A Generalized Gossip Algorithm on Convex Metric Spaces

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***Abstract*—A consensus problem consists of a group of dynamic agents who seek to agree upon certain quantities of interest. This problem can be generalized in the context of convex metric spaces that extend the standard notion of convexity. In this paper we introduce and analyze a randomized gossip algorithm for solving the generalized consensus problem on convex metric spaces, where the communication between agents is controlled by a set of Poisson counters. We study the convergence properties of the algorithm using stochastic calculus. In particular, we show that the distances between the states of the agents converge to zero with probability one and in the r^{th} mean sense. In the special case of complete connectivity and uniform Poisson counters, we give upper bounds on the dynamics of the first and second moments of the distances between the states of the agents. In addition, we introduce instances of the generalized consensus algorithm for several examples of convex metric spaces together with numerical simulations.**

***Index Terms*—Consensus, convex metric spaces, gossip algorithms, stochastic differential equations.**

I. Introduction

DISTRIBUTED algorithms are found in applications related to sensor, peer-to-peer and *ad-hoc* networks. A particular distributed algorithm is the *consensus* (or agreement) algorithm, where a group of dynamic agents seek to agree upon certain quantities of interest by exchanging information among themselves, according to a set of rules. This problem can model many phenomena involving information exchange between agents such as cooperative control of vehicles, formation control, flocking, synchronization, parallel computing, etc. Distributed computation over networks has a long history in control theory starting with the work of Borkar and Varaiya [1], Tsitsikils *et al.* [32], [33] on asynchronous agreement problems and parallel computing. A theoretical framework for solving consensus problems was introduced by Olfati-Saber and Murray in [19], [20], while Jadbabaie *et al.* [8] studied alignment problems for reaching an agreement. Relevant extensions of the consensus problem were done by Ren and Beard [25], by Moreau [16] or, more recently, by Nedic and Ozdaglar [17], [18].

Network topologies change with time (as new nodes join and old nodes leave the network) or exhibit random behavior due to link failures, packet drops, node failure, etc. This motivated the investigation of consensus algorithms under a stochastic framework [6], [11], [13], [22], [26], [27]. In addition to network variability, nodes in sensor networks operate under limited computational, communication, and energy resources. These constraints have motivated the design of "gossip" algorithms, in which a node communicates with a randomly chosen neighbor. Studies of randomized gossip consensus algorithms can be found in [2], [29]. In particular, consensus based gossip algorithms have been extensively used in the analysis and study of the performance of wireless networks, with random failures [21].

In this paper, we introduce and analyze a generalized randomized gossip algorithm for achieving consensus. The algorithm acts on *convex metric spaces*, which are metric spaces endowed with a *convex structure*. We show that under the given algorithm, the agents' states converge to consensus with probability one and in the r^{th} mean sense. The convergence study is based on analyzing the dynamics of a set of stochastic differential equations driven by Poisson counters. Additionally, for a particular network topology we investigate in more depth the rate of convergence of the first and second moment of the distances between the agents' states. We present instances of the generalized gossip algorithm for three convex metric spaces defined on the set of real numbers, the collection of compact, convex sets, and the set of discrete random variables. It is widely recognized that asymptotic agreement among agents is achieved if their states move towards the interior of the convex hull they define. This fundamental notion is explained in [16] for dynamics evolving in finite dimensional Euclidean spaces. In our work we extend these results to be applicable in convex metric spaces. Then one can define update algorithms which yield asymptotic consensus over autonomous agents, a probabilistic-gossip alternative of which is the topic of this paper. Generalizing the convex property to non-Euclidean spaces allows for dropping a number of smoothness assumptions on the dynamics. For example, the continuity assumption of the maps in [16], necessary for the stability analysis together with properties such as compactness and boundedness are no longer necessary. The present work is a continuation of our

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previous results regarding the consensus problem on convex metric space, where only deterministic, time-varying communication topologies are studied [9], [10], [12]. Compared with the aforementioned work, the stochastic framework assumed in the current paper requires a completely different approach for studying the convergence properties of the algorithm. A preliminary short version of this paper can be found in [14], where due to space limitations most of the results are introduced without proof. Here, we refine and improve the results initially introduced in [14], we include all necessary proofs together and some new examples of convex metric spaces and their corresponding agreement algorithms.

The paper is organized as follows. Section II introduces the main concepts related to convex metric spaces. Section III formulates the problem and states our main results. Sections IV and V give the proof of our main results, together with pertinent preliminary results. In Section VI, for a complete communication topology and uniform Poisson counters, we present an in-depth analysis of the rate of convergence to consensus, in the first and second moments sense. Section VII shows instances of the generalized consensus algorithm for three convex metric spaces.

Basic Notations: Given $W \in \mathbb{R}^{n\times n}$ by $[W]_{ij}$ we refer to the (i,j) element of the matrix. The *underlying graph* of W is a graph of order n without self loops, for which every edge corresponds to a *non-zero*, *off-diagonal* entry of W. We denote by $\mathbb{1}_{\{A\}}$ the indicator function of the event A. Given two symmetric matrices M_1 and M_2, by $M_1 \succ M_2$ ($M_1 \succeq M_2$) we understand that $M_1 - M_2$ is a positive definite (positive semi-definite) matrix. Additionally, by $M_1 \prec M_2$ ($M_1 \preceq M_2$) we understand that $M_2 - M_1$ is a positive definite (positive semi-definite) matrix.

II. Convex Metric Spaces

In this section, we introduce a set of definitions and basic results about convex metric spaces. Additional information about the following definitions and results can be found in [30], [31].

Definition 2.1 ([31, pp. 142]): Let $(\mathcal{X}, d)$ be a metric space. A mapping $\Psi : \mathcal{X}\times\mathcal{X}\times[0,1] \to \mathcal{X}$ is said to be a *convex structure* on $\mathcal{X}$ if

$$d\left(u, \Psi(x,y,\lambda)\right) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

$\forall x, y, u \in \mathcal{X}$ and $\forall \lambda \in [0,1]$.

Definition 2.2 ([31, pp. 142]): The metric space $(\mathcal{X}, d)$ together with the convex structure Ψ is called a *convex metric space*, and is denoted henceforth by the triplet $(\mathcal{X}, d, \Psi)$.

Definition 2.3 ([31, pp. 144]): A convex metric space $(\mathcal{X}, d, \Psi)$ is said to have *Property* (C) if every bounded decreasing net of nonempty closed convex subsets of $\mathcal{X}$ has a nonempty intersection.

Fortunately, convex matric spaces satisfying *Property* (C) are not that rare. Indeed, by Smulian's Theorem ([3, page 443]), every weakly compact convex subset of a Banach space has *Property* (C).

The following definition introduces the notion of convex set in convex metric spaces.

Definition 2.4 ([31, pp. 143]): Let $(\mathcal{X}, d, \Psi)$ be a convex metric space. A nonempty subset $K \subset \mathcal{X}$ is said to be *convex* if $\Psi(x,y,\lambda) \in K$, $\forall x, y \in K$ and $\forall \lambda \in [0,1]$.

Let $\mathcal{P}(\mathcal{X})$ be the set of all subsets of $\mathcal{X}$. We define the set valued mapping $\tilde{\Psi} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ as

$$\tilde{\Psi}(A) \triangleq \{\Psi(x,y,\lambda) | \ \forall x, y \in A, \forall \lambda \in [0,1]\}$$

where A is an arbitrary subset of $\mathcal{X}$.

In Proposition 1 of [31, p. 143] it is shown that in a convex metric space, an arbitrary intersection of convex sets is also convex, and therefore the next definition makes sense.

Definition 2.5 ([30, p. 11]): Let $(\mathcal{X}, d, \Psi)$ be a convex metric space. The *convex hull* of the set $A \subset \mathcal{X}$ is the intersection of all convex sets in $\mathcal{X}$ containing A and is denoted by $co(A)$.

Another characterization of the convex hull of a set in $\mathcal{X}$ is given in what follows. By defining $A_m \triangleq \tilde{\Psi}(A_{m-1})$ with $A_0 = A$ for some $A \subset \mathcal{X}$, it is discussed in [30] that the set sequence $\{A_m\}_{m\geq 0}$ is increasing, $\limsup_{m\to\infty} A_m$ exists, and $\limsup_{m\to\infty} A_m = \liminf_{m\to\infty} A_m = \lim_{m\to\infty} A_m = \bigcup_{m=0}^{\infty} A_m$.

Proposition 2.1 ([30, p. 12]): Let $(\mathcal{X}, d, \Psi)$ be a convex metric space. The convex hull of a set $A \subset \mathcal{X}$ is given by

$$co(A) = \lim_{m\to\infty} A_m = \bigcup_{m=0}^{\infty} A_m.$$

It follows immediately from above that if $A_{m+1} = A_m$ for some m, then $co(A) = A_m$.

We give several examples of convex metric spaces in Section VII. Among them, the most familiar convex metric space in the set of real numbers, together with the Euclidean distance and the standard convex combination operator. More interesting convex metric spaces are based on the collection of compact, convex sets on $\mathbb{R}^n$ and on the set of discrete random variables. We show that the collection of compact, convex sets endowed with the Hausdorff distance and a convex structure based on the Minkowski sum is indeed a convex metric space. This space allows us to generate set dynamics that will drive a collection of sets to the same value. Similarly, the set of discrete random variables endowed with the (expected value) of the discrete metric and a convex structure based on indicator functions is also a convex metric space. As it will be seen later, such a space allows for generating probabilistic consensus algorithm on finite, countable sets.

III. Problem Formulation and Main Results

Let $(\mathcal{X}, d, \Psi)$ be a convex metric space. We consider a set of n agents indexed by i, with states denoted by $x_i(t)$ taking values on $\mathcal{X}$, where t represents the continuous time.

A. Communication Model

The communication among agents is subject to a communication network modeled by a undirected graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is the set of agents, and $E = \{(j, i)| j$

can send information to $i\}$ is the set of edges. In addition, we denote by $\mathcal{N}_i$ the inward neighborhood of agent i, i.e.,

$$\mathcal{N}_i \triangleq \{j|\ (j,i) \in E\}$$

where by assumption node i does not belong to the set $\mathcal{N}_i$.

We make the following connectivity assumption.

Assumption 3.1: The graph $G = (V, E)$ is connected.

B. Randomized Gossip Algorithm

In the following, we describe the mechanism used by the agents to update their states. Agents can be in two modes: *sleep* mode and *update* mode. Let $N_i(t)$ be a Poisson counter associated to agent i. In the sleep mode, the agents maintain their states unchanged. An agent i exits the sleep mode and enters the update mode when the associated counter $N_i(t)$ increments its value. Let t_i be a time-instant at which the Poisson counter $N_i(t)$ increments its value. Then at t_i, agent i picks agent j with probability $p_{i,j}$, where $j \in \mathcal{N}_i$ and updates its state according to the rule

$$x_i\left(t_i^+\right) = \Psi(x_i\left(t_i\right), x_j(t_i), \lambda_i) \tag{1}$$

where $\lambda_i \in [0,\ 1)$, Ψ is the convex structure and $\sum_{j\in\mathcal{N}_i} p_{i,j} = 1$. By $x_i(t_i^+)$ we understand the value of $x_i(t)$ immediately after the instant update at time t_i, which can be also written as

$$x_i\left(t_i^+\right) = \lim_{t\to t_i,\ t>t_i} x_i(t)$$

which implies that $x_i(t)$ is a left-continuous function of t. After agent i updates its state according to the above rule, it immediately returns to the sleep mode, until the next increase in value of the counter $N_i(t)$.

Assumption 3.2: The Poisson counters $N_i(t)$ are independent and with rate μ_i, for all i.

A similar form of the above algorithm (the Poisson counters are assumed to have the same rates) was extensively studied in [2], in the case where $\mathcal{X} = \mathbb{R}$.

Let $d(x_i(t), x_j(t))$ be the distance between the states of agents i and j, at time t. We note that since the agents update their state at random times, the distances between agents are random processes. We introduce the following convergence definitions.

Definition 3.1: For given $r \geq 1$, we say that the agents converge to consensus in r^{th} *mean sense* if

$$\lim_{t\to\infty} E\left\{d\left(x_i(t), x_j(t)\right)^r\right\} = 0,\ \forall\ (i,j),\ i \neq j.$$

Definition 3.2: We say that the agents converge to consensus *with probability one* if

$$Pr\left(\lim_{t\to\infty} \max_{i,j} d\left(x_i(t), x_j(t)\right) = 0\right) = 1.$$

The following theorem states our main convergence results.

Theorem 3.1: Under Assumptions 3.1 and 3.2 and under the randomized gossip algorithm

(a) the agents converge to consensus in r^{th} mean for all $r \geq 1$, in the sense of Definition 3.1;

(b) the agents converge to consensus with probability one, in the sense of Definition 3.2;

(c) if in addition the convex metric space satisfies *Property* (C), then for any sample path ω of state processes, there exists $x^* \in \mathcal{X}$ (that may depend on ω) such that

$$\lim_{t\to\infty} d\left(x_i(t,\omega), x^*(\omega)\right) = 0.$$

The above theorem states that the agents will reach consensus in the r^{th} mean sense and almost sure sense. In addition, not only that the distances between the states of the agents will converge to zero, but in fact, all agents will converge to some common point in $\mathcal{X}$ with probability one.

We point out that although we use Poisson statistics for the activation times, other statistics can be used as well. The Poisson statistics, however, allow us to use $It\hat{o}$ calculus to derive expressions for the first and second moments of the distances between agents. Note that even though the communication graph is assumed undirected, most communications take place unidirectionally. The only situation when bidirectional links are required is at the instance an agent wakes up and needs to signal one of its neighbors that is ready to receive its latest state.

IV. Preliminary Results

In this section, we construct the stochastic dynamics of the vector of distances between agents. Let t_i be a time-instant at which counter $N_i(t)$ increments its value. According to the gossip algorithm, the distance between agents i and j at time t_i^+ is given by

$$d\left(x_i(t_i^+), x_j(t_i^+)\right) = d\left(\Psi\left(x_i(t_i), x_l(t_i), \lambda_i\right), x_j(t_i)\right) \tag{2}$$

with probability $p_{i,l}$. Let $\theta_i(t)$ be an independent and identically distributed (i.i.d.) random process, such that $Pr(\theta_i(t) = l) = p_{i,l}$ for all $l \in \mathcal{N}_i$ and for all t. It follows that (2) can be equivalently written as

$$d\left(x_i(t_i^+), x_j(t_i^+)\right) = \sum_{l\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(t_i)=l\}} d\left(\Psi\left(x_i(t_i), x_l(t_i), \lambda_i\right),\ x_j(t_i)\right)$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function. Using the inequality property of the convex structure introduced in Definition 2.1, we further get

$$d\left(x_i(t_i^+), x_j(t_i^+)\right) \leq \lambda_i d\left(x_i(t_i), x_j(t_i)\right) + \\ +(1-\lambda_i) \sum_{l\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(t_i)=l\}} d\left(x_l(t_i), x_j(t_i)\right). \tag{3}$$

Assuming that t_j is a time-instant at which the Poisson counter $N_j(t)$ increments its value, in a similar manner as above, we get that

$$d\left(x_i(t_j^+), x_j(t_j^+)\right) \leq \lambda_j d\left(x_i(t_j), x_j(t_j)\right) + \\ +(1-\lambda_j) \sum_{l\in\mathcal{N}_j} \mathbb{1}_{\{\theta_j(t_j)=l\}} d\left(x_l(t_j), x_i(t_j)\right). \tag{4}$$

Consider now the scalars $\eta_{i,j}(t)$ whose dynamics satisfy (3) and (4), but with equality, that is

$$\eta_{i,j}\left(t_i^+\right) = \lambda_i \eta_{i,j}(t_i) + (1-\lambda_i)\sum_{l\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(t_i)=l\}}\eta_{j,l}(t_i) \quad (5)$$

and

$$\eta_{i,j}\left(t_j^+\right) = \lambda_j \eta_{i,j}(t_j) + (1-\lambda_j)\sum_{l\in\mathcal{N}_j} \mathbb{1}_{\{\theta_j(t_j)=l\}}\eta_{i,l}(t_j) \quad (6)$$

with $\eta_{i,j}(0) = d(x_i(0), x_j(0))$.

Remark 4.1: Note that the index pair of η refers to the distance between two agents i and j. As a consequence, $\eta_{i,j}$ and $\eta_{j,i}$ will be considered the same objects and counted only once.

Proposition 4.1: The following inequalities are satisfied with probability one:

$$\eta_{i,j}(t) \geq 0 \quad (7)$$
$$\eta_{i,j}(t) \leq \max_{i,j} \eta_{i,j}(0) \quad (8)$$
$$d\left(x_i(t), x_j(t)\right) \leq \eta_{i,j}(t) \quad (9)$$

for all $i \neq j$ and $t \geq 0$.

Proof: Inequalities (7) and (8) follow immediately, noting that for any sample path of the Poisson counters, $\eta_{i,j}(t)$ are updated by performing convex combinations of non-negative quantities. To show inequality (9) we can use an inductive argument. Let t_i be the time instant at which the counter $N_i(t)$ increments its value and assume that $d(x_i(t_i), x_j(t_i)) \leq \eta_{i,j}(t_i)$ for all i, j. Immediately after t_i, the new value of $d(x_i(t), x_j(t))$ is given by

$$\begin{aligned} d\left(x_i(t_i^+), x_j(t_i^+)\right) &\leq \lambda_i d\left(x_i(t_i), x_j(t_i)\right) + \\ &+ (1-\lambda_i)\sum_{l\in\mathcal{N}_j} \mathbb{1}_{\{\theta_j(t_i)=l\}} d\left(x_l(t_i), x_i(t_i)\right) \leq \\ &\leq \lambda_i \eta_{i,j}(t_i) + (1-\lambda_i)\sum_{l\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(t_i)=l\}}\eta_{j,l}(t_i) = \eta_{i,j}\left(t_i^+\right). \end{aligned}$$

Therefore, after each increment of counter $N_i(t)$, we get that

$$d\left(x_i\left(t_i^+\right), x_j\left(t_i^+\right)\right) \leq \eta_{i,j}\left(t_i^+\right).$$

Using the same argument for all Poison counters, inequality (9) follows. ■

We now elaborate on the dynamics of $\eta_{i,j}(t)$. From (5) and (6) we note that $\eta_{i,j}(t)$ at time t_i and t_j must agree with the solution of a stochastic differential equation driven by Poisson counters. Namely, we have

$$\begin{aligned} d\eta_{i,j}(t) = &\left[-(1-\lambda_i)\eta_{i,j}(t) + (1-\lambda_i) \times \sum_{l\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(t)=l\}}\eta_{j,l}(t)\right] dN_i(t) + \\ &\left[-(1-\lambda_j)\eta_{i,j}(t) + (1-\lambda_j) \times \sum_{m\in\mathcal{N}_j} \mathbb{1}_{\{\theta_j(t)=m\}}\eta_{i,m}(t)\right] dN_j(t). \end{aligned} \quad (10)$$

Let us now define the $\bar{n}$ dimensional vector $\boldsymbol{\eta} = (\eta_{i,j})$, where $\bar{n} = n(n-1)/2$ (since (i,j) and (j,i) correspond to the same distance variable). Equation (10) can be compactly written as

$$\begin{aligned} d\boldsymbol{\eta}(t) = &\sum_{(i,j),i\neq j} \Phi_{i,j}\left(\theta_i(t)\right)\boldsymbol{\eta}(t) dN_i(t) \\ &+ \sum_{(i,j),i\neq j}, \Psi_{i,j}\left(\theta_j(t)\right)\boldsymbol{\eta}(t) dN_j(t). \end{aligned} \quad (11)$$

where the $\bar{n}\times\bar{n}$ dimensional matrices $\Phi_{i,j}(\theta_i(t))$ and $\Psi_{i,j}(\theta_j(t))$ are defined as

$$\Phi_{i,j}\left(\theta_i(t)\right) = \begin{cases} -(1-\lambda_i) & \text{at entry } [(i,j)(i,j)] \\ (1-\lambda_i)\mathbb{1}_{\{\theta_i(t)=l\}} & \text{at entries } [(i,j)(l,j)], \\ & l\in\mathcal{N}_i,\ l\neq j, l\neq i \\ 0 & \text{all other entries} \end{cases}$$

and

$$\Psi_{i,j}\left(\theta_j(t)\right) = \begin{cases} -(1-\lambda_j) & \text{at entry } [(i,j)(i,j)] \\ (1-\lambda_j)\mathbb{1}_{\{\theta_j(t)=m\}} & \text{at entries } [(i,j)(m,i)], \\ & m\in\mathcal{N}_j, m\neq j, m\neq i \\ 0 & \text{all other entries.} \end{cases}$$

The dynamics of the first moment of the vector $\boldsymbol{\eta}(t)$ is given by

$$\begin{aligned} \frac{d}{dt}E\left\{\boldsymbol{\eta}(t)\right\} = &\sum_{(i,j),i\neq j} E\left\{\Phi_{i,j}\left(\theta_i(t)\right)\boldsymbol{\eta}(t)\mu_i \right. \\ &\left. + \Psi_{i,j}\left(\theta_j(t)\right)\boldsymbol{\eta}(t)\mu_j\right\}. \end{aligned} \quad (12)$$

Using the independence of the random processes $\theta_i(t)$, we can further write

$$\frac{d}{dt}E\left\{\boldsymbol{\eta}(t)\right\} = \boldsymbol{W}E\left\{\boldsymbol{\eta}(t)\right\}$$

where $\boldsymbol{W}$ is a $\bar{n}\times\bar{n}$ dimensional matrix whose entries are given by

$$[\boldsymbol{W}]_{(i,j),(l,m)} = \begin{cases} -(1-\lambda_i)\mu_i - (1-\lambda_j)\mu_j & l=i \text{ and } m=j \\ (1-\lambda_i)\mu_i p_{i,l} & l\in\mathcal{N}_i,\ m=j,\ l\neq j, \\ (1-\lambda_j)\mu_j p_{j,m} & l=i,\ m\in\mathcal{N}_j,\ m\neq i, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

The elements of the matrix $\boldsymbol{W}$ are calculated from the elements of $\Phi_{i,j}$ and Ψ_{ij} after taking the expected value in (11). It is a matrix whose entries depend on the rates of the Poisson counters, on the parameters of the convex structure, on the probabilities of choosing neighbors, and on the connectivity structure of the communication graph. More importantly, it controls the dynamics of the first moment of $\boldsymbol{\eta}$.

The following Lemma studies the properties of the matrix $\boldsymbol{W}$, introduced above.

Lemma 4.1: Let $\boldsymbol{W}$ be the $\bar{n}\times\bar{n}$ dimensional matrix defined in (13). Under Assumption 3.1, the following properties hold:

(a) Let $\bar{G}$ be the directed graph (without self loops) corresponding to the matrix $\boldsymbol{W}$, that is, a link from (l,m) to (i,j) exists in $\bar{G}$ if $[\boldsymbol{W}]_{(i,j),(l,m)} > 0$. Then $\bar{G}$ is strongly connected.

(b) The row sums of matrix $\boldsymbol{W}$ are non-positive, i.e.,

$$\sum_{(l,m),l\neq m}[\boldsymbol{W}]_{(i,j)(l,m)} \leq 0,\ \forall (i,j), i\neq j.$$

(c) There exists at least one row (i^*,j^*) of $\boldsymbol{W}$ whose sum is negative, that is,

$$\sum_{(l,m),l\neq m}[\boldsymbol{W}]_{(i^*,j^*)(l,m)} < 0.$$

Proof: (a) Consider the pair of nodes (i,j). From the structure of matrix $\boldsymbol{W}$ we note that in one step (i,j) is connected to the set of nodes $\mathcal{V}^{(1)}_{(i,j)} = \{(l,j)|\ l\in \mathcal{N}_i^1\}\bigcup\{(i,m)|\ m\in\mathcal{N}_j^1\}$ (where the pairs (l,m) and (m,l) are considered equivalent and counted only once, and $l\neq m$). Fixing m, from a node (i,m) we can travel in one step to any node in the set $\{(l,m)|\ l\in\mathcal{N}_i\}$. Similarly, fixing l, from (l,j) we can travel in one step to any node in the set $\{(l,m)|\ m\in\mathcal{N}_j\}$. Therefore, in two steps, the pair (i,j) is connected to the nodes in the set $\mathcal{V}^{(2)}_{(i,j)} = \{(l,m)|\ l\in\mathcal{N}_i, m\in\mathcal{N}_j\}$. Using a simple inductive argument, from the node (i,j) of graph $\bar{G}$, we can reach in $2k$ steps the nodes in the set $\mathcal{V}^{(2k)}_{(i,j)} = \{(l,m)|\ l\in\mathcal{N}_i^k, m\in\mathcal{N}_j^k\}$, where $\mathcal{N}_i^k$ denotes all the nodes that can be reached from node i of graph G in k steps. By Assumption 3.1, we have that $\mathcal{N}_i^n=\{1,2,\ldots,n\}$, and therefore in $2n$ steps we can visit any node in graph $\bar{G}$ starting from (i,j).

(b) Consider a row (i,j). For convenience, let us define the following positive scalars:

$$\xi_i \triangleq (1-\lambda_i)\mu_i \text{ and } \xi_j \triangleq (1-\lambda_j)\mu_j. \tag{14}$$

We can express the sum of the entries of the row (i,j) as

$$\sum_{(l,m)}[\boldsymbol{W}]_{(i,j)(l,m)} = -(\xi_i+\xi_j)+\xi_i\sum_{l\in\mathcal{N}_i,l\neq j,m=j}p_{i,l} + \\ +\xi_j\sum_{m\in\mathcal{N}_j,m\neq i,l=i}p_{j,m} \leq -(\xi_i+\xi_j)+\xi_i+\xi_j=0.$$

(c) Consider an arbitrary row (i,j). The row (i,j) would sum up to zero in two cases. In the first case, $i\notin\mathcal{N}_j$ and $j\notin\mathcal{N}_i$, which implies

$$\sum_{l\in\mathcal{N}_i,l\neq j,m=j}p_{i,l}=1 \text{ and } \sum_{m\in\mathcal{N}_j,m\neq i,l=i}p_{j,m}=1$$

and therefore

$$\sum_{(l,m)}[\boldsymbol{W}]_{(i,j)(l,m)} = -(\xi_i+\xi_j)+\xi_i+\xi_j=0.$$

However, having $i\notin\mathcal{N}_j$ and $j\notin\mathcal{N}_i$ for all i and j means that the communication graph $G=(V,E)$ is not (strongly) connected, contradicting Assumption 3.1. In the second case, $i\in\mathcal{N}_j$ and $j\in\mathcal{N}_i$ and $|\mathcal{N}_i|=1$ and $|\mathcal{N}_j|=1$ (that is, node i has only one neighbor, namely j and j has only one neighbor, namely i). In this case

$$\sum_{l\in\mathcal{N}_i,l\neq j,m=j}p_{i,l}=p_{i,j}=1 \text{ and } \sum_{m\in\mathcal{N}_j,m\neq i,l=i}p_{j,m}=p_{j,i}=1$$

and consequently

$$\sum_{(l,m)}[\boldsymbol{W}]_{(i,j)(l,m)} = -(\xi_i+\xi_j)+\xi_i+\xi_j=0.$$

But this case means that the nodes i and j are separated from all other nodes in the graph $G=(V,E)$ and contradicts the connectivity Assumption 3.1. Therefore, there must exist at least one row (i^*,j^*) so that

$$\sum_{(l,m)}[\boldsymbol{W}]_{(i^*,j^*)(l,m)} < 0.$$

■

Consider now the matrix $\boldsymbol{Q}\triangleq I+\epsilon\boldsymbol{W}$, where I is the identity matrix and ϵ is a positive scalar satisfying the strict inequality

$$0<\epsilon<\frac{1}{2\max_i\{\xi_i\}}$$

where ξ_i and ξ_j were defined in (14).

The following Corollary follows from the previous Lemma and describes the properties of the matrix $\boldsymbol{Q}$.

Corollary 4.1: The matrix $\boldsymbol{Q}$ has the following properties:

(a) The directed graph (without self loops) corresponding to matrix $\boldsymbol{Q}$ (that is, a link from (l,m) to (i,j) exists if $[\boldsymbol{Q}]_{(i,j),(l,m)}>0$) is strongly connected.
(b) The matrix $\boldsymbol{Q}$ is a non-negative matrix with positive diagonal elements.
(c) The rows of $\boldsymbol{Q}$ sum up to a positive value not larger than one, that is

$$\sum_{(l,m),l\neq m}[\boldsymbol{Q}]_{(i,j)(l,m)} \leq 1,\ \forall (i,j).$$

(d) There exists at least one row (i^*,j^*) of $\boldsymbol{Q}$ which sums up to a positive value strictly smaller than one, that is

$$\sum_{(l,m),l\neq m}[\boldsymbol{Q}]_{(i^*,j^*)(l,m)} < 1.$$

Proof: Noting that the directed graph (without self loops) corresponding to the matrix $\boldsymbol{Q}$ is identical to the one corresponding to the matrix $\boldsymbol{W}$, part (a) follows. The diagonal elements of $\boldsymbol{Q}$ are given by

$$[\boldsymbol{Q}]_{(i,j)(i,j)} = 1-\epsilon(\xi_i+\xi_j).$$

Using the fact that $0<\epsilon(\xi_i+\xi_j)<1$, and the obvious observation that the non-diagonal elements are non-negative, we obtain part (b). The sum of the entries of the row (i,j) is given by

$$\sum_{(l,m)}[\boldsymbol{Q}]_{(i,j)(l,m)} = 1+\epsilon\sum_{(l,m)}[\boldsymbol{W}]_{(i,j)(l,m)}$$

and using parts (b) and (c) of Lemma 4.1, parts (c) and (d) of the current Corollary follow, respectively. ■

Remark 4.2: The above Corollary says that the matrix $\boldsymbol{Q}$ is an *irreducible*, *substochastic* matrix, with at least one row sum less than one. Therefore it's spectral radius is smaller than 1, $\rho(\boldsymbol{Q}) < 1$ by standard results in the theory of Non-Negative Matrices ([15, p. 685, problem 8.3.7]).

V. Proof of the Main Results

In this section we prove the main results presented in Section III.

A. Proof of Part (a) of Theorem 3.1

We first show that the vector $\boldsymbol{\eta}(t)$ converges to zero in mean. By Remark 4.2 we have that the spectral radius of $\boldsymbol{Q}$ is smaller than one, that is

$$\rho(\boldsymbol{Q}) < 1$$

where $\rho(\boldsymbol{Q}) = \max_{\bar{i}} |\lambda_{\bar{i},\boldsymbol{Q}}|$, with $\lambda_{\bar{i},\boldsymbol{Q}}$, $\bar{i} = 1, \ldots, \bar{n}$ being the eigenvalues of $\boldsymbol{Q}$. This also means that

$$Re(\lambda_{\bar{i},\boldsymbol{Q}}) < 1, \; \forall \bar{i}. \tag{15}$$

But since $\boldsymbol{W} = (1/\epsilon)(\boldsymbol{Q} - I)$, it follows that the real part of the eigenvalues of $\boldsymbol{W}$ are given by

$$Re(\lambda_{\bar{i},\boldsymbol{W}}) = \frac{1}{\epsilon}\left(Re(\lambda_{\bar{i},\boldsymbol{Q}}) - 1\right) < 0, \; \forall \bar{i}$$

where the last inequality follows from (15). Therefore, the solution of the linear dynamics

$$\frac{d}{dt} E\left\{\boldsymbol{\eta}(t)\right\} = \boldsymbol{W} E\left\{\boldsymbol{\eta}(t)\right\}$$

is asymptotically stable, and hence $\boldsymbol{\eta}(t)$ converges in mean to zero. A standard result in probability theory tells us that convergence to zero in mean implies (I) convergence to zero in probability, as well. In addition, from Proposition 4.1, we have that (II) $\eta_{i,j} \leq \max_{i,j} \eta_{i,j}(0)$ with probability one, for all $t \geq 0$. Using a similar argument as in the proof of Theorem 4 [5, p. 310] formulated for a sequence of random variables, and the properties (I) and (II), we can show that

$$\lim_{t\to\infty} E\left\{\eta_{i,j}(t)^r\right\} = 0, \; \forall r \geq 1.$$

Using (9) of Proposition 4.1, the result follows.

B. Proof of Part (b) of Theorem 3.1

In the following we show that $\boldsymbol{\eta}(t)$ converges to zero almost surely. Equations (5) and (6) show that with probability one $\eta_{i,j}(t)$ is non-negative and that for any $t_2 \leq t_1$, with probability one $\eta_{i,j}(t_2)$ belongs to the convex hull generated by $\{\eta_{l,m}(t_1)|$ for all pairs $(l,m)\}$. But this also implies that with probability one

$$\max_{i,j} \eta_{i,j}(t_2) \leq \max_{i,j} \eta_{i,j}(t_1). \tag{16}$$

Hence, for any sample path of the random process $\boldsymbol{\eta}(t)$, the sequence $\{\max_{i,j} \eta_{i,j}(t)\}_{t\geq 0}$ is monotone non-increasing and lower bounded. Using the monotone convergence theorem, we have that for any sample path ω, there exists $\tilde{\eta}(\omega)$ so that

$$\lim_{t\to\infty} \max_{i,j} \eta_{i,j}(t,\omega) = \tilde{\eta}(\omega)$$

or similarly

$$Pr\left(\lim_{t\to\infty} \max_{i,j} \eta_{i,j}(t) = \tilde{\eta}\right) = 1.$$

Next, we show that $\tilde{\eta}$ must be zero with probability one. We achieve this by showing that there exists a subsequence of $\{\max_{i,j} \eta_{i,j}(t)\}_{t\geq 0}$ that converges to zero with probability one. In Part (a) we proved that $\boldsymbol{\eta}(t)$ converges to zero in the rth mean sense. Therefore, for any pair (i,j) and (l,m) we have that $E\{\eta_{i,j}(t)\eta_{l,m}(t)\}$ converges to zero. Moreover, since

$$E\left\{\eta_{i,j}(t)\eta_{l,m}(t)\right\} \leq \max_{i,j} \eta_{i,j}(0) E\left\{\eta_{l,m}(t)\right\}$$

and since $E\{\eta_{l,m}(t)\}$ converges to zero exponentially fast, we have that $E\{\eta_{i,j}(t)\eta_{l,m}(t)\}$ converges to zero exponentially as well. Let $\{t_k\}_{k\geq 0}$ be a time sequence such that $t_k = kh$, for some $h > 0$. From above, it follows that $E\{\|\boldsymbol{\eta}(t_k)\|^2\}$ converges to zero geometrically. But this is enough to show that the sequence $\{\boldsymbol{\eta}(t_k)\}_{k\geq 0}$ converges to zero with probability one by using the Borel-Cantelli Lemma (Theorem 10 of [5, p. 320]). Therefore, $\tilde{\eta}$ must be zero. Using (9) of Proposition 4.1, we conclude the proof of Part (b).

C. Proof of Part (C) of Theorem 3.1

We now focus on showing that not only the distances between the states of the agents converge to zero with probability one, but in fact the states of the agents converge to some point in $\mathcal{X}$, with probability one. The essence of the proof is to show that the convex hull of the states of the agents converge to one point, for any sample path of the states processes. Let ω be a sample path of the state process and let $\{t_k\}_{k\geq 0}$ be the time instants at which the Poisson counters increase their values, corresponding to this sample path. Additionally, let A_k be the set of the agents' states at time t_k, that is $A_k = \{x_j(t_k), j = 1 \ldots n\}$. According to Definition 2.5, Proposition 2.1 and (1) of the randomized gossip algorithm, we have that

$$x_i(t_{k+1}) \in co(A_k), \; \forall i.$$

But this also implies the next convex hull's inclusion

$$co(A_{k+1}) \subseteq co(A_k).$$

From the theory of limit of sequence of sets it follows that there exists a set A_∞ such that

$$\limsup co(A_k) = \liminf co(A_k) = \lim co(A_k) = A_\infty$$

where $A_\infty = \bigcap_{k\geq 0} co(A_k)$.

Denoting the diameter of the set A_k by

$$diam(A_k) = \sup\{d(x,y) |\, x, y \in A_k\}$$

from Proposition 2 of [30], we have that

$$diam(A_k) = diam\left(co(A_k)\right).$$

Additionally, in Part (b) we showed that

$$\lim_{t\to\infty} d\left(x_i(t), x_j(t)\right) = 0,\ \forall\, (i,j)$$

with probability one and therefore, the same is true for the sample path ω, that is

$$\lim_{k\to\infty} d\left(x_i(t_k), x_j(t_k)\right) = 0,\ \forall\, (i,j).$$

But this means that

$$\lim_{k\to\infty} diam(A_k) = \lim_{k\to\infty} diam\left(co(A_k)\right) = 0$$

and therefore $diam(A_\infty) = 0$. Since the convex metric space on which the randomized gossip algorithm operates satisfies *Property* (C), and the sets A_k are bounded (they have bounded diameter) and closed (by construction), it follows that the set A_∞ is non-empty. Consequently, there exists a point x^*, which may depend on ω, so that $A_\infty = x^*$, and the result follows.

VI. Rate of Convergence of the Generalized Gossip Consensus Algorithm Under Complete and Uniform Connectivity

We note that under our general problem setup, it is difficult to get explicit formulas for the rate of convergence to consensus, in the first and second moments. We are able however to obtain explicit results for the aforementioned rates of convergence under specific assumptions on the topology of the graph, on the parameters of the Poisson counters and on the convex structure.

Assumption 6.1: The Poisson counters have the same rate, that is $\mu_i = \mu$ for all i. Additionally, the parameters used by the agents in the convex structure are equal, that is $\lambda_i = \lambda$, for all i. In the update mode, each agent i picks one of the rest $n-1$ agents uniformly, that is $\mathcal{N}_i = \mathcal{N} - \{i\}$ and $p_{i,j} = 1/n-1$, for all $j \in \mathcal{N}_i$.

The following two Propositions give upper bounds on the rate of convergence for the first and second moments of the distance between agents, under Assumption 6.1.

Proposition 6.1: Under Assumptions 3.1, 3.2, and 6.1, and under the generalized gossip algorithm, the first moment of the distances between agents' states converges exponentially to zero, that is

$$E\left\{d\left(x_i(t), x_j(t)\right)\right\} \leq c_1 e^{\alpha_1 t},\ \text{for all pairs}\ (i,j)$$

where $\alpha_1 = -2(1-\lambda)\mu/(n-1)$ and c_1 is a positive scalar depending of the initial conditions.

Proof: By Proposition 4.1, with probability one we have that for any pair (i,j) $d(x_i(t), x_j(t)) \leq \eta_{i,j}(t)$ and therefore $E\{d(x_i(t), x_j(t))\} \leq E\{\eta_{i,j}(t)\}$. But the convergence of $E\{\eta_{i,j}(t)\}$ is determined by (12) and in particular by the eigenvalues of matrix $\boldsymbol{W}$, which are studied in what follows. From (13) it immediately follows that $\boldsymbol{W}$ is a symmetric matrix and that every diagonal element is $-2(1-\lambda)\mu$. Consider an arbitrary node (i,j) and write the element of the corresponding row in the following convenient form:

$(1,2), (1,3), \ldots, (1,n)$

$(2,3), (2,4), \ldots, (2,n)$

$\ldots$

$(i-1,i), (i-1,i+1), \ldots, (i-1,n)$

$(i,i+1), \ldots, (i,n)$

$\ldots$

$(j-1,j), \ldots, (j-1,n)$

$(j,j+1), \ldots, (j,n)$

$(j+1,j+2), \ldots, (n-1,n)$

where we split it with horizontal lines in 5 segments (numbered 1 through 5 from top to bottom). Following (13) observe that excluding the diagonal, the matrix has exactly $2i-2$ positive elements in segment 1, $n-i-1$ positive elements in segment 2, $j-i-1$ positive elements in segment 3, $n-j$ positive elements in segment 4 and 0 positive elements in segment 5. Therefore, the total number of off-diagonal entries in a row is $2n-4$. Again, (13) dictates that the value in any positive element is $\mu((1-\lambda)/(n-1))$. As a consequence, we conclude that the sum of every row is $\alpha_1 = -2((1-\lambda)\mu/(n-1))$, that is obviously the eigenvalue of the right eigenvector $\mathbb{1}_{\bar{n}}$, that is the vector of all ones. Noting that $\boldsymbol{W}$ is symmetric all eigenvalues are real and by Gershgorin's theorem (Theorem 7.2.1 of [4, p. 320]) they must lie in the circle $(-2(1-\lambda)\mu, r)$ where $r = 2(1-\lambda)\mu((n-2)/(n-1))$ is the sum of the non zero, off-diagonal elements of the rows. Note that the eigenvalue α_1 lies exactly on the boundary of the circle, in the negative half plane. This leads us to conclude that this is indeed the maximum one. Therefore, there exists a positive scalar c_1 which depends on the initial conditions such that

$$E\left\{\eta_{i,j}(t)\right\} \leq c_1 e^{\alpha_1 t},\ \text{for all}\ (i,j)$$

from where the result follows. ■

Proposition 6.2: Under Assumptions 3.1, 3.2, and 6.1, and under the generalized gossip algorithm, the second moment of the distances between agents' states converges exponentially to zero, that is

$$E\left\{d\left(x_i(t), x_j(t)\right)^2\right\} \leq c_2 e^{\alpha_2 t},\ \text{for all pair}\ (i,j)$$

where $\alpha_2 = -(\mu((2(1-\lambda^2))/(n-1))$ and c_2 is a positive scalar depending of the initial distances between agents.

Proof: As before, by Proposition 4.1, with probability one we have that for any pair $(i,j) d(x_i(t), x_j(t)) \leq \eta_{i,j}(t)$ and therefore $E\{d(x_i(t), x_j(t))^2\} \leq E\{\eta_{i,j}(t)^2\}$. But $E\{\eta_{i,j}(t)^2\} \leq E\{\|\boldsymbol{\eta}(t)\|^2\}$, for any pair (i,j) and therefore is sufficient to study the convergence properties of the right-hand side of the previous inequality. Using Ito's rule, we can differentiate the quantity $\|\boldsymbol{\eta}(t)\|^2$ and obtain

$$\frac{d}{dt}\|\boldsymbol{\eta}(t)\|^2=\sum_{i,j}\boldsymbol{\eta}(t)'\left[\Phi_{i,j}(\theta_i(t))+\Phi_{i,j}(\theta_i(t))'+\Phi_{i,j}(\theta_i(t))'\Phi_{i,j}(\theta_i(t))\right]\boldsymbol{\eta}(t)dN_i(t)+\sum_{i,j}\boldsymbol{\eta}(t)'\left[\Psi_{i,j}(\theta_j(t))+\Psi_{i,j}(\theta_j(t))'+\Psi_{i,j}(\theta_j(t))'\Psi_{i,j}(\theta_j(t))\right]\boldsymbol{\eta}(t)dN_j(t)$$

from where we get

$$\frac{d}{dt}E\left\{\|\boldsymbol{\eta}(t)\|^2\right\}=\sum_{i,j}E\left\{\boldsymbol{\eta}(t)'\left[\Phi_{i,j}(\theta_i(t))+\Phi_{i,j}(\theta_i(t))'+\Phi_{i,j}(\theta_i(t))'\Phi_{i,j}(\theta_i(t))\right]\boldsymbol{\eta}(t)\right\}\mu_i+\sum_{i,j}E\left\{\boldsymbol{\eta}(t)'\left[\Psi_{i,j}(\theta_j(t)+\Psi_{i,j}(\theta_j(t))'+\Psi_{i,j}(\theta_j(t))'\Psi_{i,j}(\theta_j(t))\right]\boldsymbol{\eta}(t)\right\}\mu_j.$$

Using the independence of the random process $\theta_i(t)$ and Assumption 6.1, we can further write

$$\frac{d}{dt}E\left\{\|\boldsymbol{\eta}(t)\|^2\right\}=\mu\sum_{i,j}E\left\{\boldsymbol{\eta}(t)'\boldsymbol{H}\boldsymbol{\eta}(t)\right\}$$

where

$$\boldsymbol{H}=E\left\{\Phi_{i,j}(\theta_i(t))+\Phi_{i,j}(\theta_i(t))'+\Phi_{i,j}(\theta_i(t))'\Phi_{i,j}(\theta_i(t))++\Psi_{i,j}(\theta_j(t))+\Psi_{i,j}(\theta_j(t))'+\Psi_{i,j}(\theta_j(t))'\Psi_{i,j}(\theta_j(t))\right\}.$$

Using Assumption 6.1, we have

$$\Phi_{i,j}(\theta_i(t))+\Phi_{i,j}(\theta_i(t))'=(1-\lambda)\begin{cases}-2 & \text{at entry } (i,j)(i,j)\\ \mathbb{1}_{\{\theta_i(t)=l\}} & \text{at entries } (i,j)(l,j) \text{ and } (l,j)(i,j)\ l\in\mathcal{N}_i, l\neq j\\ 0 & \text{at all other entries,}\end{cases}$$

$$\Phi_{i,j}(\theta_i(t))'\Phi_{i,j}(\theta_i(t))=(1-\lambda)^2\begin{cases}1 & \text{at entry } (i,j)(i,j)\\ -\mathbb{1}_{\{\theta_i(t)=l\}} & \text{at entries } (i,j)(l,j) \text{ and } (l,j)(i,j)\ l\in\mathcal{N}_i, l\neq j\\ \mathbb{1}_{\{\theta_i(t)=l\}} & \text{at entries } (l,j)(l,j)\ l\in\mathcal{N}_i, l\neq j\\ 0 & \text{at all other entries,}\end{cases}$$

$$\Psi_{i,j}(\theta_j(t))+\Psi_{i,j}(\theta_j(t))'=(1-\lambda)\begin{cases}-2 & \text{at entry } (i,j)(i,j)\\ \mathbb{1}_{\{\theta_j(t)=l\}} & \text{at entries } (i,j)(i,l) \text{ and } (i,l)(i,j)\ l\in\mathcal{N}_j, l\neq i\\ 0 & \text{at all other entries,}\end{cases}$$

$$\Psi_{i,j}(\theta_j(t))'\Psi_{i,j}(\theta_j(t))=(1-\lambda)^2\begin{cases}1 & \text{at entry } (i,j)(i,j)\\ -\mathbb{1}_{\{\theta_j(t)=l\}} & \text{at entries } (i,j)(i,l) \text{ and } (i,l)(i,j)\ l\in\mathcal{N}_j, l\neq i\\ \mathbb{1}_{\{\theta_j(t)=l\}} & \text{at entries } (i,l)(i,l)\ l\in\mathcal{N}_j, l\neq i\\ 0 & \text{at all other entries.}\end{cases}$$

It follows that

$$E\left\{\Phi_{i,j}(\theta_i(t))+\Phi_{i,j}(\theta_i(t))'\right\}=(1-\lambda)\begin{cases}-2 & \text{at entry } (i,j)(i,j)\\ \frac{1}{n-1} & \text{at entries } (i,j)(l,j) \text{ and } (l,j)(i,j)\ l\in\mathcal{N}_i, l\neq j\\ 0 & \text{at all other entries,}\end{cases}$$

$$E\left\{\Phi_{i,j}(\theta_i(t))'\Phi_{i,j}(\theta_i(t))\right\}=(1-\lambda)\begin{cases}1 & \text{at entry } (i,j)(i,j)\\ -\frac{1}{n-1} & \text{at entries } (i,j)(l,j) \text{ and } (l,j)(i,j)\ l\in\mathcal{N}_i, l\neq j\\ \frac{1}{n-1} & \text{at entries } (l,j)(l,j)\ l\in\mathcal{N}_i, l\neq j\\ 0 & \text{at all other entries,}\end{cases}$$

$$E\left\{\Psi_{i,j}(\theta_j(t))+\Psi_{i,j}(\theta_j(t))'\right\}=(1-\lambda)\begin{cases}-2 & \text{at entry } (i,j)(i,j)\\ \frac{1}{n-1} & \text{at entries } (i,j)(i,l) \text{ and } (i,l)(i,j)\ l\in\mathcal{N}_j, l\neq i\\ 0 & \text{at all other entries,}\end{cases}$$

$$E\left\{\Psi_{i,j}(\theta_j(t))'\Psi_{i,j}(\theta_j(t))\right\}=(1-\lambda)^2\begin{cases}1 & \text{at entry } (i,j)(i,j)\\ -\frac{1}{n-1} & \text{at entries } (i,j)(i,l) \text{ and } (i,l)(i,j)\ l\in\mathcal{N}_j, l\neq i\\ \frac{1}{n-1} & \text{at entries } (i,l)(i,l)\ l\in\mathcal{N}_j, l\neq i\\ 0 & \text{at all other entries.}\end{cases}$$

Summing up the above matrices, we obtain that $\boldsymbol{H}$ is a symmetric matrix that has as diagonal elements quantities of the form

$$\left[-4(1-\lambda)+(1-\lambda)^2\left(2+\frac{2n-4}{n-1}\right)\right]\mu$$

and the off-diagonal, non-zero entries are given by

$$\lambda(1-\lambda)\frac{2}{n-1}\mu.$$

Counting the off-diagonal entries on a row we obtain the same result as in the case of the first moment. Namely, the number of non-zero and off-diagonal elements on each row is $2(n-2)$. Also note that the diagonal elements are negative and that the off-diagonal and non-zero elements are positive for any $n\geq 2$. Therefore, each row of $\boldsymbol{H}$ sums up to the same value and consequently $\boldsymbol{H}$ has an eigenvalue

$$\alpha_2=\left[-4(1-\lambda)+(1-\lambda)^2\left(2+\frac{2n-4}{n-1}\right)\right]\mu++2(n-2)\lambda(1-\lambda)\frac{2}{n-1}\mu=-\frac{2(1-\lambda^2)\mu}{n-1}$$

corresponding to eigenvector $\mathbf{1}_{\bar{n}}$, which is the $\bar{n}$ dimensional vector of all ones. Note that α_2 is negative for $0\leq\lambda<1$ and $n\geq 2$. In addition, by Gershgorin's theorem, we have that all eigenvalues belong to the circle centered at $[-4(1-\lambda)+(1-\lambda)^2(2+(2n-4)/(n-1))]\mu$ with radius $2(n-2)\lambda(1-\lambda)(2/(n-1))\mu$ and therefore the eigenvalue α_2 dominates the rest of the eigenvalues; eigenvalues that are real due to symmetry. Therefore, we have that

$$\boldsymbol{H}\preceq\alpha_2 I$$

and consequently

$$\frac{d}{dt}E\left\{\|\boldsymbol{\eta}(t)\|^2\right\} \leq \alpha_2 E\left\{\|\boldsymbol{\eta}(t)\|^2\right\}.$$

We can further write that

$$E\left\{\|\boldsymbol{\eta}(t)\|^2\right\} \leq e^{\alpha_2 t} E\left\{\|\boldsymbol{\eta}(t_0)\|^2\right\}$$

from where the result follows. ■

Remark 6.1: As expected, the rate of convergence of the upper bounds on the both moments increases with the rate of the Poisson counters. Interestingly, the maximum rate of convergence of the upper bounds on the both moments (that is, the minimum values for α_1 and α_2) are attained for $\lambda = 0$, meaning that an awaken agent should never pick its own value, but the value of a neighbor.

Remark 6.2: The all-to-all communication is a connectivity condition that allows for an explicit calculation of the quantities α_1, α_2 in terms of the parameters of the algorithm. Such calculations are not easy to establish under milder connectivity assumptions, such as simple connectivity. Note, however, that a numerical estimates of α_1 could be obtained from the spectral properties of the matrix $\boldsymbol{W}$ defined in (13).

VII. THE GENERALIZED GOSSIP CONSENSUS ALGORITHM FOR PARTICULAR CONVEX METRIC SPACES

In this section we present several instances of the gossip algorithm for particular examples of convex metric spaces. We consider three cases for $\mathcal{X}$: the set of real numbers, the collection of compact, convex sets and the set of discrete random variables. We endow each of these sets with a metric d and a convex structure ψ. We show the particular form the generalized gossip algorithm takes for these convex metric spaces, and give some numerical simulations of these algorithms.

A. The Set of Real Numbers

Let $(\mathcal{X}, d)$ be the standard Euclidean metric space (where for simplicity we choose $\mathcal{X} = \mathbb{R}$). It can be easily verified that $\psi(x, y, \lambda) = \lambda x + (1-\lambda)y$ is a convex structure, and therefore $(\mathcal{X}, d, \psi)$ is a convex metric space. In this case, the generalized randomized consensus algorithm takes the form shown in Algorithm 1.

Algorithm 1: Randomized Gossip Algorithm on $\mathbb{R}$

Input: $x_i(0)$, λ_i, $p_{i,j}$
for *each counting instant t_i of N_i* **do**
  Agent i enters update mode and picks a neighbor j with probability $p_{i,j}$;
  Agent i updates its state according to

$$x_i(t_i^+) = \lambda_i x_i(t_i) + (1-\lambda_i) x_j(t_i);$$

  Agent i enters sleep mode;

Note that this algorithm is exactly the randomized gossip algorithm for solving the consensus problem that was studied in [2].

B. The Collection of Compact, Convex Sets

For the following example we draw inspiration from the analysis of linear dynamics driven by compact, convex sets studied in [23], [24]. Let $\mathcal{X} = \mathrm{ComConv}(\mathbb{R}^n)$ denote the collection of convex, compact sets in $\mathbb{R}^n$. Given two sets $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^n$, the Minkowski sum between the two set is defined as $X \oplus Y = \{x + y |\ x \in X, y \in Y\}$. We also define the multiplication of a set X by a scalar by $\lambda X = \{\lambda x |\ x \in X\}$. It can be easily checked that if $X, Y \in \mathcal{X}$ then $\lambda X \oplus (1-\lambda)Y \in \mathcal{X}$, and that $\lambda X \oplus (1-\lambda)X = X$ for all $\lambda \in [0,1]$. It is well-known that $\mathrm{ComConv}(\mathbb{R}^n)$ endowed with the Hausdorff distance is a complete metric space [28], where the Hausdorff distance is defined as

$$H(L, X, Y) = \min_{\alpha}\{\alpha |\ X \subseteq Y \oplus \alpha L, Y \subseteq X \oplus \alpha L, \alpha \geq 0\} \tag{17}$$

with $L \subset \mathcal{X}$ a symmetric, nonempty set containing the origin. Let us now define the mapping $\Psi(X, Y, \lambda) = \lambda X \oplus (1-\lambda)Y$, where $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$. Using the above observations it should be clear that any set produced by the mapping Ψ belongs to $\mathcal{X}$. The following proposition shows that $\Psi(X, Y, \lambda)$ is indeed a convex structure.

Proposition 7.1: The mapping ψ is a convex structure on $\mathcal{X}$ with respect to the Hausdorff distance.

Proof: All we have to show is that the following inequality holds:

$$H(L,U,\Psi(X,Y,\lambda)) \leq \lambda H(L,U,X) + (1-\lambda) H(L,U,X) \tag{18}$$

for all $U, X, Y \in \mathcal{X}$, and $\lambda \in [0,1]$. To simplify the proof, we use the fact that the Hausdorff distance can also be represented in terms of the support function of a closed, convex set. Given that the support function at a point $z \in \mathbb{R}^n$ is given by

$$s(X, z) = \sup_x \{z'x |\ x \in X\}$$

the Hausdorff distance between two closed and convex sets X, Y can be equivalently expressed as

$$H(L, X, Y) = \|s(X, \cdot) - s(Y, \cdot)\|_\infty \tag{19}$$

where $\|\cdot\|$ is the uniform norm on the unit sphere, that is $\|f\|_\infty = \sup_z\{f(z) |\ z'z \leq 1\}$. Therefore, we have that

$$H\left(L, U, \Psi(X, Y, \lambda)\right) = \|s(U, \cdot) - s\left(\Psi(X, Y, \lambda), \cdot\right)\|_\infty. \tag{20}$$

Observing that the support function of a set $\lambda X \oplus (1-\lambda)Y$ can be expressed as

$$\begin{aligned} s\left(\lambda X \oplus (1-\lambda)Y, z\right) &= \sup_{x,y}\left\{\lambda z'x + (1-\lambda)z'y \mid y, x \right. \\ &\qquad \left. \in X, y \in Y\right\} = \\ &= \lambda s(X, z) + (1-\lambda) s(Y, z) \end{aligned}$$

we have that (20) can be further written as

$$\begin{aligned} H\left(L, U, \Psi(X, Y, \lambda)\right) &= \|s\left(\lambda U \oplus (1-\lambda)U, \cdot\right) \\ &\quad - s\left(\lambda X \oplus (1-\lambda)Y, \cdot\right)\|_\infty = \\ &= \|\lambda\left[s(U, \cdot) - s(X, \cdot)\right] \\ &\quad + (1-\lambda)\left[s(U, \cdot) - s(y, \cdot)\right]\|_\infty \leq \\ &\leq \lambda \|s(U, \cdot) - s(X, \cdot)\|_\infty \\ &\quad + (1-\lambda)\|s(U, \cdot) - s(Y, \cdot)\|_\infty = \\ &= \lambda H(L, U, X) + (1-\lambda) H(L, U, Y) \end{aligned}$$

where the last equality followed from (19), and the result follows. ■

For this convex metric space, the randomized gossip consensus algorithm is shown in Algorithm 2.

Algorithm 2: Randomized Gossip Algorithm on Compact, Convex Sets

Input: $X_i(0)$, λ_i, $p_{i,j}$
for *each counting instant* t_i *of* N_i **do**
  Agent i enters update mode and picks a neighbor j with probability $p_{i,j}$;
  Agent i updates its state according to

$$X_i(t_i^+) = \lambda_i X_i(t_i) \oplus (1-\lambda_i) X_j(t_i);$$

  Agent i enters sleep mode;

C. The Set of Discrete Random Variables

Let $S = \{s_1, s_2, \ldots, s_m, \ldots\}$ be finite or countable set of real numbers and let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. We denote by $\mathcal{X}$ the space of discrete measurable functions (random variable) on $(\Omega, \mathcal{F}, \mathcal{P})$ with values in S. We introduce the operator $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, defined as

$$d(X, Y) = E\left[\rho(X, Y)\right]$$

where $\rho : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ is the discrete metric, i.e.,

$$\rho(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

and the expectation is taken with respect to the measure $\mathcal{P}$. It is not difficult to note that the operator d can also be written as $d(X, Y) = E[\mathbb{1}_{\{X \neq Y\}}] = Pr(X \neq Y)$, where $\mathbb{1}_{\{X \neq Y\}}$ is the indicator function of the event $\{X \neq Y\}$.

We note that for all $X, Y, Z \in \mathcal{X}$, the operator d satisfies the following properties

(a) $d(X, Y) = 0$ if and only if $X = Y$ with probability one,
(b) $d(X, Z) + d(Y, Z) \geq d(X, Y)$ with probability one,
(c) $d(X, Y) = d(Y, X)$,
(d) $d(X, Y) \geq 0$,

and therefore is a metric on $\mathcal{X}$. The set $\mathcal{X}$ together with the operator d define the*metric space* $(\mathcal{X}, d)$.

Let $\gamma \in \{1, 2\}$ be an independent random variable defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with probability mass function $Pr(\gamma = 1) = \lambda$ and $Pr(\gamma = 2) = 1 - \lambda$, where $\lambda \in [0, 1]$. We define the mapping $\Psi : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ given by

$$\Psi(X_1, X_2, \lambda) = \mathbb{1}_{\{\gamma=1\}} X_1 + \mathbb{1}_{\{\gamma=2\}} X_2, \ \forall X_1, X_2 \in \mathcal{X}, \lambda \in [0, 1].$$

The following propositions shows that indeed $(\mathcal{X}, d, \Psi)$ is a convex metric space.

Proposition 7.2: The mapping Ψ is a convex structure on $\mathcal{X}$.

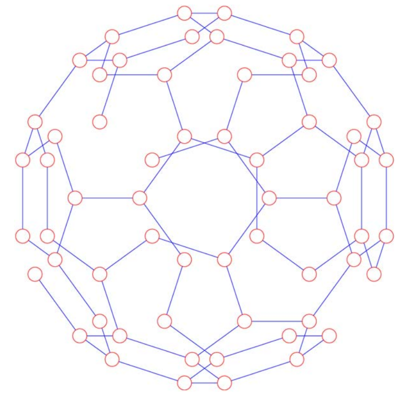

Fig. 1. Communication network with $n = 60$ nodes.

Proof: For any $U, X_1, X_2 \in \mathcal{X}$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} d\left(U, \Psi(X_1, X_2, \lambda)\right) &= E\left[\rho\left(U, \Psi(X_1, X_2, \lambda)\right)\right] \\ &= E[E[\rho(U, \Psi(X_1, X_2, \lambda)) | U, X_1, X_2]] = \\ &= E\left[E\left[\rho(U, \mathbb{1}_{\{\gamma=1\}} X_1 \right.\right. \\ &\qquad \left.\left. + \mathbb{1}_{\{\gamma=2\}} X_2)\right] | U, X_1, X_2\right] \\ &= E\left[\lambda \rho(U, X_1) + (1-\lambda)\rho(U, X_2)\right] \\ &= \lambda d(U, X_1) + (1-\lambda) d(U, X_2). \end{aligned}$$

■

For this particular convex metric space, the randomized consensus algorithm is summarized in what follows.

Algorithm 3: Randomized Gossip Algorithm on Discrete, Finite Sets

Input: $x_i(0)$, λ_i, $p_{i,j}$
for *each counting instant* t_i *of* N_i **do**
  Agent i enters update mode and picks a neighbor j with probability $p_{i,j}$;
  Agent i updates its state according to

$$x_i(t_i^+) = \begin{cases} x_i(t_i) & \text{with probability } \lambda_i \\ x_j(t_i) & \text{with probability } 1-\lambda_i \end{cases}$$

  Agent i enters sleep mode;

D. Numerical Simulations

In this subsection, we present numerical simulations of the generalized gossip algorithm in the case of the three convex metric spaces previously mentioned. We consider two networks of $n = 60$ nodes; one fully connected and one simply connected shown in Fig. 1. The Poisson counter rates were chosen to be uniformly distributed in the interval [1, 1.5]. The convex structure parameters were chosen to be uniformly distributed in the interval [0.1, 0.3]. In the connected case, when the agent i wakes up, it picks one of its neighbor with probability $(1/|\mathcal{N}_i|)$. In the fully connected case it picks an agent with probability $1/(n-1)$.

For each of the three convex metric spaces, we present three figures: the first and second figures show the values of the states for the fully connected and simply connected communication topologies, while the third figure depicts an upper bound on the normalized value of the maximum of the distances between the agents' states, that is the quantities $\eta_{i,j}(t)$. Our focus is on showing that the vector of distances converge to zero and that the states converge to the same value.

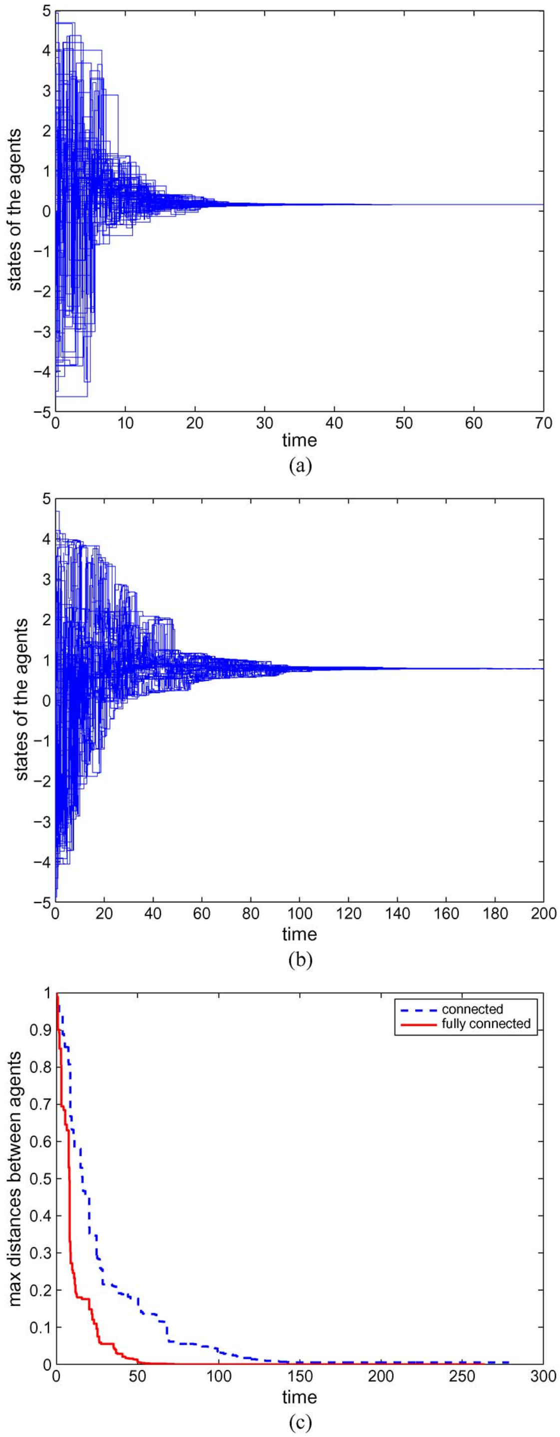

Fig. 2. Randomized Gossip Algorithm on $\mathbb{R}$: (a) the values of the states in the fully connected topology; (b) the values of the states in the simply connected topology; (c) (upper bounds on the) maximum of the distances between the states of the agents.

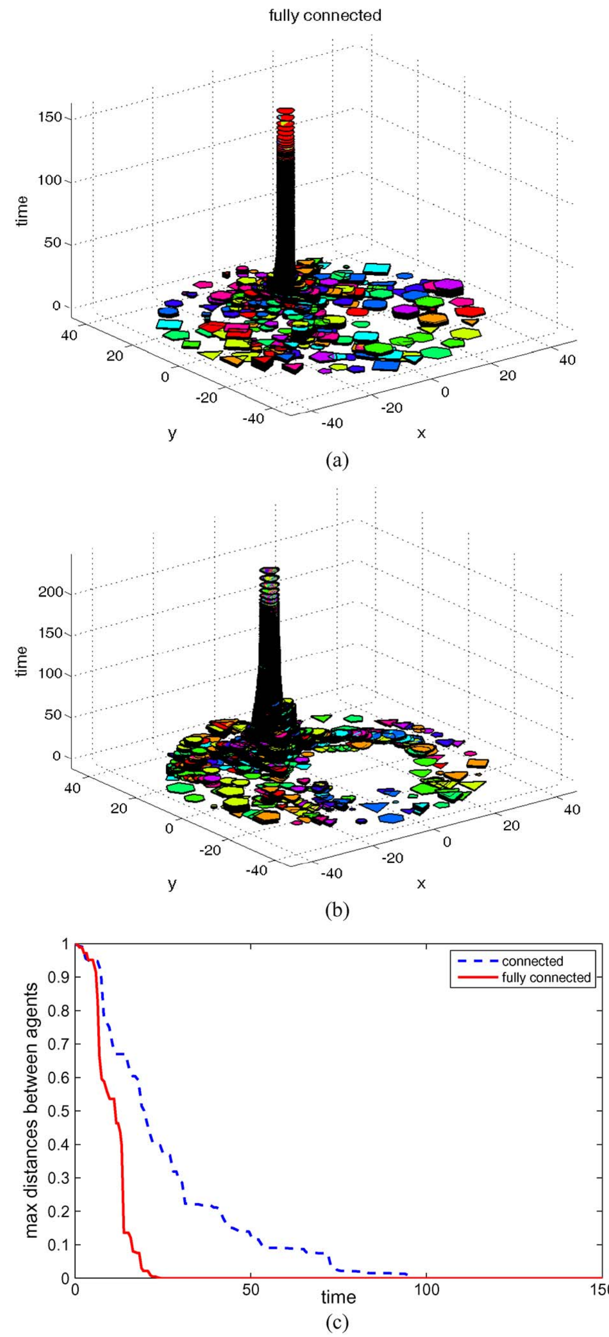

Fig. 3. Randomized Gossip Algorithm on Compact, Convex Sets: (a) the values of the states in the fully connected topology; (b) the values of the states in the simply connected topology; (c) (upper bounds on the) maximum of the distances between the states of the agents.

The Fig. 2(a)–(c) present numerical simulations of the gossip algorithm in the case of real numbers. The initial conditions are uniformly distributed in $[-5, 5]$. Fig. 3(a)–(c) show the behavior of the generalized randomized gossip algorithm applied on the collection of compact, convex sets. The initial values of the states are polytope approximations of circles with radiuses uniformly chosen from the interval [0.8, 4.8] and number of edges uniformly picked from the set $\{3, \ldots, 7\}$. Simulation results of the randomized gossip algorithm applied on a discrete set of numbers are depicted in Fig. 4(a)–(c), in which the initial conditions are uniformly chosen from the set $\{0, 1, \ldots, 10\}$. Note that since the distance on this space is defined as an expectation, the convergence speed of the distances between agents is actual lower than the converge speed of a realization of the algorithm shown in Fig. 4(a)–(b).

As expected, in all three examples the agents converge to the same value and the distances between the states of the agents converge to zero, as well. In addition, since in the fully

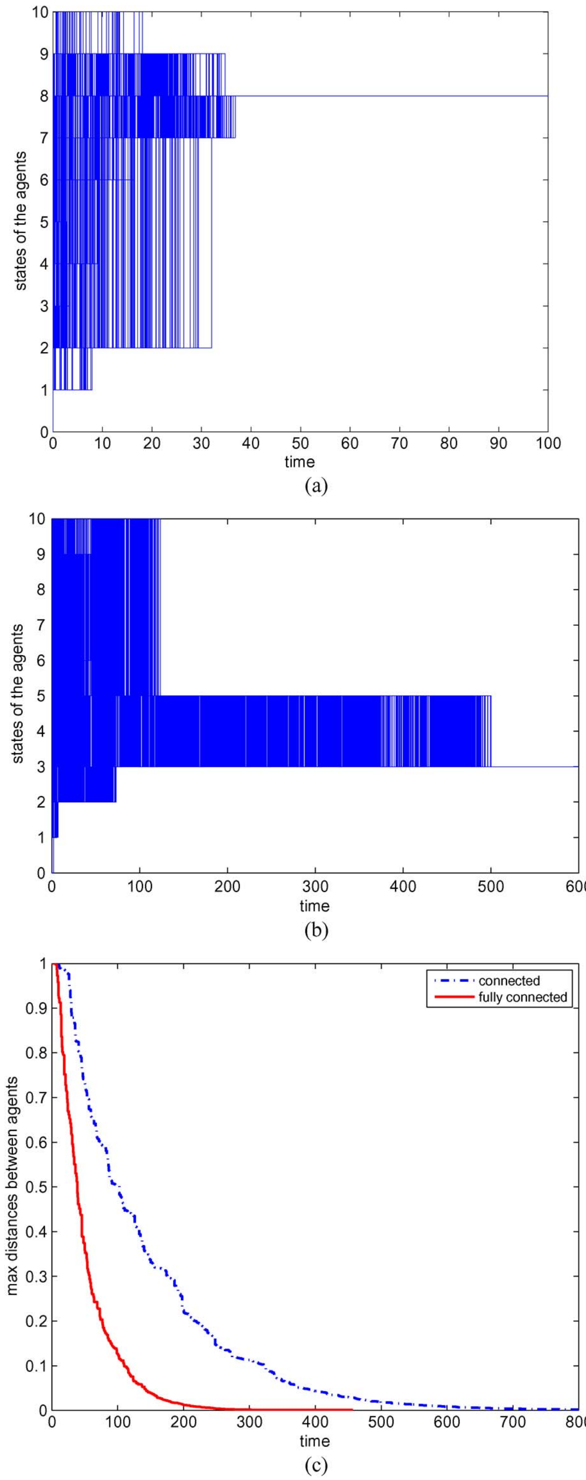

Fig. 4. Randomized Gossip Algorithm on Discrete Finite Sets: (a) the values of the states in the fully connected topology; (b) the values of the states in the simply connected topology; (c) (upper bounds on the) maximum of the distances between the states of the agents.

connected case the agents have the chance to interact with many more other agents, the convergence rate is higher. For executing the numerical simulation of the gossip algorithm on compact, convex sets, we used the Multi-Parametric toolbox [7] that provides efficient numerical algorithms for computing Minkowski sums of convex sets.

VIII. Conclusions

In this paper, we analyzed the convergence properties of a generalized randomized gossip algorithm acting on convex metric spaces. We gave convergence results in the almost sure and the r^{th} mean sense for the distances between the states of the agents. Under specific assumptions on the communication topology, we computed estimates of the rate of convergence for the first and second moments of the distances between the agents, explicitly. Additionally, we introduced instances of the generalized gossip algorithm for three particular convex metric spaces and presented numerical simulations of the algorithm. These examples show how seemingly unrelated algorithms can be put under a single umbrella: they are all instances of a generalized consensus algorithm defined on convex metric spaces. Since consensus algorithms are used as underlying tools for many distributed computation problems, this generalized framework may help with solving such problems, when they are formulated on topological structures that go beyond the standard Euclidean vector spaces.

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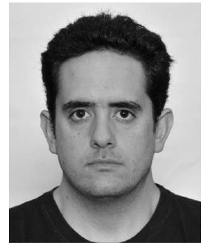

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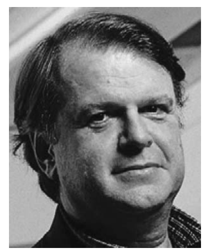

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