, 3.3 and 3.4 to hold. Consider the solution $(\mathbf{z},\mathbf{v})$ of (2) to exist on the interval $[t_0,T)$. Then $\min_{i\neq j} |z_i(t_0) - z_j(t_0)| > d$ implies that the solution is unique with $|z_i(t) - z_j(t)| > d$ for $t \in [t_0,T)$ and $i \neq j$.*

Proof By Assumption 3.4 $V(s) \to +\infty$ as $s \downarrow d$. The initial position configuration implies the existence of an enumeration $i_1, i_2, \dots, i_N$ such that $\{z_{i_l}(t_0)\}_{l\geq 1}$ are in strictly ascending order with distance at least d. Let $z_{i_l}(t_0) < z_{i_{l+1}}(t_0)$. For the $J(z_{i_l i_{l+1}})$ to change sign we must necessarily have a collision, i.e. $\{t_n\} \in [t_0,T)$ such that $z_{i_{l+1}}(t_n) - z_{i_l}(t_n) \downarrow d$ as $n \to \infty$. This case implies, in turn, the following scenario: there exists $\varepsilon > 0$ arbitrarily small such that

$$d - \varepsilon < z_{i_{l+1}}(t) - z_{i_l}(t) < d, \; t \in (t_1,t_2)$$

for some $t_0 \leq t_1 < t_2 < T$ such that $\underline{u} := \inf_{s\in(t_1,t_2)} |u_{i_{l+1}}(s) - u_{i_l}(s)| > 0$ (as agent i_l approaches i_{l+1} from below). We choose $\varepsilon > 0$ so small so that

$$V \geq \frac{\Xi + (N-1)\overline{c}\sup_{s\in(t_1,t_2)} S(\mathbf{u}(s))}{\sqrt{d}\,\underline{c}\,\underline{u}^{3/2}}$$

where $\Xi = \frac{d-\varepsilon-z_{i_{l+1}i_l}(t_1)-(t_2-t_1)u_{i_{l+1}i_l}(t_1)}{(t_2-t_1)}$. This gives an estimate on $\dot{u}_{i_{l+1}i_l} > \Xi$ throughout (t_1,t_2) and a direct calculation yields $z_{i_{l+1}i_l}(t_2) = z_{i_{l+1}i_l}(t_1) + (t_2 - t_1)v_{i_{l+1}i_l}(t_1) + \int_{t_1}^{t_2} \dot{v}_{i_{l+1}i_l}(s)\,ds > d - \varepsilon$ which contradicts the condition in (3.3.1), concluding the proof.

The aftermath of Lemma 3.7 is that the sign function depends only on the initial configuration and plays no other role throughout the solution of (2). This means that $(\mathbf{z},\mathbf{v})$ is uniquely determined in the interval of its existence which is, however, yet to be proved that it extends for all $t \geq t_0$. As a final remark we state without proof that if either $\mathbf{u}$ or $\mathbf{v}$ are bounded and satisfy $S\big(\mathbf{u}(t)\big), S\big(\mathbf{v}(t)\big) \to 0$ as $t \to \infty$, then it holds that $\mathbf{u}(t), \mathbf{v}(t)$ converge to a common value. Either of the forward limit sets are non-empty and each point $\boldsymbol{v}$ of which necessarily satisfies $S(\boldsymbol{v}) = 0$ so any solution starting from this set will remain constant.

4 Main Results

This section states and proves the pair of theorems that consist the contribution of this paper. The first is a convergence result of (1) and the second is a convergennce result of (2).

4.1 Convergence of the primary model

In the proof of the following theorem the argument of [5] was adapted.

Theorem 4.1 *Let the Assumptions 3.1 and 3.2 hold. The solution* $(\mathbf{x}, \mathbf{u})$ *of* (1) *exhibits asymptotic flocking if the initial conditions satisfy:*

$$S(\mathbf{u}^0) < N\underline{c}(\mathbf{u}^0)\int_{S(\mathbf{x}^0)}^{\infty} f(s)\, ds$$

for $\underline{c}(\mathbf{u}^0)$ *in the sense of Assumption 3.2.*

Proof At first, the initial velocity $\mathbf{u}^0$ defines the constants $\underline{c}$ and $\overline{c}$ from Assumption 3.2 and Lemma 3.6. Next it can be easily shown that throughout the solution $(\mathbf{x}, \mathbf{u})$

$$\sum_j \underline{b}_{ij}(t)\big(u_j(t)-u_i(t)\big) \leq \dot{u}_i(t) \leq \sum_j \bar{b}_{ij}(t)\big(u_j(t)-u_i(t)\big)$$

where:

1. $\underline{b}_{ij}(t) = \begin{cases} \underline{c}a_{ij}(t, \mathbf{x}(t)), & \text{if} \quad u_j(t) \geq u_i(t) \\ \overline{c}a_{ij}(t, \mathbf{x}(t)), & \text{if} \quad u_j(t) < u_i(t) \end{cases}$
2. $\bar{b}_{ij}(t) = \begin{cases} \overline{c}a_{ij}(t, \mathbf{x}(t)), & \text{if} \quad u_j(t) \geq u_i(t) \\ \underline{c}a_{ij}(t, \mathbf{x}(t)), & \text{if} \quad u_j(t) < u_i(t) \end{cases}$

for all $i \in [N]$. Next we pick $m > (N-1+\overline{c})a$ such that

$$\overline{P}(t)\mathbf{u}(t) \leq e^{-mt}\frac{d}{dt}\big(e^{mt}\mathbf{u}(t)\big) \leq \overline{P}(t)\mathbf{u}(t) \qquad (4)$$

where $\overline{P}(t) = [\overline{p}_{ij}(t)]$ with $p_{ij}(t) = \bar{b}_{ij}(t)$ and $p_{ii}(t) = \big(m - \sum_j \bar{b}_{ij}(t)\big)$ and similar for $\underline{P}(t)$. We are interested in obtaining an upper bound for $\frac{d}{dt}S(\mathbf{u})$:

$$\begin{aligned} \frac{d}{dt}S(\mathbf{u}) &= \frac{d}{dt}\left(e^{-mt}S\big(e^{mt}\mathbf{u}\big)\right) \\ &= -mS(\mathbf{u}) + e^{-mt}\frac{d}{dt}S(e^{mt}\mathbf{u}) \\ &\leq -mS(\mathbf{u}) + S\left(e^{-mt}\frac{d}{dt}(e^{mt}\mathbf{u})\right) \\ &\leq -mS(\mathbf{u}) + \big(m-\rho(t)\big)S(\mathbf{u}) = -\rho(t)S(\mathbf{u}) \end{aligned}$$

in view of Proposition 3.5. From the choice of m a direct calculation of $\rho(t)$ yields the lower bound $\rho(t) > N\underline{c}f\big(S(\mathbf{x}(t))\big)$ so that

$$\frac{d}{dt}S\big(\mathbf{u}(t)\big) \leq -N\underline{c}f\big(S(\mathbf{x}(t))\big)S\big(\mathbf{u}(t)\big) \qquad (5)$$

If $\sup_{t\geq t_0} S\big((t)\big) < S(\mathbf{x}^0)$ then from (5) we have

$$\frac{d}{dt}S\big(\mathbf{u}(t)\big) \leq -N\underline{c}f\big(S(\mathbf{x}^0)\big)S\big(\mathbf{u}(t)\big) \qquad (6)$$

i.e. exponential fast speed alignment and therefore flocking. Otherwise, consider the functional

$$W(\mathbf{x}, \mathbf{u}) = S(\mathbf{u}) + N\underline{c}\int_0^{S(\mathbf{x})} f(s)\, ds$$

the time-derivative of which along the solution $(\mathbf{x}, \mathbf{u})$ yields $\frac{d}{dt}W \leq 0$ in view of (5). Then from the imposed condition on the initial data there is r^* such that

$$S(\mathbf{u}^0) = N\underline{c}\int_{S(\mathbf{x}^0)}^{r^*} f(s)\, ds \qquad (7)$$

and since $W(t) \leq W(t_0)$ for $t \geq t_0$ we have

$$S\big(\mathbf{u}(t)\big) + N\underline{c}\int_0^{S(\mathbf{x}(t))} f(s)\, ds \leq S(\mathbf{u}^0) + N\underline{c}\int_0^{S(\mathbf{x}^0)} f(s)\, ds$$

and substituting from (7) we have that

$$0 \leq S\big(\mathbf{u}(t)\big) = N\underline{c}\bigg(\int_{S(\mathbf{x}^0)}^{r^*} + \int_0^{S(\mathbf{x}^0)} - \int_0^{S(\mathbf{x}(t))} f(s)\, ds\bigg)$$

from which it is deduced that $\int_{S(\mathbf{x}(t))}^{r^*} f(s)\, ds > 0$ and this implies that $S(\mathbf{x}(t)) \leq r^*$ a valuable upper bound for the range of the flock so that again from (5)

$$\frac{d}{dt}S\big(\mathbf{u}(t)\big) \leq -N\underline{c}f(r^*)S\big(\mathbf{u}(t)\big)$$

and that exponential speed alignment implies

$$S(\mathbf{x}(t)) \leq S(\mathbf{x}^0) + \frac{S(\mathbf{u}^0)}{N\underline{c}f(r^*)} < \infty$$

and the proof is complete.

4.2 Convergence of the collision-free model

In this section, we study the asymptotic behavior of the solution $(\mathbf{z}, \mathbf{v})$. It is reminded that the preliminary analysis does not shed light upon the existence of the solution in the large. The first step is to express the solution using a stability in variation technique. Then we create a fixed point theory argument by applying Theorem 2.1 in an appropriately defined subset of $\mathbb{B}$. In this subset, sufficient conditions for both existence in the large and stability according to Def. 2.2 are established. Based on Theorem 4.1, we will express the solution $\mathbf{v}$ in terms of $\mathbf{u}$ in the Lyapunov like form of $S(\mathbf{v})$. Our approach closely follows the methodologies presented in [8]. Then we will make a fixed point theory argument via Schauder's first fixed point theorem. We shall silently neglect the positions states $\mathbf{x}(t)$ or $\mathbf{z}(t)$ and this is due to the fact that, given the initial positions, they both depend on $\mathbf{u}$ and $\mathbf{v}$, respectively, hence they are effectively functions of $\mathbf{u}$ and $\mathbf{v}$. Contrary to the presentation followed for Theorem 4.1 here the proof precedes the statements of the result.

Fix $t_0 \leq s \leq t$ and consider the solution $\mathbf{u}\big(t, s, \mathbf{v}(s)\big)$, that is the solution $\mathbf{x}, \mathbf{u}$ that begins at s with initial data $\mathbf{x}^0, \mathbf{v}(s)$. Then differentiating $l(s) := S\big(\mathbf{u}(t, s, \mathbf{v}(s))\big)$ with respect to s we obtain

$$\frac{d}{ds}l(s) = \nabla S\left[\frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial s} + \frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial \boldsymbol{\xi}}\dot{\mathbf{v}}(s)\right]$$

where $\nabla S = \nabla S\big(\mathbf{u}(t, s, \mathbf{v}(s))\big)$ is a vector full of zeros apart from 1 at the spot of the maximum and -1 at the spot of the minimum of the vector $\mathbf{u}(t, s, \mathbf{v}(s))$, $\frac{\partial \mathbf{u}(t,s,\mathbf{v}(s))}{\partial \boldsymbol{\xi}}$ is the principal matrix solution of the linear system

$$\dot{\mathbf{y}} = A(t)\mathbf{y} \qquad (8)$$

with

$$A(t) = \begin{bmatrix} -\sum_j a_{1j} g'_{1j} & \cdots & a_{1N} g'_{1N} \\ a_{21} g'_{21} & \cdots & a_{2N} g'_{2N} \\ \vdots & \cdots & \vdots \\ a_{N1} g'_{N1} & \cdots & -\sum_j a_{Nj} g'_{Nj} \end{bmatrix}$$

evaluated at the solution $(\mathbf{x}(t), \mathbf{u}(t))$ of (1) starting at s with initial conditions $\mathbf{v}(s)$ and $\mathbf{z}(s) = \mathbf{z}^0 + \int_{t_0}^s \mathbf{v}(q)\, dq$. Since

$$\frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial s} = -\frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial \boldsymbol{\xi}} \mathbf{C}\big(s, \mathbf{v}(s)\big)$$

then for $\mathbf{v}(s)$ to satisfy (2) we conclude that

$$\frac{d}{ds} l(s) = \nabla S \frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial \boldsymbol{\xi}} \mathbf{R}(\mathbf{z}(s), \mathbf{v}(s))$$

so integrating from t_0 to t, we get the following expression for the solution $\mathbf{v}$ of (2)

$$S\big(\mathbf{v}(t, t_0, \mathbf{v}^0)\big) = S\big(\mathbf{u}(t, t_0, \mathbf{v}^0)\big) + \\ + \int_{t_0}^t \nabla S \frac{\partial \mathbf{u}}{\partial \boldsymbol{\xi}}\big(t, s, \mathbf{v}(s)\big) \mathbf{R}\big(\mathbf{z}(s), \mathbf{v}(s)\big)\, ds \tag{9}$$

Now it is important to understand how the stability of $\frac{\partial \mathbf{u}}{\partial \boldsymbol{\xi}}$ actually depends effectively on the initial data $\mathbf{z}(s)$ and $\mathbf{v}(s)$. This takes us back to Theorem 4.1 as (8) is a special case of (1). Indeed, if

$$S(\mathbf{v}(s)) < N\underline{c} \int_{S(\mathbf{z}(s))}^{\infty} f(q)\, dq \tag{10}$$

then the principal matrix solution of linearized system satisfies

$$\left| \frac{\partial \mathbf{u}(t, s, \mathbf{v}(s))}{\partial \boldsymbol{\xi}} - \mathbb{1}\mathbb{1}^T \right| = e^{-\psi(t-s)} \tag{11}$$

for $\psi = N f(\tilde{r}) \underline{c} > 0, \tilde{r} : S(\mathbf{v}(s)) = \int_{S(\mathbf{z}(s))}^{\tilde{r}} f(s)\, ds$. Let us focus on the integral $\int_{t_0}^t \nabla S \frac{\partial \mathbf{u}}{\partial \boldsymbol{\xi}}\big(t, s, \mathbf{v}(s)\big) \mathbf{R}\big(\mathbf{z}(s), \mathbf{v}(s)\big)\, ds$ and the effect of ∇S on the product which makes it equal to $\int_{t_0}^t \nabla S\big[\frac{\partial \mathbf{u}}{\partial \boldsymbol{\xi}}\big(t, s, \mathbf{v}(s)\big) - \mathbb{1}\mathbb{1}^T\big] R\big(\mathbf{x}(s), \mathbf{v}(s)\big)\, ds$ and this is upper bounded by

$$\int_{t_0}^t e^{-\psi(t-s)} S\big(\mathbf{R}(\mathbf{z}(s), \mathbf{v}(s))\big)\, ds \leq \\ \int_{t_0}^t e^{-\psi(t-s)} K \max_{i,j \in [N]} V(|z_{ij}|^2) \sqrt{|z_{ij}|} |v_{ij}|^{3/2}\, ds \leq \\ K \sqrt{\int_{t_0}^t V^2(|z_{ij}|^2) |z_{ij}| |v_{ij}|\, ds} \cdot \sqrt{\int_{t_0}^t e^{-2\psi(t-s)} |v_{ij}|^2\, ds}$$

where $K = 2(N-1)\bar{c}^{3/2}$ and after using the Cauchy-Schwarz inequality for integrals. Next we need a bound on $\int_{t_0}^t V^2(|z_{ij}|^2)|z_{ij}||v_{ij}| ds$ and this is obtained as follows: Assume that for any t close enough to t_0, the solution $(\mathbf{z}, \mathbf{v})$ exists. From Lemma 3.7 we know that z_{ij} does not change sign. Pick $i, j \in [N]$ so that $z_i(s) > z_j(s)$ for $s \in [t_0, t]$. It is easy to see that the upper bound of the integral is obtained from the worst case scenario, that of $z_i(s) - z_j(s) \downarrow d$ i.e. $v_{ij}(s) < 0$. Then we read

$$\int_{t_0}^t V^2(|z_{ij}|^2) |z_{ij}| |v_{ij}| ds = -\int_{t_0}^t V^2(|z_{ij}|^2) z_{ij} v_{ij} ds = \\ \int_{z_{ij}^2(t)}^{z_{ij}^2(t_0)} V^2(s)\, ds \leq \int_{\omega(\mathbf{v})}^{\infty} V^2(s)\, ds$$

where $\omega(\mathbf{v}) = \inf_{s \geq t_0} \min_{i,j \in [N]} z_{ij}^2(s)$ where the dependence on $\mathbf{v}$ is to remind that $\mathbf{z}$ is defined through $\mathbf{v}$. We end up with the following estimate

$$S(\mathbf{v}(t)) \leq S(\mathbf{u}(t)) + K \sqrt{\int_{\omega(\mathbf{v})}^{\infty} V^2(s)\, ds} \times \\ \times \sqrt{\int_{t_0}^t e^{-2\psi(\mathbf{v})(t-s)} S^2(\mathbf{v}(s))\, ds} \tag{12}$$

At this point we observe that a simple stability in variation argument does not work. exactly because we need a-priori estimates on the convergence rate of $\mathbf{v}$ to approximate ψ and ω. This can be achieved with the very special type of analysis to follow.

4.2.1 A fixed point argument

Given the initial data $\mathbf{v}^0, \mathbf{z}^0$ and take for brevity $d_0 := \min_{i \neq j} |z_{ij}(t_0)|$. We fix $\gamma, M > 0$ to satisfy:

$$M < \gamma \cdot (d_0 - d) \tag{13}$$

Then we set

$$\psi := N f(\tilde{r}) \underline{c} \tag{14}$$

where $\tilde{r}$ is such that

$$M = N\underline{c} \int_{S(\mathbf{z}^0) + M/\gamma}^{\tilde{r}} f(s)\, ds \tag{15}$$

Then we consider the space of functions

$$\mathbb{M} = \big\{\zeta \in C^0([t_0, \infty), \mathbb{R}_+) : \zeta(t_0) = S(\mathbf{v}^0), \\ \sup_{t \geq t_0} e^{\gamma(t-t_0)} \zeta(t) \leq M\big\}$$

Proving existence of a solution $(\mathbf{x}, \mathbf{v})$ so that $S(\mathbf{v}) \in \mathbb{M}$ implies $\sup_t S\big(\mathbf{z}(t)\big) < \infty$ because of the exponential convergence. It is vital to be able to estimate both ω and ψ in terms of $S(\mathbf{v})$. Indeed provided that $\int_0^\infty |v_{ij}(s)| ds < d_0$ then $\omega(\mathbf{v}) \geq \big(d_0 - \int_0^\infty |v_{ij}(s)| ds\big)^2$. The former condition can be met in $\mathbb{M}$ from (13). Next (14) is the second estimation since from (15) and consequently from the definition of M we have that

$$M < N\underline{c} \int_{S(\mathbf{z}^0) + M/\gamma}^{\infty} f(s)\, ds.$$

which satisfies (10) that justifies ψ. Then the stability of (12) translates to the existence of a fixed point of the following non-linear integral equation

$$\zeta(t) = e^{-\psi(t-t_0)} S(\mathbf{v}^0) + \\ + K \sqrt{\int_{(d_0 - \frac{M}{\gamma})^2}^{\infty} V^2(s)\, ds} \cdot \sqrt{\int_{t_0}^t e^{-2\psi(t-s)} \zeta^2(s)\, ds} \tag{16}$$

in $\mathbb{M}$, by applying Theorem 2.1. For any $\zeta \in \mathbb{M}$ define the operator

$$(\mathcal{P}\zeta)(t) = \zeta_{(16)}(t). \tag{17}$$

The first step is to prove that $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. Indeed this is true if $\gamma < Nf(\tilde{r})\underline{c}$ and M is chosen large enough so that

$$M > \frac{S(\mathbf{v}^0)}{1 - \frac{2(N-1)\bar{c}^{3/2}}{\sqrt{2(\psi-\gamma)}}\sqrt{\int_{(d_0-\frac{M}{\gamma})^2}^{\infty} V^2(s)\,ds}} \tag{18}$$

where we assumed that

$$1 - \frac{2(N-1)\bar{c}^{3/2}}{\sqrt{2(\psi-\gamma)}}\sqrt{\int_{(d_0-\frac{M}{\gamma})^2}^{\infty} V^2(s)\,ds} > 0 \tag{19}$$

The second step is to show that $\mathcal{P}$ is continuous in $\mathbb{M}$ endowed with the supremum norm. Indeed it is an elementary exercise to find $Q > 0$ such that

$$|(\mathcal{P}\zeta_1(t) - \mathcal{P}\zeta_2(t)| \leq Q \sup_{s\in[t_0,t]} \sqrt{|\zeta_1(s) - \zeta_2(s)|}$$

from which continuity in the supremum norm trivially follows.

The third step is to show that $\mathbb{M}$ is a compact, non-empty, convex subset of $\mathbb{B}$. Convexity follows in fairly standard way as it is easy to check that for $\zeta_1, \zeta_2 \in \mathbb{M}$ $\alpha\zeta_1 + (1-\alpha)\zeta_2$ is also in $\mathbb{M}$ for any $\alpha \in [0,1]$. $\mathbb{M}$ is clearly non-empty since M is always greater than $S(\mathbf{v}^0)$ and hence for example $e^{-\frac{\gamma}{2}(t-t_0)}S(\mathbf{v}^0)$ is a member of $\mathbb{M}$. The final and hardest step is compactness the proof of which is omitted due to space limitation. We only mention that it follows a crucial variation of Arzela-Ascoli theorem for sets of functions defined on unbounded intervals (as in our case $[t_0, \infty)$) and the proof can be found in [2]. In view of the preceding discussion we have the following result.

Theorem 4.2 *Let the Assumptions 3.1, 3.3 and 3.4 to hold. Let $(\mathbf{z}, \mathbf{v})$ to be the solution of 2 with initial data $\mathbf{z}^0$ such that $d_0 > d$. If the initial data $\mathbf{z}^0, \mathbf{v}^0$, the coupling function f and the repelling function V are such that there exist M, γ to satisfy conditions* (13)*,* (14)*,* (15)*,* (18) *and* (19) *then the solution $(\mathbf{x}, \mathbf{v})$ is unique and it exhibits asymptotic flocking in the sense of Defn. 2.2 with collision avoidance.*

5 Discussion

We introduced and examined the dynamics of two closely related non-linear flocking models via non-linear analysis techniques. Our approach is novel and it manages to deal both with the essential non-linear nature of the models and their lack of asymmetry. Our objective was to derive a relation between the initial conditions and the systems parameters for the existence of flocking solutions with or without the restriction of collision avoidance. For the latter system we would like to mention (19) cannot hold when $d_0 - M/\gamma$ is close to d if one considers Assumption 3.4 and this is a proof for collision avoidance flocking. Among the advantages of the approach we note that the model is general enough for both coupling and repelling asymmetric forces, although for the sake of simplicity we assumed a uniform repelling function V among all agents. We also mention that with extra work the connectivity assumptions can be relaxed along the lines of [13].

One of the main concerns is whether conditions (13), (14), (15), (18) and (19) can be satisfied at the same time. It is an easy yet tedious exercise, assuming d_0 large enough, to come up with examples of coupling and repelling functions which indeed satisfy all the above conditions under certain initial data and the parameters $\underline{c}$, $\bar{c}$. We omit this section due to space limitation.

Futhermore, if the coupling functions are non-summable ($\int^{\infty} f(s)\,ds = \infty$) then the condition of Theorem 4.1 is satisfied for all $\mathbf{x}^0$ and $\mathbf{u}^0$ and this is the case of unconditional flocking. Such an important feature is not an option for (2) and this is exactly because we implemented a stability in variation argument. This inability of unconditional flocking is perhaps the severest drawback of the method and it naturally paves the way for future research along these lines.

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