Fixed Point Theory Approach to Exponential Convergence in LTV Continuous Time Consensus Dynamics with Delays *

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Abstract

The problem of continuous linear time varying consensus dynamics is addressed in the presence of constant communication delays. We make a Fixed Point Theory argument with the use of contraction mappings and we state sufficient conditions for exponential convergence to a consensus value with prescribed convergence rate.

1 Introduction

Distributed computation and multi agent dynamics is one of the most prominent and active research area in the control community in the past years.

It was initiated by the seminal work of Tsitsiklis [14] and the subject was significantly reheated with the work of Jadbabaie et al. [5]. Since then, an enormous amount of works has been produced from different fields of Applied Science (Engineering, Phsysics, Mathematics) concerning types of coordination among autonomous agents who exchange information in a distributed way in many frameworks (e.g. deterministic or stochastic) and under various communication conditions (for example [8, 7, 4, 6, 12, 10, 11, 9] and references therein).

All of the proposed models are mainly based on a specific type of dynamic evolution of the agents' states known as *consensus schemes*. Each agent evolves it's state by some specific type of averaging of the states of it's 'neighbours'. So long as certain connectivity conditions hold, all agents will eventually converge to a common value.

In this work, we revisit the linear time varying (LTV) consensus model in the presence of bounded communication delays. We prove exponential convergence of the autonomous agents to a common value under conditions related to the topology of the communication graph, the nature of the time-varying weights, the maximum allowed delay and the rate of convergence of the undelayed system. Our approach differs from that of the vast majority of the works in the literature which use a Luapunov based approach. Here, we develop a Fixed Point Argument on an especially designed topological space.

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1.1 Model and related literature

Consider a finite number of agents $[N] = \{1, \dots, N\}$. The model we will discuss is of the type

(MDL)
$$\dot{x}_i(t) = \sum_j a_{ij}(t)(x_j(t-\tau) - x_i(t)), \qquad i, j = 1, \dots, N$$

Each agent evolves according to the dynamics of it's own state as well as a retarded measurement of the states of it's neighbouring agents. Surprisingly enough compared to other and more complex models, this model has not received that much attention. To the best of our knowledge we mention four relative works.

A first example was proposed and discussed in [11]. The authors introduced and analysed the LTI system

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j(t-\tau) - x_i(t-\tau))$$

with $\tau > 0$ constant and uniform for all agents, using a frequency method approach. The problem with this method is that it is over simplistic and cannot be generalized in case the weights are time varying or the delays are incommensurate.

In [1] the authors consider a discrete time version of (MDL) with, in fact, time varying delays $\tau = \tau(t)$. On condition that the delay is uniformly bounded from above, the strategy of attacking the problem is to extend the state space by adding artificial agents. This method although applicable for discrete time, it is unclear how it would work in a continuous time system, unless the latter one is discretion and solved numerically.

In [12] the authors discuss the synchronization phenomenon of the non-linear model

$$\dot{x}_i = \sum_j a_{ij} f_{ij}(x_j(t-\tau) - x_i)$$

using passivity assumptions on f_{ij} and applying invariance principles. The main setback of this approach, however, is that nothing can be said for either the rate of convergence to the consensus or synchronization space or the consensus point itself.

Finally, models similar to (MDL) were used in the so called rendezvous type of algorithms, such as in [10]. The algorithm there is of the form

$$\dot{r}_i(t) = v_i(t) \quad \dot{v}_i(t) = -cv_i(t) + \sum_j a_{ij}(r_j(t-\tau_i^j) - r_i(t))$$

and the result was proved using a Lyapunov-Krasovskii argument on the base that the delayed quantities act only as perturbations to the main dynamical equation. This is not the case here though. Moreover, little can be said about the rate of convergence of this system.

1.2 Organization of the paper. This work is organized as follows: In section (2) we introduce notations and recall definitions from the relevant theoretical frameworks which will be implemented. In section (3) we introduce the model; we pose and discuss the assumptions and we conclude by stating our main result. The proof of the main result is carried in section (5). This work is a collection of results from [13]. Due to space limitations some proofs or steps in proofs were omitted.

2 Notations and Definitions

In this section, we describe the notations and definitions which will be used in this work. By $N < \infty$ we denote the number of agents. The set of agents is denoted by $[N] := \{1, \ldots, N\}$. Each agent $i \in [N]$ is associated with a real quantity $x_i \in \mathbb{R}$ which models it's state and it is a function of time.

2.1 Euclidean Spaces The Euclidean vector space $\mathbb{R}^N$ frames the state space of the system with vectors $\mathbf{x} = (x_i, \ldots, x_N)^T$. We will endow it with the $p = 1$ norm $||\mathbf{x}|| = \sum_{i=1}^N |x_i|$ for each $\mathbf{x} \in \mathbb{R}^N$. The induced norm of a square $N \times N$ matrix A is defined as $||A|| = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}||$. By, $\mathbf{1}$, we denote the column vector of all ones. The subspace of $\mathbb{R}^N$ of interest is defined by

$$\Delta = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = \mathbf{1}c, \quad c \in \mathbb{R}\}$$

and it is called the *consensus subspace.*

2.2 Algebraic Graph Theory The mathematical object which will be used to model the communication structure among the N agents is the weighted directed graph. This is defined as the triple $G = (V, E, W)$ where V is the set of nodes (here $[N]$), E is a subset of $V \times V$ which characterizes the established communication connections and W is a set associating a positive number (the weight) with any member of E. So by a_{ij} we will denote the weight in the connection from node j to node i and this is an amount of the effect that j has on i. If $a_{ij} = 0$ then $(j, i) \notin E$.

In this work, we are interested in directed graphs with a spanning tree (for a thorough introduction the reader is refered to [2]).

Given E, each agent i has a neighbourhood of nodes, to which it is adjacent. We denote by N_i the subset of V such that $(j, i) \in E$ and by $|N_i|$ it's cardinality. The connectivity weights for each $j \in N_i$ a_{ij} are in this work considered to be time dependent. The overall network influence to i is measured by the *degree* $d_i(t) = \sum_{j \in N_i} a_{ij}(t)$ A matrix representation of G is through the adjacency matrix $A = [a_{ij}]$, the degree matrix $D = \mathrm{Diag}[d_i]$ and the Laplacian $L := D - A$, known as the (in-degree) Laplacian. The notation $\sum_{i,j}$ stands for $\sum_{i=1}^N \sum_{j \in N_i}$ and by $\mathbf{d}_t = (d_1(t), \ldots, d_N(t))$ the vector of overall degree influence.

2.3 Function Spaces and Fixed Point Theory We will establish our results on metric spaces. These are the most common type of abstract topological spaces, which are of use in applications. The topology generated in these spaces is through a non-negative valued function which determines the neighbourhoods that comprise the topology.

Definition 2.1. *A metric space is the pair $(\mathcal{M}, \rho)$ of a set $\mathcal{M}$ and a function $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, such that $x, y, z \in \mathcal{M}$ implies*

1 $\rho(y, z) \geq 0,\ \rho(y, y) = 0,\ \rho(y, z) = 0 \Rightarrow y = z$

2 $\rho(y, z) = \rho(z, y)$

3 $\rho(y, z) \leq \rho(y, x) + \rho(x, z)$

In this work, we consider only complete metric spaces. A metric space $\mathcal{M}$ is *complete* if every Cauchy sequence has a limit in $\mathcal{M}$. Another function space is the space of absolutely integrable real-valued functions defined in $[a, b]$, denoted by $L^1_{[a,b]}$.

Now, recall that given two metric spaces $(\mathcal{M}_i, \rho_{\mathcal{M}_i})$ for $i = 1, 2$ an operator $P : \mathcal{M}_1 \to \mathcal{M}_2$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $x_1, x_2 \in \mathcal{M}$ imply

$$\rho_{\mathcal{M}_2}(Px_1, Px_2) \leq \alpha \rho_{\mathcal{M}_1}(x_1, x_2) \tag{2.1}$$

The next celebrated theorem will be used in proving our main results.

THEOREM 2.1. *[Contraction Mapping Principle] Let $(\mathcal{M}, \rho)$ be a complete metric space and $P : \mathcal{M} \to \mathcal{M}$ a contraction operator. Then there is a unique $x \in \mathcal{M}$ with $Px = x$. Furthermore, if $y \in \mathcal{M}$ and if $\{y_n\}$ is defined inductively by $y_1 = Py$ and $y_{n+1} = Py_n$ then $y_n \to x$, the unique fixed point. In particular, the equation $Px = x$ has one and only one solution.*

The proof of the theorem can be found in any advanced analysis or ordinary differential equations book. We refer the reader to [3] which is closest to our work.

2.3.1 The space of solutions.

The stability problems we discuss are through contraction mappings and they are thus formulated in complete metric spaces. Let us now define this specially constructed space together with the metric function we will use in order to use Theorem (2.1).

Given $t_0 \in \mathbb{R}$, $\tau > 0$, $\phi_i(t)|_{t \in [t_0-\tau, t_0]}, d > 0$, consider the following space $\mathcal{M} = \mathcal{M}_{(\tau,\phi,k,d)}$

(CMS)
$$\mathcal{M} = \Big\{\mathbf{y} \in C([t_0 - \tau, \infty), \mathbb{R}^N) : \mathbf{y} = \boldsymbol{\phi}|_{[t_0-\tau,t_0]}, \sup_{t \geq t_0-\tau} e^{dt}||\mathbf{y}(t) - \mathbf{1}k_{\mathbf{y}}|| < \infty, |k_{\mathbf{y}}| < \infty\Big\}$$

This is the space of functions, which take values in $\mathbb{R}^N$, are identical on $[t_0 - \tau, t_0]$ to a prescribed function and converge to a common constant $k_{\mathbf{y}}$, which depends on $\mathbf{y}$ in a fashion to be analyzed in the following section. The rate of convergence is exponential with rate d. Consider also the function $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, defined by

$$\rho(\mathbf{y}_1, \mathbf{y}_2) = \sup_{t \geq t_0} e^{dt}\Big|\Big|\big[\mathbf{y}_1(t) - \mathbf{1}k_{\mathbf{y}_2}\big] - \big[\mathbf{y}_2(t) - \mathbf{1}k_{\mathbf{y}_2}\big]\Big|\Big|$$

where k_i will be a function of t to be determined. Finding a solution of our version of (MDL) in $\mathcal{M}$ using Theorem (2.1) is a de facto solution of the problem. The next result is of essence in applying Theorem 2.1.

PROPOSITION 2.1. *[13] The metric space $(\mathcal{M}, \rho)$ is complete.*

Proof. [Sketch] Consider the Definition (2.1). The function ρ readily satisfies almost all of the properties. It is not hard to see that $\rho(\mathbf{y}_1, \mathbf{y}_2) = 0$ implies $\big|\big|\big[\mathbf{y}_1(t) - \mathbf{1}k_{\mathbf{y}_2}\big] - \big[\mathbf{y}_2(t) - \mathbf{1}k_{\mathbf{y}_2}\big]\big|\big|$ for all $t \geq t_0$ hence $k_{\mathbf{y}_1} = k_{\mathbf{y}_2}$ if one takes $t = t_0$ and the result follows. The same line of arguments yields that $|k_{\mathbf{y}_1} - k_{\mathbf{y}_2}|$ is a continuous function of $\rho(\mathbf{y}_1, \mathbf{y}_3)$. Hence, if $\{\mathbf{y}_i\}$ is a Cauchy sequence in $(\mathcal{M}, \rho)$, for every $\epsilon > 0$, and fixed $t > 0$ there is $M > 0$ such that for any $m, n \geq K$

$$||\mathbf{y}_m(t) - \mathbf{y}_n(t)|| \leq ||\mathbf{1}(k_m - k_n)|| + \rho(\mathbf{y}_m, \mathbf{y}_n) < \epsilon$$

It follows that the vectors $\mathbf{y}_i(t)$ for a Cauchy sequence in $\mathbb{R}^N$ and has a limit there, which is a function of t. In the same sense $k_i(t)$ form a Cauchy sequence in $\mathbb{R}$ and converge to a function of t, as well. It is easy to show that this convergence is uniform and the limit function is continuous and bounded for bounded $k_{\{}\mathbf{y}\}$. Especially $\mathbf{y}(t)$ is also bounded and converges with the same rate in Δ.

The details of the proof can be found in [13]. In the following we will clarify the function $k(t, \mathbf{y}(t))$ and we will show that under mild assumptions it satisfies the condition of Proposition (2.1) and so does the function ρ by being is a well-defined metric, according to Definition (2.1).

3 The model, the assumptions and the statement of the results

In this section we will introduce the problem, impose the sufficient Hypotheses, discuss some first remarks and state the main result.

3.1 Formulation of the Model

Given $N < \infty$, $0 < \tau < \infty$, $t_0 \in \mathbb{R}$ and the initial functions $\phi_i(t) : [-\tau, 0] \to \mathbb{R}|_{i=1}^N$, we consider the initial value problem.

(IVP)
$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(t)(x_j^\tau - x_i), \quad t \geq t_0$$
$$x_i(t) = \phi_i(t), \quad t \in [t_0 - \tau, t_0]$$

where $x_j^\tau := x_j(t - \tau)$ and τ is the bounded delay constant.

We impose the following hypotheses:

[**H.1**] The transition matrix of the linear system

(LTV)
$$\dot{\mathbf{x}} = -L(t)\mathbf{x} \quad , \quad t > t_0$$

is denoted by $\boldsymbol{\Phi}(t_1, t_2)$ where $t_1, t_2 \geq t_0$ and satisfies the following relation: For fixed $t_0 \in \mathbb{R}$, there exist $\gamma > 0$ (independent of t_0), $\Gamma > 0$ and $\mathbf{c} \in \mathbb{R}^N$ (possibly dependent on t_0) with $\sum_{i=1}^N c_i = 1$ such that

$$\big|\big|\boldsymbol{\Phi}(t, t_0) - \mathbf{1c}^T\big|\big| \leq \Gamma e^{-\gamma(t-t_0)}$$

[**H.2**] The weights $a_{ij}(t) : \mathbb{R} \to \mathbb{R}$ are C^1, bounded functions of time, with uniform bound $|a_{ij}| \leq a$.

[**H.3**] There exists $\beta \in (0, \gamma)$ such that

$$|\dot{a}_{ij}(s)|e^{-\beta s} \in L^1_{[t_0,\infty)}$$

[**H.4**] There exist $M > 0$ and $\delta > 0$ such that $\forall i \in [N]$:

$$\left|\sum_{j \in N_i} c_i a_{ij}(t) - c_j a_{ji}(t)\right| \leq Me^{-\delta t} \quad , \quad t > t_0$$

[**H.5**] There exists $\alpha \in [0,1)$ such that

$$\tau \frac{\sup_{t\geq t_0} |\mathbf{c}^T(A(t)-A(t_0))\mathbf{1}|}{1+\tau\mathbf{c}^T A(t_0)\mathbf{1}} < \alpha$$

Set $A = \sup_{t\geq t_0}||A(t)||$, $\dot{A} = \sup_{t\geq t_0}||\dot{A}(t)||$, $L = \sup_{t\geq t_0}||L(t)||$, $\dot{D} = \sup_{t\geq t_0}||\dot{\mathbf{d}}_t||$.
[**H.6**] For fixed $d \in (\beta,\gamma)$, there exists $\alpha \in [0,1)$ such that

$$\frac{e^{d\tau}-1}{d}\left(A\left[1+\frac{M}{d+\delta}+\frac{\Gamma L}{\gamma-d}\right]+\frac{\Gamma\dot{A}}{\gamma-d}+\frac{||\mathbf{c}||\dot{D}}{d}\right) \leq \alpha$$

The imposed hypotheses, although numerous and seemingly restricting are in fact the result of the drop of symmetry assumptions usually considered in consensus dynamics. For example we do not consider symmetry in a_{ij}. Moreover very little is actually known about the weights $a_{ij}(t)$. Below, we make a few comments on the assumptions, reviewing them one by one.

3.1.1 First Remarks. Assumption $[H.1]$ describes the dynamics of (LTV) and imposes the asymptotic consensus of the agents at the rate of $\gamma > 0$. To simplify the analysis we assume not failure of connectivity (i.e. $a_{ij} > 0$ if and only if $a_{ij}(t) > 0$ for some t). In the discussion of the result we will discuss the possibility of connectivity failures. The condition $\sum_i c_i = 1$ is necessary so that Δ is an (LTV)-invariant subspace.

Assumption $[H.2]$ characterizes the dynamics of the communication weights. The boundedness of a_{ij} is imposed as a reasonable assumption based on applications of the Control & Communication area and as an expected consideration which makes independent the communication framework from the dynamics. It also follows that $|L| < \infty$ and we denote this bound by $||L||$.

Assumption $[H.3]$ characterizes the dynamics of $\dot{a}_{ij}$ and asks for certain smoothness properties. Although not very important for the stability results of (LTV), it seems that under the Fixed Point Theory Approach these dynamics are important. The assumption is readily be fulfilled if for instance $|\dot{a}_{ij}| \in L^1[t_0,\infty)$. In such case, it is implied that the transmission weights asymptotically "freeze".

Assumption $[H.4]$ is one way to bridge the gap between $\mathbf{c}$ and $a_{ij}(t)$. Indeed $[H.1]$ says nothing about the connection between the weights and the consensus value. Note that in the case of weight symmetry (i.e. $a_{ij}(t) \equiv a_{ji}(t)$), $[H.4]$ is readily fulfilled and it is thus obsolete.

Assumption $[H.5]$ is necessary to prove existence and uniqueness the function $k(t)$. It's need is due to the fact that the dynamical system is non-autonomous and thus constant information of the weights and thus the solution is needed. This assumption can be significantly relaxed if the system was autonomous (time-invariant linear or non-linear) or if it was periodic.

Assumption $[H.6]$ includes two assumptions. The reasonable one, that the rate of convergence of the (IVP) cannot be faster than (LTV) and also the crucial condition so that the solution operator P (to be introduced below) is a contraction in $(\mathcal{M},\rho)$.

3.1.2 Main Result

THEOREM 3.1. *Consider the* (IVP) *and the assumptions* $[H.1]$ *to* $[H.6]$. *Then there exists a unique* $k(t) \in C^0([t_0,\infty),\mathbb{R})$ *such that the solution of* (IVP) *converges to* $k(t)$ *in the* $p=1$ *norm, exponentially fast, with rate* d.

4 Preliminary Results

In this section we will state some first results constructing the space $\mathcal{M}$ in order to apply Theorem (2.1). The first observation is on the solutions of (IVP).

All solution of the (IVP) **are bounded.** So long as the delay is bounded , we have the following result:

PROPOSITION 4.1. *If* $\mathbf{x}(t)$ *is a solution of* (IVP) *then* $|\mathbf{x}| \leq N\max_{i\in[N]}\sup_{t\in[t_0-\tau,t_0]}|\phi_i(s)|$

Proof. Let $q = N\max_i \sup_{t\in[t_0-\tau,t_0]}|\phi_i(s)|$ and assume that the condition does not hold. Then there exists a first time of escape for some $i\in[N]$, say $\bar{t} > 0$ such that

$$\left\{x_i(\bar{t}) = q, \dot{x}_i(\bar{t}) > 0\right\} \quad \text{or} \quad \left\{x_i(\bar{t}) = -q, \dot{x}_i(\bar{t}) < 0\right\}$$

The first case, for example, yields

$$\dot{x}_i(\bar{t}) = \sum_{j\in N_i} a_{ij}(\bar{t})(x_j(\bar{t}-\tau)-c) \leq 0$$

a contradiction. A similar contradiction arises for the second case.

The result above justifies boundedness of the members in $\mathcal{M}$ which is, however, not necessarily uniformly bounded.

The function $k(t,\cdot)$ It is not a hard exercise to see that under the integrability assumption for the weights $a_{ij}(t)$, (LTV) cannot have a non-trivial periodic solution (see also [13]). The existence of a consensus point for (LTV) is imposed in $[H.1]$, (this is $\mathbf{1}\mathbf{c}^T\phi(0)$), tells us nothing about the asymptotic behaviour of (IVP). The next result is crucial in establishing the existence and uniqueness of a limit function $k(t)$ to which (IVP) converges.

DEFINITION 4.1. *Given $\mathbf{y} \in \mathcal{M}$, we define the following function*

$$k(t) = \begin{cases} \frac{\mathbf{c}^T \boldsymbol{\phi}(t_0) + \mathbf{c}^T A(t_0) \int_{t_0-\tau}^{t} \mathbf{y}(s) ds}{1 + \mathbf{c}^T A(t_0) \mathbf{1}(t - t_0 + \tau)}, & \\ \qquad t_0 - \tau \leq t \leq t_0 & \\ \frac{\mathbf{c}^T \boldsymbol{\phi}(t_0) + \mathbf{c}^T \int_{t_0}^{t} \dot{A}(s) \int_{s-\tau}^{s} [\mathbf{y}(w) - \mathbf{1} k(w)] dw ds}{1 + \tau \mathbf{c}^T A(t_0) \mathbf{1}}, & \\ \qquad t \geq t_0 & \end{cases}$$

At first, observe that the integral in the second branch is a well defined for all t in view of $[H.3]$, together with the denominator in view of $[H.5]$. This implies that the integral converges in $\mathbb{R}$ and there are no oscillatory phenomena. Next, for any fixed $t \geq t_0$ we effectively deal with a non-linear equation in k the well-posedness of the solution of which, is not readily guaranteed.

LEMMA 4.1. *[13] Assume $t_0 \in \mathbb{R}$, $\tau > 0$, $\mathbf{c} \in \mathbb{R}^N$ and $a_{ij}(t) : [t_0, \infty) \to \mathbb{R}_+$ satisfying $[H.2], [H.5]$. Define the operator $\mathbb{Q} : \mathcal{C}([t_0 - \tau, \infty), \mathbb{R}^N) \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$*

$$\begin{aligned}\mathbb{Q}(\mathbf{y}, t, k) =& \frac{\mathbf{c}^T \boldsymbol{\phi}(t_0)}{1 + \tau \mathbf{c}^T A(t_0) \mathbf{1}} + \\ &+ \frac{\mathbf{c}^T \int_{t_0}^{t} \dot{A}(s) \int_{s-\tau}^{s} [\mathbf{y}(w) - \mathbf{1} k(w)] dw ds}{1 + \tau \mathbf{c}^T A(t_0) \mathbf{1}}\end{aligned}$$

Under the assumption $[H.5]$, for any fixed $t \geq t_0$, $\mathbf{y} \in \mathcal{C}$, there exists a unique k such that

$$\mathbb{Q}(\mathbf{y}, t, k) = k$$

Proof. [Sketch] Consider the complete metric space $(\mathbb{R}, |\cdot|)$ and prove that $\mathbb{Q}$ is a contraction in $(\mathbb{R}, |\cdot|)$ with respect to k. To do this one needs hypothesis $[H.5]$.

5 Time Varying Weights and Constant delays

We restate the initial value problem and recall the assumptions $[H.-]$

$$\text{(IVP)} \qquad \begin{aligned} \dot{x}_i &= \sum_{j \in N_i} a_{ij}(t)(x_j^\tau - x_i), \quad t \geq t_0 \\ x_i(t) &= \phi_i(t), \quad t \in [t_0 - \tau, t_0] \end{aligned}$$

In vector form the equation is written as

$$\dot{\mathbf{x}}(t) = -L(t)\mathbf{x}(t) - A(t)\frac{d}{dt}\int_{t-\tau}^{t} \mathbf{x}(s) ds \tag{5.2}$$

The general solution is

(5.3)
$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\phi}(t_0) - \int_{t_0}^{t} \boldsymbol{\Phi}(t, s) A(s) \frac{d}{ds} \int_{s-\tau}^{s} \mathbf{x}(w) dw ds$$

and using integration by parts and the fundamental properties of $\boldsymbol{\Phi}(t, s)$ it reads:

(5.4)
$$\begin{aligned}\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\Big(\mathbf{x}(t_0) + A(t_0)\int_{t_0-\tau}^{t_0} \phi(s) ds\Big) - \\ - A(t)\int_{t-\tau}^{t} \mathbf{x}(s) ds + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s) L(s) A(s) \int_{s-\tau}^{s} \mathbf{x}(w) dw ds \\ + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s) \dot{A}(s) \int_{s-\tau}^{s} \mathbf{x}(w) dw ds\end{aligned}$$

Consider the metric space $(\mathcal{M}, \rho_d)$ defined in (CMS). Define the operator P by

(5.5)
$$(P\mathbf{x})(t) := \begin{cases} \boldsymbol{\phi}(t), \quad t_0 - \tau \leq t \leq t_0 \\ \boldsymbol{\Phi}(t, t_0)\big(\mathbf{x}(t_0) + A(t_0)\int_{t_0-\tau}^{t_0} \boldsymbol{\phi}(s) ds\big) - \\ -A(t)\int_{t-\tau}^{t} \mathbf{x}(s) ds + \\ + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s) L(s) A(s) \int_{s-\tau}^{s} \mathbf{x}(w) dw ds + \\ + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s) \dot{A}(s) \int_{s-\tau}^{s} \mathbf{x}(w) dw ds, \\ \qquad t > t_0 \end{cases}$$

REMARK 1. *The operator P is obviously continuous in t and it is the solution expression of (5.2) and it equivalent both to (5.3) and (5.4). While (5.3) will be used for determining the consensus point, (5.4) will be used for proving that the operator is a contraction.*

The next proposition shows that for any member of the space $\mathcal{M}$, $(P\mathbf{x})(t)$ is a function that converges to Δ and in particular, in $\mathbf{1}k_{\mathbf{x}}$.

PROPOSITION 5.1. *Given $\mathbf{x} \in \mathcal{M}$ with $\mathbf{1}k_{\mathbf{x}}$, it holds that*

$$\sup_{t \geq t_0} e^{dt} ||(P\mathbf{x})(t) - \mathbf{1}k_{\mathbf{x}}|| < \infty$$

if $d \in (\beta, \gamma)$.

Proof. At first, we show that

$$\lim_{t} (P\mathbf{x})(t) = \mathbf{1}k_{\mathbf{x}} \Rightarrow k_{(P\mathbf{x})} = k_{\mathbf{x}} \tag{5.6}$$

For $\mathbf{x} \in \mathcal{M}$ from (5.4) we have

$$\begin{aligned}(P\mathbf{x})(t) =& \boldsymbol{\Phi}(t, t_0)\mathbf{x}(t_0) \\ &- \int_{t_0}^{t} (\boldsymbol{\Phi}(t, s) - \mathbf{1}\mathbf{c}^T) A(s) \frac{d}{ds} \int_{s-\tau}^{s} \mathbf{x}(w) dw ds - \\ &- \mathbf{1} \int_{t_0}^{t} \mathbf{c}^T A(s) \frac{d}{ds} \int_{s-\tau}^{s} \mathbf{x}(w) dw ds\end{aligned}$$

As $t \to \infty$, the first term converges to $\mathbf{1}(\mathbf{c}^T \phi(t_0))$, the second term goes to zero since it's norm is bounded above by a convolution of an $L^1[t_0, \infty)$ function (that is

$\mathbf{\Phi}(t,s)-\mathbf{1c}^T)$ with a function that goes to zero. Finally, the third term reads:

$$\mathbf{1c}^T\mathbf{w}(s) = \mathbf{1}\bigg(\sum_{i=1}^{N} c_i \sum_{j\in N_i} a_{ij}(s)\frac{d}{ds}\int_{s-\tau}^{s} x_j(w)dw\bigg)$$

for $(j,i)\in E$, as $t\to\infty$

$$\begin{aligned}
&\int_{t_0}^{\infty} a_{ij}(s)\frac{d}{ds}\int_{s-\tau}^{s} x_j(w)dw = \\
&= \int_{t_0}^{\infty} a_{ij}(s)\frac{d}{ds}\int_{s-\tau}^{s}(x_j(w)-k_{\mathbf{x}})dw = \\
&- a_{ij}(t_0)\int_{t_0-\tau}^{t_0}(\phi_j(w)-k_{\mathbf{x}})dw \\
&- \int_{t_0}^{\infty}\dot a_{ij}(s)\int_{s-\tau}^{s}(x_j(w)-k_{\mathbf{x}})dwds
\end{aligned}$$

the integral

$$\int_{t_0}^{\infty}\dot a_{ij}(s)\int_{s-\tau}^{s}(x_j(w)-k_{\mathbf{x}})dwds$$

converges in view of hypothesis $[H.3]$ for $d>\beta$. To prove the limit of $(P\mathbf{x})(t)$ we look for a solution of

$$\begin{aligned}
k_{\mathbf{x}} = \mathbf{c}^T\phi(0) &+ \mathbf{c}^T A(t_0)\int_{t_0-\tau}^{t_0}(\phi(w)-\mathbf{1}k_{\mathbf{x}})dw \\
&+ \mathbf{c}^T\int_{t_0}^{\infty}\dot A(s)\int_{s-\tau}^{s}(\mathbf{x}(w)-\mathbf{1}k_{\mathbf{x}})dwds
\end{aligned}$$

The existence and uniqueness of such $k_{\mathbf{x}}$ is guaranteed by Lemma (4.1) and the first result is proved. Secondly, we show that $\sup_{t\ge t_0-\tau} e^{dt}||(P\mathbf{x})(t)-\mathbf{1}k_{\mathbf{x}}||<\infty$. We use the operator P expression as in (5.3) and use the results right above to get the estimates

$$\begin{aligned}
&||(P\mathbf{x})(t)-\mathbf{1}k_{(P\mathbf{x})}|| = ||(P\mathbf{x})(t)-\mathbf{1}k_{\mathbf{x}}|| \le \\
&\le ||(\mathbf{\Phi}(t,t_0)-\mathbf{1c}^T)\phi(0)|| + \\
&+ \bigg|\bigg|\mathbf{1c}^T A(t)\int_{t-\tau}^{t}(\mathbf{x}(w)-\mathbf{1}k_{\mathbf{x}})dw\bigg|\bigg| \\
&+ \bigg|\bigg|\mathbf{1c}^T\int_{t}^{\infty}\dot A(s)\int_{s-\tau}^{s}(\mathbf{x}(w)-\mathbf{1}k_{\mathbf{x}})dwds\bigg|\bigg|
\end{aligned}$$

The first term is bounded as $e^{-\gamma t}$ while the second and the third term is bounded by e^{-dt}. The expression is thus bounded in the weighted norm (see definition of $\mathcal{M}$) if $d<\gamma$. This concludes the proof of Proposition (5.1).

At this point we have proved that the functional space $\mathcal{M}$ is compatible enough for the solutions of the delayed LTV system and that the associated solution operator maps $\mathcal{M}$ into itself. The final step is to show that P is a contraction in $(\mathcal{M},\rho)$. To prove this we need to show that for any $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{M}$.

$$\rho((P\mathbf{x}_1),(P\mathbf{x}_2)) \le \alpha\rho(\mathbf{x}_1,\mathbf{x}_2)$$

for some $\alpha\in[0,1)$. To simplify analysis set $\mathbf{X}_{12}(s) := \big((\mathbf{x}_1(s)-\mathbf{1}k_{\mathbf{x}_1})-(\mathbf{x}_2(s)-\mathbf{1}k_{\mathbf{x}_2})\big)$. For $t>t_0$, careful calculations yield

(5.7)

$$\begin{aligned}
&\big|\big|\big[(P\mathbf{x}_1)(t)-\mathbf{1}k_{(P\mathbf{x}_1)}\big]-\big[(P\mathbf{x}_2)(t)-\mathbf{1}k_{(P\mathbf{x}_2)}\big]\big|\big| \\
&= \bigg|\bigg| - A(t)\int_{t-\tau}^{t}\mathbf{X}_{12}(s)ds + \\
&+ \int_{t_0}^{t}\big(\mathbf{\Phi}(t,s)-\mathbf{1c}^T\big)L(s)A(s)\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds \\
&+ \int_{t}^{\infty}\mathbf{1c}^T L(s)A(s)\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds \\
&+ \int_{t_0}^{t}\big(\mathbf{\Phi}(t,s)-\mathbf{1c}^T\big)\dot A(s)\cdot\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds \\
&+ \mathbf{1c}^T\int_{t}^{\infty}\dot A(s)\cdot\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg|
\end{aligned}$$

the first term is bounded as follows

$$\bigg|\bigg|A(t)\int_{t-\tau}^{t}\mathbf{X}_{12}(s)ds\bigg|\bigg| \le \frac{e^{d\tau}-1}{d}||A(t)||e^{-dt}\rho(\mathbf{x}_1,\mathbf{x}_2)$$

The second and third terms are bounded as follows (recall $[H.4]$)

$$\begin{aligned}
&\bigg|\bigg|\int_{t_0}^{t}\mathbf{\Phi}(t,s)L(s)A(s)\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg| \le \\
&\bigg|\bigg|\int_{t_0}^{t}\big(\mathbf{\Phi}(t,s)-\mathbf{1c}^T\big)L(s)A(s)\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg| + \\
&\bigg|\bigg|\int_{t}^{\infty}\mathbf{1c}^T L(s)A(s)\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg| \le \\
&\Gamma\frac{e^{d\tau}-1}{d(\gamma-d)}\sup_t|L(t)A(t)|_1(1-e^{-(\gamma-d)(t-t_0)})e^{-dt}\rho(\mathbf{x}_1,\mathbf{x}_2) + \\
&M\sup_t||A(t)||\frac{e^{d\tau}-1}{d(d+\delta)}e^{-dt}\rho(\mathbf{x}_1,\mathbf{x}_2)
\end{aligned}$$

Finally the last two terms in (5.7) are bounded as follows:

$$\begin{aligned}
&\bigg|\bigg|\int_{t_0}^{t}\big(\mathbf{\Phi}(t,s)-\mathbf{1c}^T\big)\dot A(s)\cdot\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg| \\
&\bigg|\bigg|\mathbf{1c}^T\int_{t}^{\infty}\dot A(s)\cdot\int_{s-\tau}^{s}\mathbf{X}_{12}(w)dwds\bigg|\bigg| \le \\
&\frac{e^{d\tau}-1}{d(\gamma-d)}\sup_t|\dot A(t)|_1\Gamma(1-e^{-(\gamma-d)(t-t_0)})e^{-dt}\rho(\mathbf{x}_1,\mathbf{x}_2) \\
&+ \frac{e^{d\tau-1}}{d^2}||\mathbf{c}||\cdot\sup_t||\dot{\mathbf{d}}_t||e^{-dt}\rho(\mathbf{x}_1,\mathbf{x}_2)
\end{aligned}$$

Set $A = \sup_{t\geq t_0} ||A(t)||$, $\dot{A} = \sup_{t\geq t_0} ||\dot{A}(t)||$, $L = \sup_{t\geq t_0} ||L(t)||$, $\dot{D} = \sup_{t\geq t_0} ||\dot{\mathbf{d}}_t||$. Gathering up all these results we obtain:

$$\rho((P\mathbf{x}_1),(P\mathbf{x}_2)) \leq$$
$$\frac{e^{d\tau}-1}{d}\left(A\left[1+\frac{M}{d+\delta}+\frac{\Gamma L}{\gamma-d}\right]+\frac{\Gamma\dot{A}}{\gamma-d}+\frac{|\mathbf{c}|_1\dot{D}}{d}\right)\rho(\mathbf{x}_1,\mathbf{x}_2)$$

Under condition $[H.6]$ the operator P is indeed a contraction and the Theorem 3.1 is proved.

6 Discussion and concluding remarks

We conclude this paper with some important remarks.

6.1 On the overall approach and the assumptions Contrary to a Lyapunov based approach, the main novelty of this work is the use of Fixed Point Theory. Our effort was to bypass the main problem of Lyapunov theory, which is to come up with a good candidate function. The main advantage of contraction mappings is that one needs not to worry about it. Indeed the more asymmetric such a multi-agent system is the more difficult is the construction of this a Lyapunov candidate function is even more difficult (if not impossible) in the case of multi-agent dynamics. In [13] we discuss the difficulties of approaching (MDL) type of systems using the Lyapunov method. The system proposed here is of utmost assymetry, both in (time varying weights) and in the delays. The Fixed Point Argument, worked but at significant cost, most of it due at assumption $[H.6]$. It should be noted that the more asymmetrical assumptions one makes for the delays, the stronger (and more restricting) the assumptions get.

6.1.1 Multiple delays The convergence of the following system is discussed in [13]. In case τ is replaced on (IVP) with τ_j^i one understands the propagation delay in the exchange of information of agent j to agent i. One writes

$$\dot{\mathbf{x}}(t) = -L(t)\mathbf{x}(t) - \sum_{i=1}^{N}\sum_{j\in N_i} A_i^j(s)\int_{t-\tau_j^i}^{t}\mathbf{x}_j(s)ds \tag{6.8}$$

where $A_i^j(t)$ is the $N \times N$ square matrix with elements $e_{n,m} = a_{ij}(t)\delta_{n,m}^{i,j}$, where $\delta_{i,j}^{n,m} = 1$ if and only if $(n = i, m = j)$. The analysis only requires more careful algebra in view of the fact that one can take $\tau = \max_{i,j}\tau_j^i$. Other restricting factors is that we essentially exploited the heritage of the undelayed dynamical system (LTV) and used a very similar space of functions asking for the assumptions for equivalent behaviour of (5.2). Perhaps a more suitable choice of $\mathcal{M}$ would encourage less strict assumptions.

6.1.2 The LTI case In [13] we make a thorough analysis of the LTI case. The results in this case are obviously smoother. Not only because assumptions like $[H.3]$ or $[H.5]$ are obsolete, but because the state space of the solutions of the undelayed system can be represented by the elementary linear algebra methods. So for example γ stands for $\lambda_2(L)$ and $||L||$ stands for $\lambda_n(L)$, the second smallest and the largest eigenvalue of the (time invariant) Laplacian, respectively.

6.2 On the connectivity Assumption $[H.1]$ implicitly considers sufficient connectivity conditions among the agents so that consensus is reached. The overall connectivity assumption is implied in the sense that there may be failures of communication at times and weights $a_{ij}(t)$ can vanish to zero or even become negative (if that makes any sense at all), on condition that this happens in a smooth way so that the overall hypotheses set is valid certain continuity properties are not violated. In this case however one needs to consider time varying neighbourhoods $N_i = N_i(t)$ and this costs more careful analysis.

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