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LTI Continuous Time Consensus Dynamics with Multiple and Distributed Delays

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Abstract— **We study linear time invariant (LTI) continuous time consensus dynamics in the presence of bounded communication delays. Contrary to traditional Lyapunov based methods, we approach the problem using Fixed Point Theory. This method, allows us to create an appropriate complete functional metric space and through contraction mappings to establish the existence and uniqueness of a solution of this model. We explore the case of constant as well as distributed delays.**

I. Introduction

Distributed consensus dynamics have, over the past decade, carried the beacon of research in the control community. Starting from the seminal work of Tsitsiklis [1] the subject was reheated with the work of Jadbabaie et al. [2] who gave a rigorous proof of the leaderless co-ordination in a flocking model proposed by Viscek et al. [3].

Since then, an enormous amount of work has been produced from different fields of Applied Science concerning types of coordination among autonomous agents who exchange information in a distributed way, under several different frameworks (e.g. deterministic or stochastic) and communication conditions. See for example [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and references therein. All of the proposed models are based on a specific type of dynamic evolution of the agents' states known as *consensus algorithm*. Each agent evolves it's state by some type of convex averaging of the states of it's 'neighbours'. Each new state lies on the convex hull of the previous averaged ones so that the limit value is common to all the agents on condition that certain communication criteria hold (see for example [4]).

In this work, we revisit the classic linear time invariant consensus dynamics problem but in this case, we assume delays in the communication between agents. Based on results from current literature (as well as rigorous attempts presented in [14]), we believe that a Lyapunov-based approach for the stability of the network to the convergence subspace is not only restricting on the assumptions for the communication graph; but also does not shed light upon the critical quantities associated with the asymptotic behavior of the system as it is for instance, the consensus point or the rate of convergence.

On the other hand, a Fixed Point Theory approach appears to fit better to these types of problems where robust results can be obtained in the price of extensive analysis and, perhaps, more conservative assumptions.

A. Introduction to the model and related literature

The model we will discuss in this work is of the form

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j(t-\tau) - x_i(t)) \tag{MDL}$$

where the letter τ will be used here representing a generic type of bounded delay. Each agent evolves according to the dynamics of it's own state as well as a retarded measurement of the states of it's neighbouring agents. Surprisingly enough, this simple model has not received that much attention , compared to other and more complex models. To the best of our knowledge, we mention four relative works.

A simple delayed consensus algorithm was proposed and discussed in the work of Olfati-Saber et. al [12] where the model

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j(t-\tau) - x_i(t-\tau)) \tag{1}$$

With $\tau > 0$ constant and uniform for all agents, a frequency method analysis was carried through. The problem with this method is that it is over simplistic and cannot be generalized in case the weights are time varying or the delays are incommensurate.

In [1], [4] the authors consider a discrete time version of (MDL) with, in fact, time varying delays $\tau = \tau(t)$. On condition that the delay is uniformly bounded from above, the strategy of attacking the problem is to extend the state space by adding artificial agents which played no actual role in the dynamics other than transmitting a pre-described delayed version of an agent's state. This method although applicable in the discrete time, it is unclear how it would work in a continuous time system, unless the latter one is discretized and solved numerically.

In [10] the authors discuss the convergence properties of a non-linear model which has the form

$$\dot{x}_i = \sum_j a_{ij} f_{ij}(x_j(t-\tau) - x_i) \tag{2}$$

Using passivity assumptions on f_{ij} they apply invariance principles to derive delay-independent convergence results both in static and switching topologies. The main setback of this approach, also noted by the authors, is that nothing can be said about either the consensus point or either the rate of convergence to it.

The last type of models, have to do with rendezvouz type of algorithms. For example in [11] the authors propose a second order consensus based algorithms, where agents asymptotically meet in a common place as their speed vanishes to zero. This algorithm is of the form

$$\dot{v}_i(t) = -cv_i(t) + \sum_{i=1}^{N} a_{ij}(r_j(t-\tau) - r_i(t)) \tag{3}$$

The authors make a Lyapunov-Krasovskii argument on the base that the delayed quantities act only as perturbations to the main dynamical equation. Again, little can be said about the rate of convergence of this system.

B. Structure of the paper

This work is organized as follows. In section 2, we introduce the main notations and definitions that we will use throughout the paper. In section 3 we introduce our models, pose the sufficient assumptions and the main results. In section 4, we take a digression to a Lyapunov-based approach outlining the difficulties of such an attempt. In section 5, we introduce the family of metric spaces we are interested in and prove an important result which will help us make use of Contraction Mappings. The analysis of linear time invariant weights will be carried out in section (V) using undirected connected network communication (i.e. symmetric weights $a_{ij} = a_{ji}$). In section (V-A) we will briefly explain how these results can be adapted in the case of asymmetric weighted graphs. In section (VI) we discuss the case of distributed delays and some final comments are made in Section (VII).

II. Notations and Definitions

In this section, we will explain the preliminary notations and definitions which will be used in this work. By $N < \infty$ we denote the number of agents. The set of agents is denoted by $[N] := \{1,\ldots,N\}$. Each agent $i \in [N]$ is associated with a real quantity $x_i \in \mathbb{R}$ which models the state of agent i.

The Euclidean vector space $\mathbb{R}^N$ frames the state space of the system with state vectors $\mathbf{x} = (x_i,\ldots,x_N)^T$ and it is equipped with the norm $|\cdot|_1$, (i.e. $|\mathbf{x}|_1 = \sum_{i=1}^{N} |x_i|$). For a square $N \times N$ matrix A the induced norm is defined as $|A|_1 = \sup_{|\mathbf{x}|_1=1} |A\mathbf{x}|_1$. By $\mathbf{1}$ we understand the N dimensional column vector of all ones. The subspace of $\mathbb{R}^N$ of interest is defined by

$$\Delta = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = \mathbf{1}k \quad \text{for some } k \in \mathbb{R}\}$$

and it is traditionally called the *consensus space*. By Δ^c we understand the complementary space of Δ. By L^1 we denote the space of absolutely integrable functions.

A. Algebraic Graph Theory

In this subsection we review some tools from algebraic and spectral graph theory. For more information on the subject the interested reader is refered to [15], [16], [6].

The mathematical object which will be used to model the communication structure among the N agents is the weighted directed graph. This is defined as the triple $G = (V, E, W)$ where V is the set of nodes (here $[N]$), E is a subset of $V \times V$ which characterizes the established communication connections and W is a set associating a positive number (the weight) with any member of E. So by a_{ij} we will denote the weight in the connection from node j to node i and this is amount of the effect that j has on i. If $a_{ij} = 0$ then $(j,i) \notin E$. This is a directed graph and in this work we will be interested in the family routed out branching graphs or strongly connected directed graphs. A rooted out branching graph is a graph that contains a spanning tree (i.e. at least one node is a root) and a strongly connected graph is a graph that each node is a root. Moreover, in case of symmetric communicating weights $a_{ij} = a_{ji}$ the graph is called undirected; hence simple connectivity suffices for our results. Given E, each agent i has a neighbourhood of nodes, to which it is adjacent. We denote by N_i the subset of V such that $(j,i) \in E$ and by $|N_i|$ it's cardinality. The degree of any node i, denoted by d_i, is the sum of the weights with which each of it's adjacent nodes affects him, i.e. $d_i = \sum_{j\in N_i} a_{ij}$. In the analysis to follow, $\sum_{i,j}$ stands for $\sum_{i=1}\sum_{j\in N_i}$.

A matrix representation of G is through the adjacency matrix $A = [a_{ij}]$, the degree matrix $D = \mathrm{Diag}[d_i]$ and the Laplacian $L := D - A$. If G is directed we name it as in-degree Laplacian. If G is undirected it is simply known as graph Laplacian. The spectral properties of L are of interest. In case of undirected network the L is a symmetric positive semi-definite matrix. So there is an eigensystem of real eigenvalues and mutually orthogonal eigenvectors such that

$$0 = \lambda_1(L) \le \lambda_2(L) \le \ldots \le \lambda_N(L)$$

and $\mathbf{u}_i|_{i=1}^{N}$ is the family of eigenvectors such that $\mathbf{u}_j^T \mathbf{u}_i = \delta_{ij}$. An important result is that a (directed) graph is assumed to be (strongly) connected if and only if $\lambda_2(L) > 0$

B. Fixed Point Theory

By the pair $(\mathcal{M},\rho)$ we define a metric space. In this work we consider only complete metric spaces and one way to rigorously define them is through a compactness argument of a subset of a Banach space.

For $0 < \tau < \infty$ consider the vector space $\mathcal{B} = C([-\tau,\infty),\mathbb{R}^N)$ of continuous bounded functions and the norm $|\phi| = \sup_{t\ge-\tau} |\phi(t)|_1$ for $\phi \in \mathcal{B}$. It is well-known that the pair $(\mathcal{B},|\cdot|)$ consists a Banach space, [17]. A subset of $\mathcal{B}$, $\mathcal{M}$, is said to be *complete* if every Cauchy sequence of $\mathcal{M}$ converges in $\mathcal{M}$. Such a subset is together with the metric

$$\rho(\phi_1,\phi_2) = |\phi_1 - \phi_2| = \sup_{t\ge-\tau} |\phi_1(t) - \phi_2(t)|_1 \tag{4}$$

constitutes a *complete metric space*. These objects lie in the center of our attention as it is in such desirable metric spaces where the contraction mappings will be defined, to guarantee the existence and the uniqueness of solutions.

Recall that given two metric spaces $(\mathcal{M}_i,\rho_{\mathcal{M}_i})$ for $i = 1,2$ an operator $P : \mathcal{M}_1 \to \mathcal{M}_2$ is a *contraction* if there exists a constant $\alpha \in [0,1)$ such that $x_1, x_2 \in \mathcal{M}_1$ imply

$$\rho_{\mathcal{M}_2}(Px_1,Px_2) \le \alpha\rho_{\mathcal{M}_1}(x_1,x_2) \tag{5}$$

The next, well-known theorem, will be used in proving our main results.

Theorem 2.1: [Contraction Mapping Principle] Let $(\mathcal{M},\rho)$ be a complete metric space and $P : \mathcal{M} \to \mathcal{M}$ a contraction operator. Then there is a unique $x \in \mathcal{M}$ with $Px = x$.

The proof of the theorem can be found in any advanced analysis or ordinary differential equations book. (See for example [18] which is closest to our work).

Some proofs or steps in the proofs were omitted due to space limitations. The reader is kindly referred to the technical report that accompanies this paper, [14].

III. The model, the assumptions and the statement of the results

In this section we will introduce the models and state the two main results of this work. These are two LTI consensus systems, one with multiple constant delays and another with distributed delays.

Given $N < \infty$, $0 < \tau < \infty$ and the initial functions $\phi_i(t) : [-\tau, 0] \to \mathbb{R}|_{i=1}^N$, we consider the following models:

A. Linear Time Invariant Weights with Constant Delays

$$\begin{aligned} \dot{x}_i &= \sum_{j\in N_i} a_{ij}(x_j^i - x_i), \qquad t \geq 0 \\ x_i(t) &= \phi_i(t), \qquad t \in [-\tau, 0] \end{aligned} \tag{IVP1}$$

where $x_j^i := x_j(t - \tau_i^j)$ for some constants τ_i^j. The notation merely signifies the delay with which agent i receives the signal from agent j. We impose the following Assumptions:

Assumption 3.1 (H.1): $\forall i, j$ we have $\tau_i^j \geq 0$ such that $\tau = \max_{i,j} \tau_i^j$.

Assumption 3.2 (H.2): The initial functions ϕ_i are given, continuous functions of time.

Assumption 3.3 (H.3): The weights a_{ij} are non-negative constants such that $a_{ij} = a_{ji}$. The associated graph is simply connected.

Theorem 3.1: Consider an undirected connected graph G with the associated combinatorial Laplacian L and it's spectrum. Let $\tilde{A} := \sum_{i,j} a_{ij}$ denote the sum of all the communication weights. Then there exist constants $k \in \mathbb{R}$ and $d > 0$ such that under assumptions $(H.1-3)$,(IVP1) converges to a common value, k, exponentially fast with rate d if the additional two conditions hold:

$$d < \lambda_2(L) \tag{H.4}$$

$$\tilde{A}\frac{e^{d\tau} - 1}{d}\left(1 + \sqrt{N}\frac{\lambda_N(L)}{\lambda_2(L) - d}\right) \leq \alpha \tag{H.5}$$

for some $0 \leq \alpha < 1$.

Remark 3.1: A first comment on the assumptions $(H.4, H.5)$ is that at one hand the rate of convergence of the delayed system cannot be faster than the rate of convergence of the un-delayed system while at the other hand, $(H.5)$ establishes a stability condition associated with the topological connectivity of the graph, with the weights, the rate of convergence and the maximum allowed delay.

Remark 3.2: The assumption $(H.5)$ is rather restrictive since A, is the sum of all the weights. This is the price one pays for not using the Lyapunov approach for this model. This assumption can be significantly improved if assumptions on the symmetry of the delays are taken. For example, if $\tau_i^j \equiv \tau > 0$ then A can be replaced with $|A|_1$, that is the induced norm of the adjacency matrix of the communication graph G.

Remark 3.3: The consensus point, k, has an analytical representation and is defined in (CNS1). It is, as expected, a function of $N, a_{ij}, \tau_i^j, \phi_i$.

B. Linear Time Invariant Weights with Distributed Delays

$$\begin{aligned} \dot{x}_i &= -\sum_{j\in N_i} a_{ij}x_i(t) + \sum_{j\in N_i} a_{ij}\int_{t-\tau}^{t} p_{ij}(s-t)x_j(s)ds, t > 0 \\ x_i &= \phi_i, \qquad t \in [-\tau, 0] \end{aligned} \tag{IVP2}$$

where it holds that for all and $i \sim j$

$$\int_{-\tau}^{0} p_{ij}(s)ds = 1 \tag{7}$$

are given distribution functions. Consider the sum $\tilde{B} = \sum_{i=1}^N \sum_{j\in N_i} \tilde{b}_{ij}$ where $\tilde{b}_{ij}$ are defined by

$$\tilde{b}_{ij} = \begin{cases} 0 & \text{if} j \notin N_i \\ a_{ij}\int_{-\tau}^{0} |p_{i,j}(s)|(e^{-ds} - 1)ds & \text{if } j \in N_i \end{cases} \tag{8}$$

Theorem 3.2: Under assumptions $(H.2, 3, 4)$ if there exists, $\alpha \in [0, 1)$ such that

$$\tilde{B}\frac{e^{d\tau} - 1}{d}\left(1 + \sqrt{N}\frac{\lambda_N(L)}{\lambda_2(L) - d}\right) \leq \alpha \tag{H.6}$$

then (IVP2) converges to a constant value k exponentially fast with rate d.

IV. Preliminary results

In this section, we review preliminary results which will be used as tools for the analysis to follow for both (IVP1) and (IVP2).

A. The undelayed dynamics

Equations (IVP1) without delays is a simple and well-studied system (see for example [6]). What is of importance to recall for this work is that the solution kernel e^{-Lt} takes any vector $\mathbf{z} \in \mathbb{R}^N$ which can be uniquely decomposed as $\mathbf{z} = \mathbf{z}_{//} + \mathbf{z}_c := \mathbf{1}\frac{1}{N}z_i + \mathbf{z}_c$ for some $\mathbf{z}_c \in \Delta^c$ and "suppress" the "magnitude" of $\mathbf{z}_c$ by $e^{-\lambda_2(L)t}$ so that $\lim_t e^{-Lt}\mathbf{z} = \mathbf{1}\frac{1}{N}z_i$. Another interesting view is that the quantity $\mathbb{I}(t) := \frac{1}{N}\sum_{i=1}^N x_i(t)$ is an integral of motion.

Next we state two technical lemmas to be used in the proof of the main result. We only prove the first one due to space limitations.

Lemma 4.1: Let $\mathbf{z}(t) \in \mathbb{R}^N$ such that $\lim_{t\to\infty}\mathbf{z}(t)$ exists and is finite. Then for L the combinatorial Laplacian of an undirected connected graph we have:

$$\lim_{t\to\infty}\mathbf{z}(t) - \int_0^t Le^{-L(t-s)}\mathbf{z}(s)ds = \mathbf{1}\frac{1}{N}\sum_{i=1}^N z_i(\infty) \tag{9}$$

Proof: Write $\mathbf{z}(t)$ as the sum of the vector projected onto the consensus subspace and it's complement, i.e.

$$\mathbf{z}(t) := \mathbf{z}_{//}(t) + \mathbf{z}_c(t) = \mathbf{1}\frac{1}{N}\sum_i z_i(t) + \mathbf{z}_c(t)$$

Then, since L and e^{sL} commute, the integral is equal to

$$\begin{aligned}\int_0^t Le^{Ls}\mathbf{z}(s)ds &= \int_0^t e^{Ls}L\mathbf{z}(s)ds \\ &= \int_0^t e^{Ls}L\mathbf{z}_c(s)ds = \int_0^t Le^{Ls}\mathbf{z}_c(s)ds\end{aligned}$$

and integration by parts yields

$$\int_0^t d(e^{Ls})\mathbf{z}_c(s) = e^{Lt}\mathbf{z}_c(t) - \mathbf{z}_c(0) - \int_0^t e^{Ls}\dot{\mathbf{z}}_c(s)ds$$

In view of the whole expression the integral

$$\begin{aligned}Q := &\int_0^t e^{-L(t-s)}\dot{\mathbf{z}}_c(s)ds \\ = &\int_0^t \Big(e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N}\Big)\dot{\mathbf{z}}_c(s)ds \to 0\end{aligned} \tag{10}$$

by the standard argument that the convolution of an L^1 function (that is, $(e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N})$) with a function that goes to zero (that is, $\dot{\mathbf{z}}(t)$), vanishes as well. So the whole expression converges to

$$\mathbf{z}_{//}(t) + \mathbf{z}_c(t) - \mathbf{z}_c(t) + e^{-Lt}\mathbf{z}_c(0) + Q \to \mathbf{z}_{//}(t)$$

■

Lemma 4.2 (Bounds): For any $r \le s \le t$

$$\begin{aligned}&\left|e^{-L(t-s)}L\right|_1 \le \sqrt{N}\lambda_N(L)e^{-\lambda_2(L)(t-s)} \\ &\int_r^t \left|e^{-L(t-s)} - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right|_1 ds \le \frac{\sqrt{N}}{\lambda_2(L)}(1 - e^{-\lambda_2(L)(t-r)})\end{aligned} \tag{11}$$

Proof: [Sketch of Proof] Among many approaches, one may choose to exploit the symmetry and positive semi-definiteness of L. These properties allow an orthonormal decomposition of $\mathbb{R}^N$ into invariant eigenspaces. The $\sqrt{N}$ term comes out of the norm equivalence in $\mathbb{R}^N$ between $|\cdot|_1$ and $\|\cdot\|_2$ (the Euclidean norm). ■

a) Consensus Point: At this moment, we will make an **Ansatz**:

All solutions of (IVP1) *and* (IVP2) *tend to some constants in* Δ, $\mathbf{1}k_{(IVP1)}$ *and* $\mathbf{1}k_{(IVP2)}$, *respectively.*

In view of this educated guess,we take the limit $t \to \infty$ so that $x_i(t) \to k$ for all i and solve for k to obtain (CNS1).

Proposition 4.1: An integral of motion for (IVP1) and (IVP2) are

$$\mathbb{I}_{(\text{IVP1})}(t) = \sum_{i=1}^N x_i(t) + \sum_{i,j} a_{ij}\int_{t-\tau_i^j}^t x_j(s)ds \tag{12}$$

and

$$\mathbb{I}_{(\text{IVP2})}(t) = \sum_{i=1}^N x_i(t) + \sum_{i,j} a_{ij}\int_{-\tau}^0 p_{ij}(s)\int_{t-\tau_i^j}^t x_j(w)dwds \tag{13}$$

respectively.

Proof: For (IVP1) take

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N x_i &= \sum_{i=1}^N\sum_{j\in N_i} a_{ij}(x_j - x_i) \\ &\quad - \frac{d}{dt}\sum_{i=1}^N\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^t x_j(s)ds \\ &= -\frac{d}{dt}\sum_{i=1}^N\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^t x_j(s)ds \Rightarrow\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^N x_i(t) - \sum_{i=1}^N x_i(0) = &-\sum_{i=1}^N\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^t x_j(s)ds \\ &+ \sum_{i=1}^N\sum_{j\in N_i} a_{ij}\int_{-\tau_i^j}^0 \phi_j(s)ds\end{aligned}$$

to derive $\mathbb{I}_{(\text{IVP1})}(t)$. The procedure for $\mathbb{I}_{(\text{IVP2})}(t)$ is identical. ■

Moreover, two critical points in Δ occur if one, for a moment, assumes that the solutions of (IVP1) and (IVP2) tend to finite consensus values. These points should be

$$k_{(\text{IVP1})} := \frac{\sum_i^N \phi_i(0) + \sum_{i,j} a_{ij}\int_{-\tau_i^j}^0 \phi_j(s)ds}{N + \sum_{i,j} a_{ij}\tau_i^j} \tag{CNS1}$$

$$k_{(\text{IVP2})} := \frac{\sum_i \phi_i(0) + \sum_{i,j} a_{ij}\int_{-\tau}^0 p_{ij}(s)\int_s^0 \phi_j(w)dwds}{N + \sum_{i,j} a_{ij}\int_{-\tau}^0 p_{ij}(s)(-s)ds} \tag{CNS2}$$

B. Fixed Point Theory

The stability problems we discuss is through contraction mappings and it is thus formulated in complete metric spaces. We begin defining a Banach space $(\mathcal{B}, |\cdot|)$ in the background and prove that a subset of interest $\mathcal{M}$ of $\mathcal{B}$ is closed and thus constitutes a complete metric space. So, a crucial step is to describe our complete metric space on which the contraction mapping principle (Theorem 2.1) will be applied.

1) Completeness of the metric space $(\mathcal{M}, \rho)$*:* In this section we will introduce and discuss the metric space with such desired properties so that by "finding" a solution of (IVP1), (IVP2) in there we will have, de-facto, solved the problem. Given $\tau > 0$, $\phi(t), t \in [-\tau, 0], d > 0, k > 0$, define the functional space $\mathcal{M} = \mathcal{M}_{(\tau,k,\phi,d)}$

$$\begin{aligned}\mathcal{M} := \{\mathbf{y} \in C([-\tau, \infty], \mathbb{R}^n) : \mathbf{y} = \phi|_{[-\tau,0]}, \\ \sup_{t\ge-\tau} e^{dt}|\mathbf{y}(t) - \mathbf{1}k|_1 < \infty\}\end{aligned} \tag{CMS}$$

and the weighted metric

$$\rho_d(\mathbf{y}_1, \mathbf{y}_2) := \sup_{t\ge0} e^{dt}|\mathbf{y}_1(t) - \mathbf{y}_2(t)|_1 \tag{14}$$

Proposition 4.2: The metric space $(\mathcal{M}, \rho_d)$ is complete.

Proof: [Sketch of Proof] The proof proceeds by applying straightforwardly the definition of completeness. (see [17] and for rigorous proof [14]) ■

V. Dynamics With Constant delays

In this section we will prove convergence to the consensus point and the rate at which this occurs for the case of linear time invariant symmetric communication weights. We rewrite (IVP1) as follows

$$\dot{x}_i(t) = \sum_{j\in N_i} a_{ij}(x_j - x_i) - \frac{d}{dt}\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^{t} x_j(s)ds$$

In the vector form, we get for $\mathbf{x} = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$

$$\dot{\mathbf{x}} = -L\mathbf{x} - \frac{d}{dt}\sum_{i,j}\int_{t-\tau_i^j}^{t} A_i^j \mathbf{x}(s)ds \tag{15}$$

where L is the Laplacian matrix and $A_i^j = [A_i^j]_{m,n} = [a_{m,n}\delta_{i,j}]$ are matrices with zero elements that are not in the i,j position.

The general solution using the variation of constants formula and integration by parts becomes:

$$\begin{aligned}
\mathbf{x}(t) &= e^{-Lt}\mathbf{x}(0) - \int_0^t e^{-L(t-s)}\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)duds \\
&= e^{-Lt}\mathbf{x}(0) - \sum_{i,j}\int_{t-\tau_i^j}^{t} A_i^j \mathbf{x}(u)du \\
&+ e^{-Lt}\sum_{i,j}\int_{-\tau_i^j}^{0} A_i^j \phi(u)du \\
&+ e^{-Lt}\int_0^t Le^{Ls}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)duds \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t)
\end{aligned}$$

We consider now the weighted metric space ($\mathcal{M}$,ρ_d) as it was defined in (CMS). We will also use a weighted norm denoted as $|\cdot|_d$.

The fixed point argument is the implementation of the Contraction Mapping Principle (Theorem (2.1)) and consists of the following steps: We define an appropriate operator with prescribed smoothness properties, we show that this operator maps $\mathcal{M}$ onto itself and we show that this operator is a contraction in ($\mathcal{M}$,ρ_d). So for $\mathbf{y} \in \mathcal{M}$ and $t \geq -\tau$, we define the operator P by

$$(P\mathbf{y})(t) := \begin{cases} \phi(t), \qquad -\tau \leq t \leq 0 \\[2ex] e^{-Lt}(\phi(0) + \sum_{i,j}\int_{-\tau_i^j}^{0} A_i^j \phi(u)du) - \\ -\sum_{i,j}\int_{t-\tau_i^j}^{t} A_i^j \mathbf{y}(u)du + \\ +e^{-Lt}\int_0^t Le^{Ls}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{y}(u)duds, t > 0 \end{cases}$$

For convenience, set $\mathbf{z}(t) := -\sum_{i,j}\int_{t-\tau_j^i}^{t} A_i^j \mathbf{x}(u)du$.

Proposition 5.1: The operator P possesses the following properties:

1) P is a continuous function of time for any $t > 0$.
2) $P : \mathcal{M} \to \mathcal{M}$ under assumption (H.4).
3) P is a contraction under assumption (H.5).

Proof: The first statement follows trivially by the definition. The second requires to prove that that P converges to a value $\mathbf{1}k \in \Delta$ with rate faster than d. Since by hypothesis the communication graph is simply connected, the terms of P for $t > 0$ converge as follows:

$$\lim_t (I_1 + I_3)(t) = \mathbf{1}\frac{1}{N}\Big(\sum_{i=1}^{N}\phi_i(0) + \sum_{i,j} a_{ij}\int_{-\tau_i^j}^{0}\phi_j(s)ds\Big)$$

by the discussion in section (IV.A), and by Lemma (4.1)

$$\lim_t (I_2 + I_4)(t) = -\mathbf{1}\frac{1}{N}\sum_{i,j} a_{ij}\tau_j^i k$$

if $\mathbf{x}(t) \to \mathbf{1}k$. Combine the results above to conclude that the operator $\mathcal{P}$ indeed converges to $\mathbf{1}k$ just like all the members of $\mathcal{M}$ only if k is defined as in (CNS1). Then another useful expression for k for any $\mathbf{x} \in \mathcal{M}$ comes from the variation of constants formula:

$$\mathbf{1}k = \mathbf{1}\frac{1}{N}\sum_i \phi_i(0) + \frac{\mathbf{1}\mathbf{1}^T}{N}\int_0^\infty \frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)duds \tag{16}$$

It remains to show that $|(P\mathbf{y}) - \mathbf{1}k|_d < \infty$. We break the components of k in the form of (16) and use the triangular inequality to obtain the estimates.

$$\begin{aligned}
&|(P\mathbf{y})(t) - \mathbf{1}k|_1 \leq \left|e^{-Lt} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right|_1 |\phi(0)|_1 \\
&+ \int_0^t \left|e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right|_1 \cdot \left|\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)du\right|_1 ds \\
&+ \int_t^\infty \left|\frac{\mathbf{1}\mathbf{1}^T}{N}\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)du\right|_1 ds
\end{aligned}$$

It is an easy exercise to see that the first two parts converge to 0 with rate $\lambda_2(L)$ whereas the last part converges with rate d.

So the claim $|(P\mathbf{y})(t)|_d < \infty$ follows under the assumption that

$$d < \lambda_2(L) \tag{H.4}$$

The last part is to show that $(P\mathbf{y})(t)$ is a contraction in ($\mathcal{M}$,ρ_d). Recall the notations from sections (III) and (IV) and consider $t > 0$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{M}$. Then

$$\begin{aligned}
&|(P\mathbf{y}_1)(t) - (P\mathbf{y}_2)(t)|_1 \leq \\
&\leq \left|\sum_{i,j} A_i^j \int_{t-\tau_i^j}^{t}(\mathbf{y}_1(s) - \mathbf{y}_2(s))ds\right|_1 + \\
&+ \left|e^{-Lt}\int_0^t Le^{Ls}\sum_{i,j} A_i^j \int_{s-\tau_i^j}^{s}(\mathbf{y}_1(w) - \mathbf{y}_2(w))dwds\right|_1
\end{aligned}$$

The first term can be bounded as follows:

$$\begin{aligned}
&\left|\sum_{i,j} A_i^j \int_{t-\tau_i}^{t}(\mathbf{y}_1(s) - \mathbf{y}_2(s))ds\right|_1 \leq \\
&\leq \sum_{i,j}|a_{ij}|\int_{t-\tau}^{t}|(\mathbf{y}_1(s) - \mathbf{y}_2(s))|_1 ds \\
&\leq e^{-dt}\frac{e^{d\tau} - 1}{d}\Big(\sum_{i,j} a_{ij}\Big)\rho_d(\mathbf{y}_1, \mathbf{y}_2)
\end{aligned}$$

The second term can be bounded as follows:

$$\Big| \int_0^t e^{-L(t-s)} L \sum_{i,j} A_i^j \int_{s-\tau_i^j}^{s} (\mathbf{y}_1(w) - \mathbf{y}_2(w)) dw ds \Big|_1 \leq$$

$$\leq \sum_{i,j} a_{ij} \int_0^t |e^{-L(t-s)} L|_1 \int_{s-\tau}^{s} |\mathbf{y}_1 - \mathbf{y}_2|_1 ds$$

$$\leq e^{-dt} \frac{e^{d\tau}-1}{d} \sum_{i,j} a_{ij} \Big[\sum_{n=2}^{N} \frac{N}{\lambda_n(L) - d} \Big] \rho_d(\mathbf{y}_1, \mathbf{y}_2)$$

The result follows on condition that there exists $0 \leq \alpha < 1$ such that

$$\tilde{A} \frac{e^{d\tau}-1}{d} \Big(1 + \sqrt{N} \frac{\lambda_N(L)}{\lambda_2(L) - d} \Big) \leq \alpha \tag{H.5}$$

for some $d < \lambda_2(L)$ according to (H.4). ■

A. The non-symmetric case:

The above result can be generalized for the case of non-symmetric constant weights at the expense of stronger assumptions. This is the case of directed networks and the sufficient assumption is for the corresponding graph to contain a routed out sub-graph. The integral of motion for this system is $\mathbf{q}^T \mathbf{x}$ where $\mathbf{q}$ is the left eigenvector of the (in-degree) Laplacian L (see [6]). The analysis differs from the symmetric case in two points:

1) The consensus point in this case is

$$k := \frac{\sum_i^N q_i \phi_i(0) + \sum_{i=1}^N \sum_{j \in N_i} q_i a_{ij} \int_{-\tau_i^j}^0 \phi_j(s) ds}{1 + \sum_{i=1}^N \sum_{j \in N_i} q_i a_{ij} \tau_i^j}$$

2) The bound in Lemma (4.2) needs to be modified (see [14]).

VI. Dynamics With Distributed delays

The analysis does not substantially change from the one just carried out, so all but the last step of the Fixed Point Argument will be omitted. We follow the same harmless perturbation technique and bring the equations in the vector form of (15). and write

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) - \frac{d}{dt} \sum_{j \in N_i} a_{ij} \int_{-\tau}^0 p_{ij}(s) \int_{t+s}^t x_j(w) dw ds \tag{17}$$

which in vector form is

$$\dot{\mathbf{x}} = -L\mathbf{x} - \frac{d}{dt} \sum_{i,j} \int_{-\tau}^0 B_i^j(s) \int_{t+s}^t \mathbf{x}(w) dw ds \tag{18}$$

where B_{ij} is the $N \times N$ matrix with elements $[a_{ij} p_{ij}(s) \delta_{m,n}]_{mn}$.

The operator solution is:

$$\mathbf{x}(t) = e^{-Lt} \Big(\mathbf{x}_0 + \sum_{i,j} \int_{-\tau}^0 B_i^j(s) \int_{-s}^0 \phi(w) dw ds \Big) -$$

$$\sum_{i,j} \int_{-\tau}^0 B_i^j(s) \int_{t+s}^t \mathbf{x}(w) dw ds +$$

$$+ \int_0^t L e^{-L(t-s)} \sum_{i,j} \int_{-\tau}^0 B_i^j(\hat{s}) \int_{s+\hat{s}}^s \mathbf{x}(w) dw d\hat{s} ds$$

So for $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}$ we follow the same procedure as above, but with a bit more care:

The first one can be bounded as follows:

$$\Big| \sum_{i,j} \int_{-\tau}^0 B_i^j(s) \int_{t+s}^t \mathbf{x}_1(w) - \mathbf{x}_2(w) dw ds \Big|_1$$

$$\leq \sum_{i,j} \int_{-\tau}^0 a_{i,j} |p_{i,j}(s)| \int_{t+s}^t |\mathbf{x}_1(w) - \mathbf{x}_2(w)|_1 dw ds$$

$$\leq \sum_{i,j} \int_{-\tau}^0 a_{i,j} |p_{i,j}(s)| \int_{s+t}^t e^{-dw} dw ds \rho_d(\mathbf{x}_1, \mathbf{x}_2)$$

$$\leq \Big(\sum_{i,j} a_{i,j} \int_{-\tau}^0 |p_{i,j}(s)| (e^{-ds} - 1) ds \Big) e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2)$$

and respective bound for the second term just like the multiple delays case. So here the difference is not only on the consensus point (defined in (CNS2)) but also in the assumption (H.5), which from the analysis above, needs to be replaced by (H.6).

VII. Discussion and Concluding Remarks

The crucial factor when one models the dynamics of multiple agents, is the amount of symmetry the designer is willing to sustain. The more symmetrical the proposed model is, the easier the mathematical manipulation is and the stronger (or more elegant) the results are. The price for this is the distance from Reality. The more symmetrical a system is, the more ideal hence the less realistic is.

An excellent example of this general principle is the Consensus Dynamics, especially in the LTI case. Although the step from symmetric to asymmetric weights a_{ij} effects only the consensus point; the step from synchronous to asynchronous communication can be really hard to analyse using a Lyapunov method. The interested reader is refered to [14] where we discuss these difficulties for the system of this paper.

The Fixed Point Theory comes to fulfil the gap since it does not require a, usually too insightful, construction of a Lyapunov candidate, at the expense of harder analysis and perhaps a bit strong assumptions. The main advantage of this approach is that it cannot lead to a dead-end. Here the researcher is free to choose his own space of function and search for an existence of a solution of the model he proposes. In this work, we considered simple consensus models. We exploited the heritage of the solutions of the synchronous version (i.e. convergence to a common value with exponential rate) and we asked whether similar behavior can be found in the delayed case and at what cost. What is in, our opinion, is very interesting, is that through the procedure of constructing a contraction mapping, we were able to understand the interplay between symmetry and sufficient assumptions. This is depicted, in the case of constant delays for instance, in hypothesis (H.5).

This major condition becomes stricter (i.e. restricts the maximum delay τ) mainly in two cases: The first is this of weak connectivity, which interprets to small $\lambda_2(L) > 0$

and the second is the magnitude of asymmetry, which is represented by A. Indeed in this work we considered the most asymmetrical case for the delays, that is each agent i has a possibly different delay in it's communication with j τ_i^j with in principal $i \neq j$. This freedom costs a large value for A. The reader should rest assured that A would be much smaller in case the $\tau_i^j = \tau_i \ \forall j \in N_i$ or even more symmetrical $\tau_i^j \equiv \tau$.

Another interesting feature of assumptions (H.5) and (H.6) is this of d and τ which is yet to be investigated more. In [14] we study the case where the weight vary with time. There, the trade-off between symmetry and assumptions is even more distinct.

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