

# Emergent Behaviors Over Signed Random Dynamical Networks: State-Flipping Model

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**Abstract**—Recent studies from social, biological, and engineering network systems have drawn attention to the dynamics over signed networks, where each link is associated with a positive/negative sign indicating trustful/mistrustful, activator/inhibitor, or secure/malicious interactions. We study asymptotic dynamical patterns that emerge among a set of nodes that interact in a dynamically evolving signed random network. Node interactions take place at random on a sequence of deterministic signed graphs. Each node receives positive or negative recommendations from its neighbors depending on the sign of the interaction arcs, and updates its state accordingly. Recommendations along a positive arc follow the standard consensus update. As in the work by Altafini, negative recommendations use an update where the sign of the neighbor state is flipped. Nodes may weight positive and negative recommendations differently, and random processes are introduced to model the time-varying attention that nodes pay to these recommendations. Conditions for almost sure convergence and divergence of the node states are established. We show that under this so-called state-flipping model, all links contribute to a consensus of the absolute values of the nodes, even under switching sign patterns and a dynamically changing environment. A no-survivor property is established, indicating that every node state diverges almost surely if the maximum network state diverges.

**Index Terms**—Consensus dynamics, random graphs, signed networks.

## I. INTRODUCTION

### A. Motivation

THE need to model, analyze, and engineer large complex networks appears in a wide spectrum of scientific disciplines, ranging from social sciences and biology to physics and engineering [1]–[3]. In many cases, these networks are composed of relatively simple agents that interact locally with their neighbors based on a very limited knowledge about the

system state. Despite the simple local interactions, the resulting networks can display a rich set of emergent behaviors, including certain forms of intelligence and learning [4], [5].

Consensus problems, where the aim is to compute a weighted average of the initial values held by a collection of nodes, play a fundamental role in the study of node dynamics over complex networks. Early work [1] focused on understanding how opinions evolve in a network of agents, and showed that a simple deterministic opinion update based on the mutual trust and the differences in belief between interacting agents could lead to global convergence of the beliefs. Consensus dynamics has since then been widely adopted for describing opinion dynamics in social networks, for example, [5]–[7]. In engineering sciences, a huge amount of literature has studied these algorithms for distributed averaging, formation forming, and load balancing between collaborative agents under fixed or time-varying interaction networks [8]–[15]. Randomized consensus seeking has also been widely studied, motivated by the random nature of interactions and updates in real complex networks [16]–[21], [23]–[25].

Interactions in large-scale networks are not always collaborative since nodes take on different, or even opposing, roles. A convenient framework for modeling different roles and relationships between agents is to use signed graphs. Signed graphs were introduced in the classical work by Heider in 1946 [28] to model the structure of social networks, where a positive link represents a friendly relation between two agents, and a negative link an unfriendly one. In [29], a dynamic model based on a signed graph with positive links between nodes (representing nations) belonging to the same coalition and negative otherwise, was introduced to study the stability of world politics. In biology, sign patterns have been used to describe activator–inhibitor interactions between pairs of chemicals [30], neural networks for vision and learning [31], and gene regulatory networks [32]. In all of these examples, the state updates that occur when two nodes interact depend on the sign of the arc between the nodes in the underlying graph. The understanding of the emergent dynamical behaviors in networks with agents having different roles is much more limited than our knowledge about collaborative agents performing consensus algorithms.

It is intriguing to investigate what happens when two types of dynamics are coupled in a single network. Naturally we ask: how should we model the dynamics of positive and negative interactions? When do behaviors, such as consensus, swarming, and clustering emerge, and how does the structure of the sign patterns influence these behaviors? In this paper, continuing the previous efforts in [36] and [37], we answer these questions

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for a general model of opinion formation in dynamic signed random networks.

## B. Contributions

In this paper, we study a scheme of randomized node interaction over a signed network of nodes, and show how the nodes' states asymptotically evolve under these positive or negative interactions. A sequence of deterministic signed graphs defines the dynamics of the network. Random node interactions take place under independent, but not necessarily identically distributed, random sampling of the environment. Once interaction relations have been realized, each node receives a positive recommendation consistent with the standard consensus algorithm from its positive neighbors. Nodes receive negative recommendations from their negative neighbors. In this paper, we investigate a model where neighbors construct negative recommendations by flipping the sign of their true state during the interaction. This definition of negative interaction was introduced in [36]. After receiving these recommendations, each node puts a (deterministic) weight to each recommendation, and then encodes these weighted recommendations in its state update through stochastic attentions defined by two Bernoulli random variables.

Our model is general and covers many of the existing node interaction models, for example, consensus over the Erdős-Rényi graph [16], pairwise randomized gossiping [17], random link failure [19], etc. We allow the sign of each link to be time-varying as well in a dynamically changing environment. We establish conditions for almost sure convergence and divergence of the node states. We show that under the state-flipping model, all links contribute to a consensus of the absolute values of the nodes, even under switching sign patterns. We also show that strong structural balance [39] is crucial for belief clustering, which is consistent with the results derived in [36]. In the almost sure divergence analysis, we establish that the deterministic weights nodes placed on negative recommendations play a crucial role in driving the divergence of the network. A no-survivor property is established, indicating that every node state diverges almost surely given that the maximum network state diverges. Our analysis does not rely on a spectrum analysis as that used in [36], but instead we study the asymptotic behavior of the node states using a sample-path analysis.

## C. Organization

In Section II, we present the network dynamics and the node update rules. The state-flipping model is defined for the negative recommendations. Section III presents our main results on the state-flipping model, and the detailed proofs are presented in Section IV. Finally, some concluding remarks are drawn in Section V.

*Notations:* A simple directed graph (digraph)  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  consists of a finite set  $\mathcal{V}$  of nodes and an arc set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , where  $e = (i, j) \in \mathcal{E}$  denotes an *arc* from node  $i \in \mathcal{V}$  to  $j \in \mathcal{V}$  with  $(i, i) \notin \mathcal{E}$  for all  $i \in \mathcal{V}$ . We say that node  $j$  is *reachable* from node  $i$  if there is a directed path from  $i$  to  $j$ , with the additional convention that every node is reachable from

itself. A node  $v$  from which every node in  $\mathcal{V}$  is reachable is called a *center node* (or a root). A digraph  $\mathcal{G}$  is *strongly connected* if every two nodes are mutually reachable;  $\mathcal{G}$  has a spanning tree if it has a center node;  $\mathcal{G}$  is *weakly connected* if a connected undirected graph can be obtained by removing all of the directions of the arcs in  $\mathcal{E}$ . A subgraph of  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is a graph on the same node set  $\mathcal{V}$  whose arc set is a subset of  $\mathcal{E}$ . The induced graph of  $\mathcal{V}_i \subseteq \mathcal{V}$  on  $\mathcal{G}$ , denoted as  $\mathcal{G}|_{\mathcal{V}_i}$ , is the graph  $(\mathcal{V}_i, \mathcal{E}_i)$  with  $\mathcal{E}_i = (\mathcal{V}_i \times \mathcal{V}_i) \cap \mathcal{E}$ . A weakly connected component of  $\mathcal{G}$  is a maximal weakly connected induced graph of  $\mathcal{G}$ . If each arc  $(i, j) \in \mathcal{E}$  is associated with a sign, either “+” or “−,”  $\mathcal{G}$  is called a signed graph and the sign of  $(i, j) \in \mathcal{E}$  is denoted as  $\sigma_{ij}$ . The positive and negative subgraphs containing the positive and negative arcs of  $\mathcal{G}$  are denoted as  $\mathcal{G}^+ = (\mathcal{V}, \mathcal{E}^+)$  and  $\mathcal{G}^- = (\mathcal{V}, \mathcal{E}^-)$ , respectively.

Depending on the argument,  $|\cdot|$  stands for the absolute value of a real number, the Euclidean norm of a vector, or the cardinality of a set. The  $\sigma$ -algebra generated by a random variable  $X$  is denoted as  $\sigma(X)$ .

## II. RANDOM NETWORK MODEL AND NODE UPDATES

We consider a dynamic network where each node holds and updates its belief or state when interacting with other nodes. In this section, we present a general model specifying the network dynamics and the way nodes interact.

### A. Dynamic Signed Graphs

We consider a network with a set  $\mathcal{V} = \{1, \dots, n\}$  of  $n$  nodes, with  $n \geq 3$ . Time is slotted and at each slot  $t = 0, 1, \dots$ , each node can interact with its neighbors in a simple directed graph  $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$ . The graph evolves over time in an arbitrary and deterministic manner. We assume  $\mathcal{G}_t$  is a signed graph, and we denote by  $\sigma_{ij}(t)$  the sign of arc  $(i, j) \in \mathcal{E}_t$ . The sign of arc  $(i, j)$  indicates whether  $i$  is a friend ( $\sigma_{ij}(t) = +$ ), or an enemy ( $\sigma_{ij}(t) = -$ ) of node  $j$ . The positive and negative subgraphs containing the positive and negative arcs of  $\mathcal{G}_t$  are denoted by  $\mathcal{G}_t^+ = (\mathcal{V}, \mathcal{E}_t^+)$  and  $\mathcal{G}_t^- = (\mathcal{V}, \mathcal{E}_t^-)$ , respectively. We say that the sequence of graphs  $\{\mathcal{G}_t\}_{t \geq 0}$  is *sign consistent* if the sign of any arc  $(i, j)$  does not evolve over time, that is, if for any  $s, t \geq 0$

$$(i, j) \in \mathcal{E}_s \cap \mathcal{E}_t \implies \sigma_{ij}(s) = \sigma_{ij}(t).$$

We also define  $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$  with  $\mathcal{E}_* = \bigcup_{t=0}^{\infty} \mathcal{E}_t$  as the total graph of the network. If  $\{\mathcal{G}_t\}_{t \geq 0}$  is sign consistent, then the sign of each arc  $\mathcal{E}_*$  never changes and in that case,  $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$  is a well-defined signed graph.

*Remark 1:* Note that  $\mathcal{G}_t$  is defined over directed graphs. The only requirement on  $\mathcal{G}_t^+$  and  $\mathcal{G}_t^-$  is that they should be disjoint, so the signed graph model under consideration is quite general. In particular, we allow for the two possible edge directions to coexist between a pair of nodes and that the two directions can have different signs.

Next, we introduce the notion of positive cluster in a signed digraph, which will play an important role in the analysis of the belief dynamics (see Fig. 1).

*Definition 1:* Let  $\mathcal{G}$  be a signed digraph with positive subgraph  $\mathcal{G}^+$ . A subset  $\mathcal{V}_*$  of the set of nodes  $\mathcal{V}$  is a positive

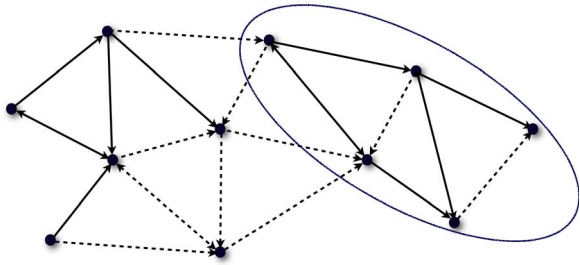


Fig. 1. Signed network and one of its three positive clusters encircled. The positive arcs are solid, and the negative arcs are dashed. Note that negative arcs are allowed within each positive cluster.

cluster if  $\mathcal{V}_*$  constitutes a weakly connected component of  $\mathcal{G}^+$ . A positive cluster partition of  $\mathcal{G}$  is a partition of  $\mathcal{V}$  into  $T_p \geq 1$  positive clusters  $\mathcal{V}_i, i = 1, \dots, T_p$ , such that  $\mathcal{V} = \text{cup}_{i=1}^{T_p} \mathcal{V}_i$ .

Note that negative arcs may exist between the nodes of a positive cluster. Therefore, a positive-cluster partition of  $\mathcal{G}$  can be seen as an extension of the classical notion of weak structural balance for which negative links are strictly forbidden inside each positive cluster [40]. From the above definition, it is clear that for any signed graph  $\mathcal{G}$ , there is a unique positive cluster partition  $\mathcal{V} = \text{cup}_{i=1}^{T_p} \mathcal{V}_i$ , where  $T_p$  is the number of maximal positive clusters covering the entire set of nodes.

### B. Random Interactions

Time is discrete and at time  $t$ , node  $i$  may only interact with its neighboring nodes in  $\mathcal{G}_t$ . We consider a general model for the random node interactions. At time  $t$ , some pairs of nodes are randomly selected for interaction. We denote by  $E_t \subset \mathcal{E}_t$  the random subset of arcs corresponding to interacting node pairs at time  $t$ . More precisely,  $E_t$  is sampled from the distribution  $\mu_t$  defined over the set  $\Omega_t$  of all subsets of arcs in  $\mathcal{E}_t$ . We assume that  $E_0, E_1, \dots$  form a sequence of independent sets of arcs. Formally, we introduce the probability space  $(\Theta, \mathcal{F}, \mathbb{P})$  obtained by taking the product of the probability spaces  $(\Omega_t, \mathcal{S}_t, \mu_t)$ , where  $\mathcal{S}_t$  is the discrete  $\sigma$ -algebra on  $\Omega_t$ :  $\Theta = \prod_{t \geq 0} \Omega_t$ ,  $\mathcal{F}$  is the product of  $\sigma$ -algebras  $\mathcal{S}_t, t \geq 0$ , and  $\mathbb{P}$  is the product probability measure of  $\mu_t, t \geq 0$ . We denote by  $G_t = (\mathcal{V}, E_t)$  the random subgraph of  $\mathcal{G}_t$  corresponding to the random set  $E_t$  of arcs. The disjoint sets  $E_t^+$  and  $E_t^-$  denote the positive and negative arc sets of  $E_t$ , respectively. Finally, we split the random set of nodes interacting with node  $i$  at time  $t$  depending on the sign of the corresponding arc: for node  $i$ , the set of positive neighbors is defined as  $N_i^+(t) := \{j : (j, i) \in E_t^+\}$ , whereas similarly, the set of negative neighbors is  $N_i^-(t) := \{j : (j, i) \in E_t^-\}$ .

*Remark 2:* The above model is quite general. It includes as special cases the classical Erdős-Rényi random graph [26], gossiping models where a single pair of nodes is chosen at random for interaction [7], [17], or where all nodes interact with their neighbors at a given time [1], [5], [9], [18]. Independence is the only hard requirement in our random graph process, which is imposed in most existing works on randomized consensus dynamics, for example, [7] and [17]–[19]. Nonindependent random graph models for randomized consensus were discussed in [22]–[24], and [41].

### C. Node Updates

Next, we explain how nodes update their states. Each node  $i$  holds a state  $s_i(t) \in \mathbb{R}$  at  $t = 0, 1, \dots$ . To update its state at time  $t$ , node  $i$  considers recommendations received from positive and negative neighbors:

- a) The positive recommendation node  $i$  received at time  $t$  is

$$h_i^+(t) := - \sum_{j \in N_i^+(t)} (s_i(t) - s_j(t)).$$

- b) The negative recommendation node  $i$  received at time  $t$  is

$$h_i^-(t) := - \sum_{j \in N_i^-(t)} (s_i(t) + s_j(t)).$$

In the above expressions, we use the convention that summing over empty sets yields a recommendation equal to zero, for example, when node  $i$  has no positive neighbors, then  $h_i^+(t) = 0$ .

Now let  $\{B_t\}_{t \geq 0}$  and  $\{D_t\}_{t \geq 0}$  be two sequences of independent Bernoulli random variables. We further assume that  $\{B_t\}_{t \geq 0}$ ,  $\{D_t\}_{t \geq 0}$ , and  $\{G_t\}_{t \geq 0}$  define independent processes. For any  $t \geq 0$ , define  $b_t = \mathbb{E}\{B_t\}$  and  $d_t = \mathbb{E}\{D_t\}$ . The processes  $\{B_t\}_{t \geq 0}$  and  $\{D_t\}_{t \geq 0}$  represent the attention that node  $i$  pays to the positive and negative recommendations, respectively.

Node  $i$  updates its state as

$$s_i(t+1) = s_i(t) + \alpha B_t h_i^+(t) + \beta D_t h_i^-(t) \quad (1)$$

where  $\alpha, \beta > 0$  are two positive constants marking the weight that each node puts on the positive and negative recommendations.

The role of  $h_i^+(t)$  in (1) is consistent with the classical DeGroot's social learning model [1] along trustful interactions. In view of the definition of  $h_i^-(t)$ , in contrast to  $h_i^+(t)$ , the model is referred to as the state-flipping model.

*Remark 3:* The state-flipping model can be interpreted as a situation where the neighbors connected by a negative link provide false values of their states to each node by flipping their true sign [36]. Under this interpretation, it is the *head* node along each negative arc that knows the sign of that arc. However, the tail node does not see the sign of the arc associated with the recommendations it receives. The weights and attentions of recommendations, represented by  $\alpha/\beta$  and  $B_t/D_t$ , respectively, are then descriptions of each node's possible prior knowledge of the signs of its neighbors.

*Remark 4:* In standard consensus algorithms, nodes communicate relative states. In other words, nodes hold no absolute state information. For the state-flipping model to make sense, there must exist a global *origin* (state equal to 0) known by each node so that sign flipping is possible in the negative interactions.

Let  $s(t) = (s_1(t) \dots s_n(t))^T$  be the random vector representing the network state at time  $t$ . The main objective of this paper is to analyze the behavior of the stochastic process  $\{s(t)\}_{t \geq 0}$ . In the following text, we denote by  $\mathbb{P}$  the probability measure capturing all random components driving the evolution of the network state.

### III. MAIN RESULTS

In this section, we present our main results. We begin by stating two natural assumptions on the way nodes are selected for updates, and on the graph dynamics. In the first assumption, we impose that at time  $t$ , any arc is selected with positive probability. The second assumption states that the unions of the graphs  $\mathcal{G}_t$  over time windows of fixed duration are strongly connected.

**A1.** There is a constant  $p_* \in (0, 1)$  such that for all  $t \geq 0$  and  $i, j \in \mathcal{V}$ ,  $\mathbb{P}((i, j) \in E_t) \geq p_*$  if  $(i, j) \in \mathcal{E}_t$ .

**A2.** There is an integer  $K \geq 1$  such that the union graph  $\mathcal{G}([t, t + K - 1]) = (\mathcal{V}, \cup_{\tau \in [t, t + K - 1]} \mathcal{E}_\tau)$  is strongly connected for all  $t \geq 0$ .

The following theorem provides conditions under which the system dynamics converges almost surely. Surprisingly, these conditions are mild: we just require that the sum of the updating parameters  $\alpha$  and  $\beta$  is small enough, and that node updates occur with constant probabilities, that is,  $\mathbb{E}\{B_t\}$  and  $\mathbb{E}\{D_t\}$  do not evolve over time. In particular, the state of each node converges almost surely even if the signs of the arcs change over time.

*Theorem 1:* Assume that A1 and A2 hold, and that  $\alpha, \beta > 0$  are such that  $\alpha + \beta < 1/(n - 1)$ . Further assume that for any  $t \geq 0$ ,  $b_t = b$  and  $d_t = d$  for some  $b, d \in (0, 1)$ . Then under the state-flipping model, we have, for all  $i \in \mathcal{V}$  and all initial state  $s(0)$ ,  $\mathbb{P}(\lim_{t \rightarrow \infty} s_i(t) \text{ exists}) = 1$ .

In the above theorem, we say that  $\lim_{t \rightarrow \infty} s_i(t)$  exists if  $s_i(t)$  converges to a finite limit as  $t$  tends to infinity.

*Remark 5:* Theorem 1 shows an interesting property of the state-flipping model: negative updates, together with the positive updates, contribute to the convergence of the node states whenever it holds that  $\alpha + \beta < 1/(n - 1)$ .

Characterizing the limiting states is, in general, challenging. There are, however, scenarios where this can be done, which require the notion of structural balance [39].

*Definition 2:* Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a signed digraph.  $\mathcal{G}$  is strongly balanced if we can divide  $\mathcal{V}$  into two disjoint nonempty subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  where negative arcs exist only between these two subsets.

To predict the limiting system behavior, we make the following assumption.

**A3.**  $\{\mathcal{G}_t\}_{t \geq 0}$  is sign consistent.

Recall that  $\mathcal{G}_*$  denotes the total graph. The following theorem holds.

*Theorem 2:* Assume that A1, A2, and A3 hold, and that  $\alpha, \beta > 0$  are such that  $\alpha + \beta < 1/(n - 1)$ . Suppose  $\mathcal{G}_*$  contains at least one negative arc and that every negative arc in  $\mathcal{G}_*$  appears infinitely often in  $\{\mathcal{G}_t\}_{t \geq 0}$ . Further assume that for any  $t \geq 0$ ,  $b_t = b$  and  $d_t = d$  for some  $b, d \in (0, 1)$ . Then, under the state-flipping model, we have, for any initial state  $s(0)$ :

- If  $\mathcal{G}_*$  is strongly balanced, then there is a random variable  $y_*$ , with  $y_* \leq \|s(0)\|_1$  almost surely, such that  $\mathbb{P}(\lim_{t \rightarrow \infty} s_i(t) = y_*, \forall i \in \mathcal{V}_1; \lim_{t \rightarrow \infty} s_i(t) = -y_*, \forall i \in \mathcal{V}_2) = 1$ .
- If  $\mathcal{G}_*$  is not strongly balanced, then  $\mathbb{P}(\lim_{t \rightarrow \infty} s_i(t) = 0, \forall i \in \mathcal{V}) = 1$ .

Theorem 2 states that strong structural balance is crucial to ensure convergence to nontrivial clustering states, which

is consistent with the result of [36] derived for fixed graphs under continuous-time node updates. To establish the result, we do not rely on a spectral analysis as in [36], but rather study the asymptotic behavior for each sample path. Using similar arguments as in [36], the value of  $y_*$  can be described as the limit of a random consensus process with the help of a gauge transformation.

Next, we are interested in determining whether the states could diverge depending on the values of the updating parameters  $\alpha$  and  $\beta$ . We show that by increasing  $\beta$ , that is, the strength of the negative recommendations, one may observe such divergence. To this aim, we make the following assumptions.

**A4.** There is an integer  $K \geq 1$  such that the union graph  $\mathcal{G}^+([t, t + K]) = (\mathcal{V}, \cup_{\tau \in [t, t + K - 1]} \mathcal{E}_\tau^+)$  is strongly connected for all  $t \geq 0$ .

**A5.** There is an integer  $K \geq 1$  such that the union graph  $\mathcal{G}^-([t, t + K]) = (\mathcal{V}, \cup_{\tau \in [t, t + K - 1]} \mathcal{E}_\tau^-)$  is strongly connected for all  $t \geq 0$ .

**A6.** The events  $\{(i, j) \in E_t\}$ ,  $i, j \in \mathcal{V}$ ,  $t = 0, 1, \dots$  are independent and there is a constant  $p^* \in (0, 1)$  such that for all  $t \geq 0$  and  $i, j \in \mathcal{V}$ ,  $\mathbb{P}((i, j) \in E_t) \leq p^*$  whenever  $(i, j) \in \mathcal{E}_t$ .

*Proposition 1:* Assume that A1, A4, A5, and A6 hold, and that for any  $t \geq 0$ ,  $b_t = b$  and  $d_t = d$  for some  $b, d \in (0, 1)$ . Fix  $\alpha \in [0, (4n)^{-1}]$ . Then, under the state-flipping model, there is  $\beta_* > 0$  such that whenever  $\beta > \beta_*$ , we have  $\mathbb{P}(\lim_{t \rightarrow \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty) = 1$  for almost all initial states  $s(0)$ .

Proposition 1 shows that under appropriate conditions,  $\max_{i \in \mathcal{V}} |s_i(t)|$  diverges almost surely if the negative updating parameter  $\beta$  is sufficiently large. We can, in fact, derive an explicit value for  $\beta_*$ .

*Remark 6:* The main difficulties of establishing Proposition 1 lie in the fact that we need on one hand to establish an absolute bound for the way  $\max_{i \in \mathcal{V}} |s_i(t)|$  decreases (which is obtained by a constructive proof) and, on the other hand, to establish a probabilistic lower bound for the possible increase of  $\max_{i \in \mathcal{V}} |s_i(t)|$  (which is obtained by combining A4–A6 and by constructing and analyzing sample paths). These constructive derivations are rather conservative since we consider general random graph processes, but they nevertheless establish a positive drift for  $\{\max_{i \in \mathcal{V}} |s_i(t)|\}$  with an explicit  $\beta_*$  so that almost sure divergence is guaranteed.

*Remark 7:* We also remark from Proposition 1 that a large deterministic weight on negative recommendations leads to the divergence of the node states. It can also be seen from the forthcoming Lemma 1 that if these weights on the recommendations are sufficiently small,  $\max_{i \in \mathcal{V}} |s_i(t)|$  always converges regardless of how the random attentions  $\{B_t\}$  and  $\{D_t\}$  are selected.

Actually, one may even prove that when  $\max_{i \in \mathcal{V}} |s_i(t)|$  grows large when  $t \rightarrow \infty$ , the state of any node diverges. This result is referred to as the *no-survivor* property, and is formally stated in the following result.

*Theorem 3:* Assume that A1, A2, and A6 hold, and that for any  $t \geq 0$ ,  $b_t \equiv b$  and  $d_t \equiv d$  for some  $b, d \in (0, 1)$ . Fix the initial state  $s(0)$ . Then, under the state-flipping model, we have

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} |s_i(t)| = \infty, i \in \mathcal{V} \mid \limsup_{t \rightarrow \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty\right) = 1.$$

*Remark 8:* A similar kind of no-survivor property was first established in [38] under the model of repulsive negative dynamics for pairwise node interactions. Theorem 3 establishes the same property for the considered state-flipping model, but for the general random graph process. From the proof of Theorem 3, it is clear that it is the arc independence (assumption A6, see [41]), rather than synchronous or asynchronous node interactions, that directly result in the no-survivor divergence property for dynamics over signed random networks.

In all of the aforementioned results, it can be seen from their proofs that extensions to time-varying  $\{b_t\}_{\geq 0}$  and  $\{d_t\}_{\geq 0}$  are straightforward under mild assumptions. The resulting expressions are, however, more involved. We omit those discussions here to shorten the presentation.

#### IV. PROOFS

In this section, we present the detailed proofs of the results stated in the previous section. We first establish some technical lemmas, and then the proofs of each result.

##### A. Supporting Lemmas

For any  $t \geq 0$ , we define  $M(t) = \max_{i \in \mathcal{V}} |s_i(t)|$  and  $Y_i(t) = \alpha B_t |N_i^+(t)| + \beta D_t |N_i^-(t)|$ , which will be used throughout the rest of this paper.

*Lemma 1:* Suppose  $\alpha + \beta \leq 1/(n-1)$ . Then  $M(t+1) \leq M(t)$ .

*Proof:* Observe that  $|N_i^+(t)| + |N_i^-(t)| \leq n-1$ . Hence,  $Y_i(t) \in [0, 1]$  as long as  $\alpha + \beta \leq 1/(n-1)$ . Now for any  $i \in \mathcal{V}$

$$\begin{aligned} |s_i(t+1)| &= \left| (1 - Y_i(t)) s_i(t) + \alpha B_t \sum_{j \in N_i^+(t)} s_j(t) \right. \\ &\quad \left. - \beta D_t \sum_{j \in N_i^-(t)} s_j(t) \right| \\ &\leq (|1 - Y_i(t)| + Y_i(t)) \max_{j \in \mathcal{V}} |s_j(t)| \\ &= \max_{j \in \mathcal{V}} |s_j(t)| \end{aligned}$$

which completes the proof.  $\blacksquare$

*Remark 9:* Lemma 1 establishes the nonexpansiveness property of the considered model. It is clear from its proof that the condition  $\alpha + \beta \leq 1/(n-1)$  in Lemma 1 can be relaxed to  $\alpha + \beta \leq 1/\text{Deg}(\mathcal{G}_t)$ , where  $\text{Deg}(\mathcal{G}_t)$  denotes the maximum degree of the graph  $\mathcal{G}_t$ . Here, for convenience, we use the current statement since  $\text{Deg}(\mathcal{G}_t) \leq n-1$  for all  $t$ .

*Lemma 2:* Assume that  $\alpha + \beta \leq 1/(n-1)$ . Let  $i \in \mathcal{V}$  and assume that  $|s_i(t)| \leq \zeta_0 M(t)$  for some  $0 < \zeta_0 < 1$ . Then

$$|s_i(t+k)| \leq (1 - (1 - \zeta_0)\gamma_*^k) M(t), \quad k = 0, 1, \dots$$

where  $\gamma_* = 1 - (\alpha + \beta)(n-1)$ .

*Proof:* We have

$$\begin{aligned} |s_i(t+1)| &\leq (1 - Y_i(t)) |s_i(t)| + Y_i(t) M(t) \\ &\leq (1 - Y_i(t)) \zeta_0 M(t) + Y_i(t) M(t) \\ &\leq (1 - (\alpha + \beta)(n-1)) \zeta_0 M(t) + (\alpha + \beta)(n-1) M(t) \\ &= (1 - (1 - \zeta_0)\gamma_*) M(t). \end{aligned}$$

The lemma is then obtained by applying a simple induction argument.  $\blacksquare$

*Lemma 3:* Assume that  $\alpha + \beta \leq 1/(n-1)$ . Let  $i \in \mathcal{V}$  and assume that  $|s_i(t)| \leq \zeta_0 M(t)$  for some  $0 < \zeta_0 < 1$ . Let  $(i, j) \in \mathcal{E}_t$ . Then conditioned on  $B_t = 1$  if  $(i, j) \in \mathcal{E}_t^+$ ,  $D_t = 1$  if  $(i, j) \in \mathcal{E}_t^-$ , we have

$$|s_j(t+1)| \leq (1 - (1 - \zeta_0) \min\{\alpha, \beta\}) M(t).$$

*Proof:* Suppose  $(i, j) \in \mathcal{E}_t^+$  with  $B_t = 1$ . Then we have

$$\begin{aligned} |s_j(t+1)| &= \left| s_j(t) - \alpha B_t \sum_{k \in N_j^+(t)} (s_j(t) - s_k(t)) \right. \\ &\quad \left. - \beta D_t \sum_{k \in N_j^-(t)} (s_j(t) + s_k(t)) \right| \\ &= \left| \alpha s_i(t) + (1 - Y_j(t)) s_j(t) \right. \\ &\quad \left. + \alpha B_t \sum_{k \in N_j^+(t) \setminus \{i\}} s_k(t) - \beta D_t \sum_{k \in N_j^-(t)} s_k(t) \right| \\ &\leq \alpha |s_i(t)| + (1 - Y_j(t) + Y_j(t) - \alpha) M(t) \\ &\leq \min\{\alpha, \beta\} |s_i(t)| + (1 - \min\{\alpha, \beta\}) M(t) \end{aligned} \quad (2)$$

where in the last inequality we have used the fact that  $|s_i(t)| \leq M(t)$ . It is straightforward to see that (2) continues to hold with  $D_t = 1$  if  $(i, j) \in \mathcal{E}_t^-$ . Plugging in the assumption that  $|s_i(t)| \leq \zeta_0 M(t)$  into (2), one gets the desired inequality. This proves the lemma.  $\blacksquare$

Note that if the conditions in Lemmas 2 and 3 are replaced by  $|s_i(t)| < \zeta_0 M(t)$ , then we have the same conclusions but with strict inequalities. Moreover, in view of Lemma 1, the following limit is well defined:  $M_* = \lim_{t \rightarrow \infty} M(t)$ .

*Lemma 4:* Assume that A1 and A2 hold  $\alpha, \beta > 0$  and  $\alpha + \beta < 1/(n-1)$ . Further assume that for any  $t \geq 0$ ,  $b_t \equiv b$  and  $d_t \equiv d$  for some  $b, d \in (0, 1)$ . Then, for any initial state  $s(0)$ , we have  $\mathbb{P}(\lim_{t \rightarrow \infty} |s_i(t)| = M_*, \forall i \in \mathcal{V}) = 1$ .

*Proof:* We prove this lemma using sample path arguments by contradiction. Let us assume that:

H1. There exists  $i_0 \in \mathcal{V}$  and  $\delta, q_* \in (0, 1)$  such that  $\mathbb{P}(\liminf_{t \rightarrow \infty} |s_{i_0}(t)| < \delta M_*) \geq q_*$ .

Let  $\epsilon > 0$ . Define  $T(\epsilon) := \inf_{k \geq 0} \{M(t) \leq (1 + \epsilon) M_*, \forall t \geq k\}$  and  $T^* := \inf_{t \geq T(\epsilon)} \{s_{i_0}(t) < \delta M_*\}$ . Note that  $T(\epsilon)$  is a stopping time, and the monotonicity of  $M(t)$  guarantees that  $T$  is bounded almost surely [27]. Moreover,  $T^*$  is also a stopping time, and it is bounded with a probability of at least  $q_*$  in view of H1. Next, we use Lemmas 2 and 3 to obtain a contradiction. Plugging in the fact that  $M(T^*) \leq M_*(1 + \epsilon)$  and invoking Lemma 2, conditioned on  $\{T^* < \infty\}$ , we have that for all  $k = 0, 1, \dots$

$$|s_{i_0}(T^* + k)| < (1 - (1 - \delta)\gamma_*^k) M_*(1 + \epsilon). \quad (3)$$

Now consider the time interval  $[T^*, T^* + K - 1]$ . The independence of  $\{B_t\}_{t \geq 0}$ ,  $\{D_t\}_{t \geq 0}$ , and  $\{G_t\}_{t \geq 0}$  guarantees that  $(G_{T^*}, B_{T^*}, D_{T^*}), (G_{T^*+1}, B_{T^*+1}, D_{T^*+1}), \dots$  are independent random variables, and they are independent of  $\mathcal{F}_{T^*-1}$  [27, Theor. 4.1.3]). From their definitions, we also know that  $(B_{T^*}, D_{T^*}), (B_{T^*+1}, D_{T^*+1}), \dots$  are i.i.d. with the same distribution as  $(B_0, D_0)$ , and Assumption A2 guarantees that  $\mathcal{G}([T^*, T^* + K - 1]) = (\mathcal{V}, \cup_{\tau \in [T^*, T^* + K - 1]} \mathcal{E}_\tau)$  is strongly connected. Therefore, there exists a node  $i_1 \neq i_0$  and  $\tau_1 \leq K$  such that  $(i_0, i_1) \in \mathcal{E}_{T^* + \tau_1}$  (note that  $i_1$  and  $\tau_1$  are random variables, but they are independent with  $\mathcal{F}_{T^*-1}$  since  $T^*$  is a stopping time). Hence, we can apply Lemma 3 and conclude that

$$|s_{i_1}(T^* + \tau_1)| < (1 - (1 - \delta)\gamma_*^{\tau_1} \min\{\alpha, \beta\}) M_*(1 + \epsilon)$$

with a probability of at least  $p_* \min\{b, d\}$ . Taking  $\zeta_0 = 1 - (1 - \delta)\gamma_*^{\tau_1} \min\{\alpha, \beta\}$  for the  $\zeta_0$  introduced in Lemma 2, we have

$$1 - (1 - \zeta_0)\gamma_*^k = 1 - (1 - \delta)\gamma_*^{\tau_1 + k} \min\{\alpha, \beta\}.$$

Therefore, applying Lemma 2 (note that we can replace  $M(t)$  with  $M_*(1 + \epsilon)$  in Lemma 2), we have that for all  $k = K, K + 1, \dots$

$$|s_{i_1}(T^* + k)| < (1 - (1 - \delta)\gamma_*^k \min\{\alpha, \beta\}) M_*(1 + \epsilon).$$

We can repeat the same argument over time intervals  $[T^* + K, T^* + 2K - 1], \dots, [T^* + (n - 2)K, T^* + (n - 1)K - 1]$ . Assuming that the node set  $I_k := \{i_0, \dots, i_{k-1}\}$  is selected, it follows from the strong connectivity assumption A2 that there exists an arc from  $I_k$  to  $\mathcal{V} \setminus I_k$  in the union graph of the corresponding interval. In this way, we add the tail node of such an arc into  $I_k$  and obtain  $I_{k+1}$  for  $k = 1, \dots, n - 1$ . We can thus recursively find  $i_2, \dots, i_{n-1}$  with  $\mathcal{V} = \{i_0, \dots, i_{n-1}\}$  and bound the absolute values of their states. Finally, we obtain

$$\begin{aligned} \mathbb{P}(M(T^* + (n - 1)K) < [1 - \gamma_*^{(n-1)K} (\min\{\alpha, \beta\})^{n-1} \\ \times (1 - \delta)] M_*(1 + \epsilon) | T^* < \infty) \geq (p_* \min\{b, d\})^{n-1}. \end{aligned} \quad (4)$$

Now select  $\epsilon$  sufficiently small so that  $\theta := (1 - (1 - \delta)\gamma_*^{(n-1)K} (\min\{\alpha, \beta\})^{n-2})(1 + \epsilon) < 1$ . Using the monotonicity of  $M(t)$  established in Lemma 1, we deduce from (4)

$$\mathbb{P}(M_* < \theta M_* | T^* < \infty) \geq (p_* \min\{b_*, d_*\})^{n-1}$$

which is impossible and, hence, H1 is not true. We have proved that

$$\mathbb{P}\left(\liminf_{t \rightarrow \infty} |s_i(t)| = M_*, \forall i \in \mathcal{V}\right) = 1.$$

The claim then follows easily from Lemma 1.  $\blacksquare$

*Remark 10:* It is easy to see from the proof that Lemma 4 continues to hold if we relax the requirement of  $b_t, d_t$  to  $0 < b \leq b_t \leq 1$  and  $0 < d \leq d_t \leq 1$  for some  $b, d \in (0, 1)$ . Lemma 4 indicates that with sufficient connectivity on the graphs defining the dynamical environment (Assumption A2), the absolute values of the nodes states will eventually converge to a consensus

with probability one under quite general conditions on how the random interactions take place in the environment. Noting that A2 is imposed on the overall underlying graph, this concludes that the positive and negative links contribute to the node states' consensus in absolute value.

*Lemma 5:* Let  $\alpha < (4n)^{-1}$  and  $\beta > 16n^{n-1}$ . Then,  $M(t + 1) \geq (2n)^{-1}M(t)$  defines a sure event.

*Proof:* Let us first assume that  $D_t = 0$ . Let  $i \in \mathcal{V}$  such that  $|s_i(t)| = M(t)$ . Then, with  $\alpha < (4n)^{-1}$ , we have

$$\begin{aligned} M(t + 1) &\geq |s_i(t + 1)| \\ &\geq |1 - \alpha B_t |N_i^+(t)|| \cdot |s_i(t)| - \alpha B_t |N_i^+(t)| \cdot M(t) \\ &\geq |1 - 2\alpha B_t |N_i^+(t)|| \cdot M(t) \\ &\geq \frac{1}{n} M(t). \end{aligned}$$

Now assume that  $D_t = 1$ . We first prove the following claim.

*Claim:* Suppose there exists  $i_1 \in \mathcal{V}$  such that  $|s_{i_1}(t)| \in [(1 - Z_2)M(t), (1 - Z_1)M(t)]$  with  $0 \leq Z_1 < Z_2 < nZ_2 < 1/4$  and  $\beta Z_2 \geq 2$ . Then,  $\mathcal{H}_1 \cup \mathcal{H}_2$  is a sure event, where

$$\mathcal{H}_1 = \{M(t + 1) \geq M(t)/4\}$$

and

$$\mathcal{H}_2 = \{\exists i_2 : |s_{i_2}(t)| \in [(1 - nZ_2)M(t), (1 - Z_2)M(t)]\}.$$

To prove this claim, we distinguish three cases:

Case 1) Let  $s_{i_1}(t) \in [(1 - Z_2)M(t), (1 - Z_1)M(t)]$  and assume that there exists  $j_* \in \mathcal{V}$  such that  $j_* \in N_{i_1}^-(t)$  and  $s_{j_*}(t) \in [-(1 - nZ_2)M(t), M(t)]$ . Then,  $s_{i_1}(t) + s_{j_*}(t) \geq (n - 1)Z_2M(t) \geq 0$  and  $s_{i_1}(t) + s_j(t) \geq -Z_2M(t)$  for all  $j \in \mathcal{V} \setminus \{i_1, j_*\}$ . Thus, taking out the term  $s_{i_1}(t) + s_{j_*}(t)$  in  $h_{i_1}^-(t)$  from (1), some simple algebra leads to

$$\begin{aligned} M(t + 1) &\geq |s_{i_1}(t + 1)| \\ &\geq \beta |s_{i_1}(t) + s_{j_*}(t)| - M(t) - 2\alpha(n - 1)M(t) \\ &\quad - \beta(n - 2)Z_2M(t) \\ &\geq |\beta Z_2 - 1 - (n - 1)(2n)^{-1}| \cdot M(t) \\ &\geq \frac{1}{2} M(t) \end{aligned} \quad (5)$$

where in the last inequality we have used the assumption that  $\beta Z_2 \geq 2$ .

Case 2) Let  $s_{i_1}(t) \in [(1 - Z_2)M(t), (1 - Z_1)M(t)]$  and assume that  $s_j(t) \in [-M(t), -(1 - Z_1)M(t)]$  for all  $j \in N_{i_1}^-(t)$ , and, more generally,  $s_{i_1}(t) + s_j(t) \leq 0$  for all  $j \in N_{i_1}^-(t)$ , which implies that  $h_{i_1}^-(t) \geq 0$ . Observing that  $s_{i_1}(t) \geq 0$ , we obtain

$$\begin{aligned} M(t + 1) &\geq |s_{i_1}(t + 1)| \\ &\geq |s_{i_1}(t)| - 2\alpha(n - 1)M(t) \\ &\geq |1 - Z_2 - (n - 1)(2n)^{-1}| \cdot M(t) \\ &\geq \frac{1}{4} M(t). \end{aligned} \quad (6)$$

Case 3) Let  $m_1, \dots, m_\ell \in \mathcal{V}$  and  $w_1, \dots, w_j \in \mathcal{V}$  satisfy

$$s_{m_\varrho}(t) \in [(1 - Z_2)M(t), (1 - Z_1)M(t)], \quad \varrho = 1, \dots, \ell$$

and

$$s_{w_\varrho}(t) \in [-(1 - Z_1)M(t), -(1 - Z_2)M(t)], \quad \varrho = 1, \dots, j$$

respectively. Without loss of generality, we assume the existence of such  $m_\varrho$  and  $w_\varrho$  since otherwise the desired conclusions immediately fall to Cases 1) and 2).

Now without loss of generality, suppose  $s_{m_1}(t) = \min_{\varrho=1, \dots, \ell} s_{m_\varrho}(t)$ . From Cases 1) and 2), the desired claim can possibly be violated only when there exists  $w_{\varrho^*} \in N_{m_1}^-(t)$  with  $s_{w_{\varrho^*}}(t) + s_{m_1}(t) > 0$  for some  $\varrho^* \in \{1, \dots, j\}$  [otherwise we can bound  $|s_{m_1}(t+1)|$  from Case 2)]. While due to the choice of  $m_1$ , it holds that

$$s_{w_{\varrho^*}}(t) + s_{m_1}(t) > 0, \quad \varrho = 1, \dots, \ell.$$

We can therefore denote  $y_i(t) = -s_i(t)$ ,  $i \in \mathcal{V}$  and obtain  $y_{w_{\varrho^*}}(t) \in [(1 - Z_2)M(t), (1 - Z_1)M(t)]$  and  $y_{w_{\varrho^*}}(t) + y_{m_1}(t) < 0$ ,  $\varrho = 1, \dots, \ell$ . We can thus establish the bound for  $|y_{w_{\varrho^*}}(t)| = |s_{w_{\varrho^*}}(t)|$ , again applying Case 2).

From the aforementioned three cases, we deduce that if  $\mathcal{H}_2$  does not hold, then  $\mathcal{H}_1$  must be true. This proves the claim.

Finally, we complete the proof of the lemma using the claim we just established. Take  $\epsilon = 8^{-1}n^{-n-1}$  and  $\beta = 16n^{n+1}$ . We proceed in steps.

Step 1) Let  $m_1 \in \mathcal{V}$  with  $|s_{m_1}(t)| = M(t)$ . Applying the claim with  $Z_1 = 0$  and  $Z_2 = \epsilon$ , we deduce that either the lemma holds or there is another node  $m_2 \in \mathcal{V}$  such that  $|s_{m_2}(t)| \in [(1 - n\epsilon)M(t), (1 - \epsilon)M(t)]$ .

Step 2) If in the first step, we could not conclude that the lemma holds, we can apply the claim to  $m_2$  and then obtain that either the lemma holds, or there is a node  $m_3$  such that  $|s_{m_3}(t)| \in [(1 - n^2\epsilon)M(t), (1 - n\epsilon)M(t)]$ .

The argument can be repeated for  $m_3, \dots$  applying the claim adapting the value of  $\epsilon$  and  $\beta$ . Since there are a total of  $n$  nodes, the aforementioned repeated procedure necessarily ends, so the lemma holds. ■

*Remark 11:* The purpose of Lemma 5 is to establish an absolute lower bound regarding the possible decrease of  $M(t)$ . This lower bound is absolute in the sense that it does not depend on the random graph processes, and requires the constructive conditions  $\alpha < (4n)^{-1}$  and  $\beta > 16n^{n+1}$  to hold. These conditions are certainly rather conservative for a particular node interaction process.

## B. Proof of Theorem 1

From Lemma 4, we know that for any  $i \in \mathcal{V}$ , one of the following events happens almost surely:  $\{\lim_{t \rightarrow \infty} s_i(t) = M_*\}$ ,  $\{\lim_{t \rightarrow \infty} s_i(t) = -M_*\}$ ,  $\{\liminf_{t \rightarrow \infty} s_i(t) = -M_*\}$ , and

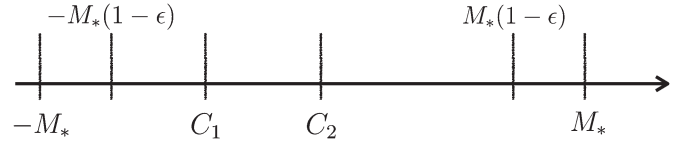


Fig. 2. Illustration of the existence of  $\hat{t}_m$  in the proof of Theorem 1. Here,  $C_1 := -\gamma_* M_*(1 - \epsilon) + (1 - \gamma_*) M_*(1 + \epsilon)$ ,  $C_2 := -\gamma_*^2 M_*(1 - \epsilon) + (1 - \gamma_*^2) M_*(1 + \epsilon)$ . By (7),  $s_i$  starting from the interval  $[-M_*, -M_*(1 - \epsilon)]$  entering  $[M_*(1 - \epsilon), \infty)$  must go through the interval  $[C_1, C_2]$ .

$\limsup_{t \rightarrow \infty} s_i(t) = M_*\}$ . Therefore, we just need to rule out the last case. We actually prove that  $\mathbb{P}(M_* > 0, \liminf_{t \rightarrow \infty} s_i(t) = -M_*, \limsup_{t \rightarrow \infty} s_i(t) = M_*, \lim_{t \rightarrow \infty} |s_i(t)| = M_*) = 0$ .

The following claim holds.

*Claim:* Suppose  $\alpha + \beta \leq 1/(n - 1)$ . Then,  $s_i(t + k) \leq \gamma_*^k s_i(t) + (1 - \gamma_*^k)M(t)$  for all  $k \geq 0$ , where  $\gamma_* = 1 - (\alpha + \beta)(n - 1)$  was introduced in Lemma 2.

First, we note that

$$\begin{aligned} s_i(t + 1) &\leq (1 - Y_i(t)) s_i(t) + Y_i(t)M(t) \\ &\leq \gamma_* s_i(t) + (1 - \gamma_*)M(t) \end{aligned}$$

where the last inequality holds from the fact that  $1 - Y_i(t) \geq 1 - (\alpha + \beta)(n - 1) = \gamma_*$  and that  $s_i(t) \leq M(t)$ . A simple recursive analysis leads to the claim immediately.

Now take  $\epsilon > 0$  and define  $T_1(\epsilon) := \inf\{k : M(k) \leq M_*(1 + \epsilon)\}$ . Note that  $T_1(\epsilon)$  is a stopping time due to the monotonicity of  $M(t)$  established in Lemma 1. In light of the aforementioned claim, we obtain

$$s_i(t + k) \leq \gamma_*^k s_i(t) + (1 - \gamma_*^k)M_*(1 + \epsilon) \quad (7)$$

for all  $k = 0, 1, \dots$  and  $t \geq T_1$ .

Let  $M_* > 0$ . Assume that  $\liminf_{t \rightarrow \infty} s_i(t) = -M_*$ . Then, for the given  $\epsilon$ , we can find an infinite sequence  $T_1(\epsilon) < t_1 < t_2 < \dots$  such that  $s_i(t_m) \leq -M_*(1 - \epsilon)$ . Now, if  $\limsup_{t \rightarrow \infty} s_i(t) = M_*$ , for any  $t_m$ , we can find  $\bar{t}_m > t_m$  with  $s_i(\bar{t}_m) \geq M_*(1 - \epsilon)$ . Then, based on (7), there must be  $\hat{t}_m \in [t_m, \bar{t}_m]$  such that (see Fig. 2)

$$\begin{aligned} -\gamma_* M_*(1 - \epsilon) + (1 - \gamma_*) M_*(1 + \epsilon) &\leq s_i(\hat{t}_m) \\ &\leq -\gamma_*^2 M_*(1 - \epsilon) + (1 - \gamma_*^2) M_*(1 + \epsilon). \end{aligned} \quad (8)$$

We then deduce that for all  $m = 1, 2, \dots$

$$\begin{aligned} |s_i(\hat{t}_m)| &\leq \max\{ |-\gamma_* M_*(1 - \epsilon) + (1 - \gamma_*) M_*(1 + \epsilon)|, \\ &\quad |-\gamma_*^2 M_*(1 - \epsilon) + (1 - \gamma_*^2) M_*(1 + \epsilon)| \} \\ &= M_* \max\{ |1 - 2\gamma_* + \epsilon|, |1 - 2\gamma_*^2 + \epsilon| \} \\ &\leq M_* (\epsilon + \max\{ |1 - 2\gamma_*|, |1 - 2\gamma_*^2| \}) \\ &\leq M_* (1 + \max\{ |1 - 2\gamma_*|, |1 - 2\gamma_*^2| \}) / 2 \end{aligned} \quad (9)$$

if we choose  $\epsilon$  sufficiently small so that

$$\epsilon < (1 - \max\{ |1 - 2\gamma_*|, |1 - 2\gamma_*^2| \}) / 2.$$

Now we see that (9) contradicts  $\lim_{t \rightarrow \infty} |s_i(t)| = M_*$  since

$$0 < (1 + \max\{|1 - 2\gamma_*|, |1 - 2\gamma_*^2|\}) / 2 < 1$$

when  $0 < \gamma_* < 1$ . We have completed the proof.

### C. Proof of Theorem 2

In view of Theorem 1, with probability one, we can divide the node set  $\mathcal{V}$  into the following two subsets of nodes  $\mathcal{V}_1^* := \{i \in \mathcal{V} : \lim_{t \rightarrow \infty} s_i(t) = -M_*\}$  and  $\mathcal{V}_2^* := \{i \in \mathcal{V} : \lim_{t \rightarrow \infty} s_i(t) = M_*\}$ . Apparently at least one of the sets is nonempty. Without loss of generality, we assume  $\mathcal{V}_1^* \neq \emptyset$  and  $\mathbb{P}(M_* > 0) > 0$  for the rest of the proof. With Assumption A3, each arc  $(i, j) \in \mathcal{E}_*$  is associated with a unique sign. We therefore denote the sign of  $(i, j) \in \mathcal{E}_*$  as  $\sigma_{ij}^*$ . To establish the desired conclusion, we first show the following claim holds.

*Claim:* If  $(i, j) \in \mathcal{E}_*$  with  $i, j \in \mathcal{V}_1^*$ , then  $\sigma_{ij}^* = +$ .

The above claim indicates that the arcs among nodes in  $\mathcal{V}_1^*$  are necessarily positive. We now prove this claim by using a similar sample-path analysis as the proof of Theorem 1 by a contradiction argument. Suppose there exists  $i^\dagger, j^\dagger \in \mathcal{V}_1^*$  such that  $(i^\dagger, j^\dagger) \in \mathcal{E}_*$  with  $\sigma_{i^\dagger j^\dagger}^* = -$ . By our assumption, the arc  $(i^\dagger, j^\dagger)$  appears infinitely often in  $\{\mathcal{G}_t\}_{t \geq 0}$ . This means that there exists an infinite subsequence  $\{t_m\}_{m=0}^\infty$  such that  $(i^\dagger, j^\dagger) \in \mathcal{E}_{t_m}$  for all  $m \geq 0$ . We assume without loss of generality that  $(i^\dagger, j^\dagger) \in \mathcal{E}_t$  for all  $t \geq 0$  since the following analysis can indeed be carried out along the subsequence  $\{t_m\}_{m=0}^\infty$ .

Based on the definition of  $\mathcal{V}_1^*$  and  $\mathcal{V}_2^*$ , for any  $\epsilon > 0$ , we can define

$$T_1^\dagger(\epsilon) := \inf \{k : s_i(k) \in [-M_*(1+\epsilon), -M_*(1-\epsilon)], i \in \mathcal{V}_1^*; \\ \times s_i(k) \in [M_*(1-\epsilon), M_*(1+\epsilon)], i \in \mathcal{V}_2^*\}.$$

Since  $(i^\dagger, j^\dagger) \in \mathcal{E}_t$  for all  $t \geq 0$  with  $\sigma_{i^\dagger j^\dagger}^* = -$ , we conclude that

$$\begin{aligned} s_{j^\dagger}(T_1^\dagger + 1) &= \left(1 - Y_{j^\dagger}(T_1^\dagger)\right) s_{j^\dagger}(T_1^\dagger) - \beta D_{T_1^\dagger} s_{i^\dagger}(T_1^\dagger) \\ &\quad + \alpha B_{T_1^\dagger} \sum_{k \in N_{j^\dagger}^+(T_1^\dagger)} s_k(T_1^\dagger) \\ &\quad - \beta D_{T_1^\dagger} \sum_{k \in N_{j^\dagger}^-(T_1^\dagger) \setminus \{i^\dagger\}} s_k(T_1^\dagger) \\ &\geq -\left(1 - Y_{j^\dagger}(T_1^\dagger)\right) M_*(1+\epsilon) + \beta M_*(1-\epsilon) \\ &\quad - Y_{j^\dagger}(T_1^\dagger) M_*(1+\epsilon) \\ &= -[1 - \beta + \epsilon(1 + \beta)] M_* \\ &> -(1 - \epsilon) M_* \end{aligned} \quad (10)$$

if  $D_{T_1^\dagger} = 1$  and  $\epsilon$  is chosen to satisfy  $\epsilon < (1 - \beta)/(2(1 + \beta))$ .

We can recursively define

$$\begin{aligned} T_{m+1}^\dagger(\epsilon) \\ := \inf \{k \geq T_m^\dagger : s_i(k) \in [-M_*(1+\epsilon), -M_*(1-\epsilon)], \\ i \in \mathcal{V}_1^*; s_i(k) \in [M_*(1-\epsilon), M_*(1+\epsilon)], i \in \mathcal{V}_2^*\}. \end{aligned}$$

Repeating the aforementioned analysis, we have

$$s_{j^\dagger}(T_m^\dagger + 1) > -(1 - \epsilon) M_* \quad (11)$$

for each  $m \geq 1$  conditioned on that  $D_{T_m^\dagger} = 1$ . Note that  $\{D_{T_m^\dagger}\}_1^\infty$  defines a sequence of independent random variables since  $\{T_m^\dagger\}$  are a sequence of stopping times. We can therefore invoke the second Borel-Cantelli Lemma (e.g., [27, Theor. 2.3.6]) to conclude that (11) holds for infinitely many  $m$ . In other words, we have established that

$$\liminf_{t \rightarrow \infty} s_{j^\dagger}(T_m^\dagger + 1) > -(1 - \epsilon) M_*$$

for any  $\epsilon < (1 - \beta)/(2(1 + \beta))$ . This contradicts the fact that  $j^\dagger \in \mathcal{V}_1^*$ . We have now proved the given claim.

Having established the above claim, the rest of the argument becomes straightforward. Next, the following analysis will be carried out for the two cases in the theorem statement.

- The total graph  $\mathcal{G}_*$  is strongly balanced with nonempty  $\mathcal{V}_1$  and  $\mathcal{V}_2$  and, hence,  $\mathcal{V}_1^*$  is for example included in  $\mathcal{V}_1$  which, in turn, implies that  $\mathcal{V}_2^* \neq \emptyset$ . Again, there are only positive arcs among nodes of  $\mathcal{V}_2^*$ . We simply deduce that  $\{\mathcal{V}_1, \mathcal{V}_2\} = \{\mathcal{V}_1^*, \mathcal{V}_2^*\}$ . Thus, the required  $y_*$  is exactly  $M_*$ .
- We take a contradiction argument. Since  $\mathbb{P}(M_* > 0) > 0$ , we have  $\mathcal{V}_1^* \cap \mathcal{V}_2^* = \emptyset$ . Again, arcs between nodes in the same set from  $\mathcal{V}_i^*, i = 1, 2$  are necessarily positive. However, there is at least one negative link in  $\mathcal{G}_*$  by assumption, which can only be an arc between  $\mathcal{V}_1^*$  and  $\mathcal{V}_2^*$ . Thus,  $\mathcal{V}_1^*$  and  $\mathcal{V}_2^*$  are nonempty, which implies that  $\mathcal{G}_*$  must be strongly balanced.

The proof is complete.

### D. Proof of Proposition 1

Let  $\beta > 16n^{n+1}$  so the conditions of Lemma 5 hold. Let us fix  $t \geq 0$  and assume that  $|s_{i_0}(t)| = M(t)$  for some  $i_0 \in \mathcal{V}$ . By symmetry, we can also assume without loss of generality that  $s_{i_0}(t) = M(t)$ . Let  $i_* \in \mathcal{V} \setminus \{i_0\}$ . Under Assumptions A4 and A6, we prove the following claim.

*Claim:* There is an integer  $N_0 \geq 1$  and  $q_0 > 0$  such that

$$\mathbb{P}(s_{i_0}(t + N_0K) = M(t), s_{i_*}(t + N_0K) \geq M(t)/2) \geq q_0.$$

In view of the connectivity assumption A4 and the arc independence assumption A6, the event

$$\{s_{i_0}(t + N_0K) = M(t), s_{i_*}(t + N_0K - 1) \geq M(t)/2\}$$

given  $s_{i_0}(t) = M(t)$  can be easily constructed by selecting a proper sequence of positive arcs for time slots  $t, t + 1, \dots, t + N_0K - 1$ , and by imposing that  $B_\tau = 1, D_\tau = 0, \tau = t, t + 1, \dots, t + N_0K - 1$ . This analysis follows standard arguments to analyze basic consensus algorithms (e.g., [37]), and we omit the details. The given claim therefore holds by computing the probability of the selection of the above sequence of arcs and the event  $\{B_\tau = 1, D_\tau = 0, \tau = t, t + 1, \dots, t + N_0K - 1\}$ . Note that  $N_0$  and  $q_0$  depend on  $\alpha, b_*, d_*, p_*, p^*, n$  but do not depend on  $\beta$ .



In addition, in view of Assumption A5, we can select a node  $i_* \neq i_0$  satisfying  $(i_*, i_0) \in \cup_{\tau \in [t+N_0K, t+(N_0+1)K-1]} \mathcal{E}_\tau^-$ . Consider the following sequence of events:

$$\begin{aligned} & \{s_{i_0}(t+N_0K) = M(t), s_{i_*}(t+N_0K) \geq M(t)/2\}; \\ & \{\exists \tau \in [t+N_0K, t+(N_0+1)K-1] \text{ s.t. } (i_*, i_0) \in E_\tau^-\}; \\ & \{D_\tau = 1\}; \\ & \{B_m = D_m = 0, m \neq \tau \in [t+N_0K, t+(N_0+1)K-1]\}. \end{aligned}$$

If they all happen, then

$$\begin{aligned} & |s_{i_0}(t+(N_0+1)K)| \\ & = \left| s_{i_0}(t+N_0K) - \beta(s_{i_0}(t+N_0K) + s_{i_*}(t+N_0K)) \right. \\ & \quad - \beta \sum_{j \in N_{i_0}^-(\tau) \setminus \{i_*\}} (s_{i_0}(t+N_0K) + s_j(\tau)) \\ & \quad \left. + \alpha \sum_{j \in N_{i_0}^+(\tau)} (s_j(\tau) - s_{i_0}(t+N_0K)) \right| \\ & \geq \left( \frac{3}{2}\beta - 1 - \frac{n-2}{n} \right) M(t) \end{aligned} \quad (12)$$

where the last inequality is obtained from the facts that  $s_{i_0}(t+N_0K) = M(t)$ ,  $\beta(s_{i_0}(t+N_0K) + s_{i_*}(t+N_0K)) \geq (3/2)\beta M(t)$ ,  $\beta \sum_{j \in N_{i_0}^-(\tau) \setminus \{i_*\}} (s_{i_0}(t+N_0K) + s_j(\tau)) \geq 0$ , and

$$\left| \alpha \sum_{j \in N_{i_0}^+(\tau)} (s_j(\tau) - s_{i_0}(t+N_0K)) \right| \leq \frac{(n-2)}{2n} \times 2M(t)$$

in light of  $|N_{i_0}^+(\tau)| \leq n-2$  and  $\alpha \leq (4n)^{-1}$ .

It then follows that:

$$\mathbb{P}\left(|s_{i_0}(t+(N_0+1)K)| \geq \left(\frac{3}{2}\beta - 1 - \frac{n-2}{n}\right) M(t)\right) \geq \vartheta_0$$

where  $\vartheta_0 = q_0 p_* d((1-d)(1-b))^{K-1}$ . This implies

$$\mathbb{P}(M(t+(N_0+1)K) \geq (3\beta-4)M(t)/2) \geq \vartheta_0. \quad (13)$$

Now assume that  $M(0) > 0$  so that  $U(m) = \log(M(m(N_0+1)K))$  for  $m \geq 0$  is well defined. Note that from Lemma 5 and (13), we have

$$\mathbb{E}\{U(m+1) - U(m)\} \geq -N_0K \log(2n) + \vartheta_0 \log((3\beta-4)/2).$$

For  $\beta$  large enough, the right-hand side in the above inequality is strictly positive. We can then easily conclude, using classical arguments in random walks, that the process  $U(m)$  has a strictly positive drift, from which it can be deduced that  $\mathbb{P}(\liminf_{m \rightarrow \infty} M(mN_0K) = \infty) = 1$  (for  $\beta$  large enough). Using Lemma 5, one can easily conclude the desired theorem.

### E. Proof of Theorem 3

The argument is based on the intuition that when one of the node states diverges, there is always a realization of edges with nonzero probability that ‘‘pulls’’ another node toward divergence since these pulling actions happen infinitely often due to the Borel-Cantelli Lemma. Then suitable connectivity of the interaction graphs recursively leads to the desired no-survivor property.

Assume that for some  $q_* > 0$ , we have  $\mathbb{P}(\limsup_{t \rightarrow \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty) \geq q_*$ . There must be a node  $i_0$  satisfying  $\mathbb{P}(\limsup_{t \rightarrow \infty} |s_{i_0}(t)| = \infty) \geq q_*/n$ . Let  $C_0 > 0$ , and define  $T_1^* := \inf_t \{|s_{i_0}(t)| \geq C_0\}$ .  $T_1^*$  is a stopping time. Let  $K > 0$  be an integer. We can further recursively define  $T_2^*, \dots, T_m^*, \dots$  by

$$T_{m+1}^* := \inf_{t \geq T_m^* + K} \{|s_{i_0}(t)| \geq C_0\}.$$

Based on [27, Theor. 4.1.3], each  $T_m^*$  is a stopping time for all  $m \geq 0$  and  $(G_{T_1^*}, B_{T_1^*}, D_{T_1^*}), \dots, (G_{T_1^*+K-1}, B_{T_1^*+K-1}, D_{T_1^*+K-1}); (G_{T_2^*}, B_{T_2^*}, D_{T_2^*}), \dots, (G_{T_2^*+K-1}, B_{T_2^*+K-1}, D_{T_2^*+K-1}); \dots$  are independent random variables that are also independent of  $\mathcal{F}_{T_1^*-1}$ . In addition, we have  $\mathbb{P}(T_m^* < \infty, m = 1, 2, \dots) \geq q_*/n$ . Under Assumption A2,  $\mathcal{G}([T_1^*, T_1^* + K - 1])$  being strongly connected is a sure event. As a result, there exists another node  $i_1 \in \mathcal{V} \setminus i_0$  and  $\tau_0 \in [T_1^*, T_1^* + K - 1]$  such that  $(i_0, i_1) \in \mathcal{E}_{\tau_0}$ . Assume the event  $s_{i_0}(\tau_0) = s_{i_0}(T_1^*)$  (whose probability can be lower bounded by  $B_t = D_t = 0, t \in [T_1^*, \tau_0 - 1]$  and is then used to derive the  $\chi_0$  below). We treat two cases:  $\sigma_{i_0 i_1} = -$  and  $\sigma_{i_0 i_1} = +$ .

a)  $\sigma_{i_0 i_1} = -$ .

- If  $\beta = 1$ , then  $|\beta s_{i_0}(\tau_0) + (1-\beta)s_{i_1}(\tau_0)| = |\beta s_{i_0}(\tau_0)| = |s_{i_0}(T_1^*)| \geq C_0$ .
- If  $\beta \neq 1$  and  $|s_{i_1}(\tau_0)| < \beta C_0 / (2|1-\beta|)$ , then  $|\beta s_{i_0}(\tau_0) + (1-\beta)s_{i_1}(\tau_0)| \geq \beta C_0 - (1-\beta)|s_{i_1}(\tau_0)| \geq \beta C_0 / 2$ .

b)  $\sigma_{i_0 i_1} = +$ .

- If  $\alpha = 1$ , then  $|\alpha s_{i_0}(\tau_0) + (1-\alpha)s_{i_1}(\tau_0)| = C_0$ .
- If  $\alpha \neq 1$  and  $|s_{i_1}(\tau_0)| < \alpha C_0 / (2|1-\alpha|)$ , then  $|\alpha s_{i_0}(\tau_0) + (1-\alpha)s_{i_1}(\tau_0)| \geq \alpha C_0 / 2$ .

Now  $s_{i_1}(\tau_0+1) = -\beta s_{i_0}(\tau_0) + (1-\beta)s_{i_1}(\tau_0)$  when  $i_0$  is the unique node in  $N_{i_1}^-(\tau_0)$  and  $D_{\tau_0} = 1$ . Also observe that  $s_{i_1}(\tau_0+1) = \alpha s_{i_0}(\tau_0) + (1-\alpha)s_{i_1}(\tau_0)$  when  $i_0$  is the unique node in  $N_{i_1}^+(\tau_0)$  and  $B_{\tau_0} = 1$ . Stationarity ensures that  $(B_{T_1^*}, D_{T_1^*}), \dots, (B_{T_1^*+K-1}, D_{T_1^*+K-1})$  have the same distribution as  $(B_0, D_0)$ . We can therefore simply bound the probabilities of the above events and establish

$$\mathbb{P}(\exists i_1 \in \mathcal{V} \setminus \{i_0\} : |s_{i_1}(T_1^* + K)| \geq \phi C_0) \geq \chi_0$$

where  $\chi_0 = ((1-b)(1-d))^{2K-1} \min\{b, d\} p_* (1-p^*)^{n-2}$  and  $\phi = \min\{[\alpha/(2|1-\alpha|)], \alpha/2, [\beta/(2|1-\beta|)], \beta/2, 1\}$  (we use  $[\cdot]$  to indicate that the corresponding term is taken into account

in the min only if it is well defined). Repeating the analysis on  $T_2^*, \dots$ , we obtain

$$\mathbb{P}(\exists i_m \in \mathcal{V} \setminus \{i_0\} : |s_{i_m}(T_m^* + K)| \geq \phi C_0) \geq \chi_0.$$

Since we have a finite number of nodes, independence allows us to invoke the second Borel-Cantelli Lemma [27, Theor. 2.3.6] and conclude that

$$\mathbb{P}\left(\exists i_1 \in \mathcal{V} \setminus \{i_0\} : \limsup_{t \rightarrow \infty} |s_{i_1}(t)| \geq \phi C_0 |T_m^*| < \infty, m = 1, \dots\right) = 1. \quad (14)$$

Note that  $C_0$  can be chosen arbitrarily and, hence, (14) implies that there exists  $i_1 \in \mathcal{V} \setminus \{i_0\}$  such that

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} |s_{i_1}(t)| = \infty \mid \limsup_{t \rightarrow \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty\right) = 1. \quad (15)$$

We can apply the same argument recursively, to show that (15) holds for any node  $i_1$  in the network. This concludes the proof.

## V. CONCLUSION

Inspired by examples from social, biological, and engineering networks, the emerging behaviors of node states evolving over signed random networks in a dynamical environment were studied. Each node received positive and negative recommendations from its neighbors determined by the sign of the arcs. The positive recommendations were consistent with the standard consensus dynamics, while the negative recommendations flip the sign of node states in the local interactions as introduced by Altafini in [36]. After receiving recommendations, each node puts a deterministic weight and a random attention on each recommendation and then updates its state. Various conditions were derived regarding the almost sure convergence and divergence of this model. These results have significantly extended the analysis of the results of [36] to more general models and detailed results. The corresponding relative-state flipping model [37], [38] under this general random graph model will be investigated in our future work. Some other interesting future directions include the co-evolution of the signs of the interaction links along with the node states, as well as the optimal placement of negative links with the aim of breaking the effect of positive updates as much as possible.

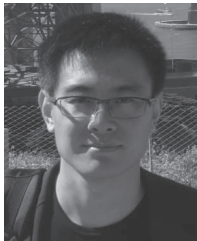
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## REFERENCES

- [1] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, pp. 118–121, 1974.
- [2] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transitions in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, pp. 1226–1229, 1995.
- [3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.
- [4] E. Bonabeau, M. Dorigo, and G. Theraulaz, *Swarm Intelligence: From Natural to Artificial System*. New York, USA: Oxford University Press, 1999.
- [5] B. Golub and M. O. Jackson, "Naive learning in social networks and the wisdom of crowds," *Amer. Econ. J.: Microecon.*, vol. 2, pp. 112–149, 2010.
- [6] P. M. DeMarzo, D. Vayanos, and J. Zwiebel, "Persuasion bias, social influence, unidimensional opinions," *Quart. J. Econ.*, vol. 118, no. 3, pp. 909–968, 2003.
- [7] D. Acemoglu, A. Ozdaglar, and A. ParandehGheibi, "Spread of (Mis)information in social networks," *Games Econ. Behav.*, vol. 70, no. 2, pp. 194–227, 2010.
- [8] J. N. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control*, vol. AC-31, no. 9, pp. 803–812, Sep. 1986.
- [9] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Syst. Control Lett.*, vol. 53, pp. 65–78, 2004.
- [10] V. Blondel, J. M. Hendrickx, A. Olshevsky, and J. Tsitsiklis, "Convergence in multiagent coordination, consensus, flocking," in *Proc. IEEE Conf. Dec. Control*, 2005, pp. 2996–3000.
- [11] L. Moreau, "Stability of multi-agent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.
- [12] W. Ren and R. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.
- [13] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," in *Proc. IEEE*, Jan. 2007, vol. 95, no. 1, pp. 215–233.
- [14] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: A graphical approach," *SIAM J. Control Optimiz.*, vol. 47, no. 2, pp. 575–600, 2008.
- [15] J. M. Hendrickx and J. N. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 214–218, Jan. 2013.
- [16] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1867–1872, Nov. 2005.
- [17] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2508–2530, Jun. 2006.
- [18] F. Fagnani and S. Zampieri, "Randomized consensus algorithms over large scale networks," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 4, pp. 634–649, May 2008.
- [19] S. Kar and J. M. F. Moura, "Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise," *IEEE Trans. Signal Process.*, vol. 57, no. 5, pp. 355–369, May 2009.
- [20] B. Touri and A. Nedić, "On ergodicity, infinite flow and consensus in random models," *IEEE Trans. Autom. Control*, vol. 56, no. 7, pp. 1593–1605, Jul. 2011.
- [21] A. Tahbaz-Salehi and A. Jadbabaie, "A necessary and sufficient condition for consensus over random networks," *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 791–795, Apr. 2008.
- [22] A. Fazeli and A. Jadbabaie, "Consensus in martingale graph processes," in *Proc. Amer. Control Conf.*, 2012, pp. 845–850.
- [23] T. C. Aysal and K. E. Barner, "Convergence of consensus models with stochastic disturbances," *IEEE Trans. Inf. Theory*, vol. 56, no. 8, pp. 4101–4113, Aug. 2010.
- [24] I. Matei, J. S. Baras, and C. Somarakis, "Convergence results for the linear consensus problem under markovian random graphs," *SIAM J. Control Optimiz.*, vol. 51, no. 2, pp. 1574–1591, 2013.
- [25] D. W. Soh, W. P. Tay, and T. Q. S. Quek, "Randomized information dissemination in dynamic environments," *IEEE/ACM Trans. Netw.*, vol. 21, no. 3, pp. 681–691, Jun. 2013.
- [26] P. Erdős and A. Rényi, "On the evolution of random graphs," *Publ. Math. Inst. Hungarian Acad. Sci.*, pp. 17–61, 1960.
- [27] R. Durrett, *Probability Theory: Theory and Examples*, 4th ed. Cambridge, U.K.: Cambridge University Press, 2010.

- [28] F. Heider, "Attitudes and cognitive organization," *J. Psychol.*, vol. 21, pp. 107–112, 1946.
- [29] S. Galam, "Fragmentation versus stability in bimodal coalitions," *Phys. A.*, vol. 230, pp. 174–188, 1996.
- [30] L. Edelstein-Keshet, *Mathematical Models in Biology*. New York, USA: McGraw-Hill, 1987.
- [31] S. Grossberg, "Adaptive resonance theory," *Scholarpedia*, 2013. [Online]. Available: [http://www.scholarpedia.org/article/Adaptive\\_resonance\\_theory](http://www.scholarpedia.org/article/Adaptive_resonance_theory)
- [32] N. Yosef, A. K. Shalek, J. T. Gaublot, H. Jin, Y. Lee, A. Awasthi *et al.*, "Dynamic regulatory network controlling TH17 cell differentiation," *Nature*, vol. 496, pp. 461–468, 2013.
- [33] B. T. Polyak, *Introduction to Optimization*. New York, Optimization Software, USA, 1987.
- [34] S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz, "Continuous-time model of structural balance," *PNAS*, vol. 108, no. 5, pp. 1751–1752, 2011.
- [35] G. Facchetti, G. Iacono, and C. Altafini, "Computing global structural balance in large-scale signed social networks," *PNAS*, vol. 108, no. 52, pp. 20953–20958, 2011.
- [36] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.
- [37] G. Shi, M. Johansson, and K. H. Johansson, "How agreement and disagreement evolve over random dynamic networks," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 6, pp. 1061–1071, Jun. 2013.
- [38] G. Shi, A. Proutiere, M. Johansson, J. S. Baras, and K. H. Johansson, *The Evolution of Beliefs Over Signed Social Networks*, 2013, arXiv:1307.0539.
- [39] D. Cartwright and F. Harary, "Structural balance: A generalization of Heider's theory," *Psychol Rev.*, vol. 63, pp. 277–293, 1956.
- [40] J. A. Davis, "Structural balance, mechanical solidarity, interpersonal relations," *Amer. J. Sociol.*, vol. 68, pp. 444–462, 1963.
- [41] G. Shi and K. H. Johansson, *Consensus Computations Over Random Graph Processes*, 2011, arXiv:1112.1336.



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