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Reachability Analysis for Linear Hybrid Set-Dynamics Driven by Random Convex Compact Sets

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Abstract— **This paper studies linear set-dynamics driven by random convex compact sets (RCCSs), where the parameter matrix evolves according to an underlying Markovian random process taking values in a finite set. We derive dynamics of the expectations of the associated reach sets. We establish that such expectations evolve according to coupled deterministic set–dynamics. We provide sufficient conditions for the convergence of the reach sets expectations. We also give conditions under which the reach sets remain asymptotically bounded with probability one. As an illustrative example, we apply our results to evaluate the expectations of the reach sets associated to the position of a quadrotor.**

I. Introduction

Reachability analysis determines the set of states that a system can possibly visit. Its relevance stems from the close relationship with optimal control, set–membership state estimation, safety verification and control synthesis under uncertainty. The analysis of uncertain constrained dynamics based on the concepts of reachability leads to *a-priori* guarantees of robustness properties such as robust constraint satisfaction, robust stability and convergence and recursive robust feasibility. The main research topics in reachability analysis include both the characterization and computation of the exact and approximate reachable sets and tubes [1]–[3]. Reachability analysis within the deterministic set–membership setting was addressed in [2], [4], [5], where the derived results are valid as long as the constraint sets are known exactly. An extension of the deterministic setting to the random case was presented in [6], where the sets of possible initial states and the disturbance constraint sets are random; this was done by considering the associated linear set–dynamics driven by RCCSs. In particular, the set–dynamics of the associated expected reach sets as well as the dynamics of the corresponding covariance functions were derived. The derived reachability notions in the stochastic case were based on the theory of random sets [7]–[14], on the set–dynamics framework for the set invariance under output feedback [15], and on the theory of the minimal invariant sets [16].

In this paper, we extend the reachability analysis provided in [6] to the case where the system parameters are also uncertain. In particular, we assume that the matrix of parameters evolves according to an underlying homogeneous, irreducible Markov chain. We derive the dynamics for the expectations of the associated reach sets, and study the convergence properties of such dynamics. The type of random process governing the evolution of the matrix of parameters accommodates a rich class of stochastic time-varying systems. Such systems can be used to represent random abrupt changes in structure, intermittent faults, or random time-delays. An alternative interpretation of the results presented here is in the context of uncertainty quantification. Namely, we demonstrate how the uncertainty in system parameters propagates through the system dynamics, as depicted by the evolution of the reach sets, in addition to the uncertainty in the initial states and disturbance.

Paper structure: Section II provides the problem setup and paper objectives. Section III introduces the mathematical background concerning random sets. Section IV presents the set–dynamics corresponding to the reach sets expectations together with the convergence analysis for such dynamics. We introduce here also a result that characterizes the limiting behavior of the reach sets in the almost sure sence. Section V presents numerical simulations for the reach set expectations associated to a quadrotor position, under different scenario regarding the random process driving the matrix of parameters.

Basic Notations and Definitions: Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X, \ y \in Y\}$, and we write $x \oplus X$ instead of $\{x\} \oplus X$. The Cartesian product of X and Y is a set $Z \subset \mathbb{R}^{2n}$, with $Z = X \otimes Y = \{z \ : \ z^T = [x^T, y^T], x \in X, \ y \in Y\}$. Given the sequence of sets $\{X_i \subset \mathbb{R}^n\}_{i=a}^b, \ a \in \mathbb{N}, \ b \in \mathbb{N}, b > a$, we denote $\bigoplus_{i=a}^b X_i := X_a \oplus \cdots \oplus X_b$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) the image of X under M is denoted by $MX := \{Mx \ : \ x \in X\}$. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ denotes the spectral radius of M, that is, the largest absolute value of its eigenvalues. A set $X \subset \mathbb{R}^n$ is *a C set* if it is compact (closed and bounded), convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is *a proper C set* if it is a C set and has non–empty interior. We say that a set $X \subseteq \mathbb{R}^n$ is a symmetric set w.r.t. $0 \in \mathbb{R}^n$ if $X = -X$. The collection of non–empty compact sets in $\mathbb{R}^n$ is denoted by $\mathrm{Com}(\mathbb{R}^n)$. The collection of non–empty compact, convex, sets in $\mathbb{R}^n$ is denoted by $\mathrm{ComConv}(\mathbb{R}^n)$. The convex hull of a set $X \subset \mathbb{R}^n$ is denoted by $\mathrm{co}(X)$. The support function $\mathrm{s}(X, \cdot)$ of a non–empty closed convex set $X \subset \mathbb{R}^n$ is given by

$$\mathrm{s}(X, y) := \sup_x \{y^T x \ : \ x \in X\} \text{ for } y \in \mathbb{R}^n.$$

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Given a PC–set L in $\mathbb{R}^n$, the function $\mathrm{g}(L,\cdot)$ given by

$$\mathrm{g}(L,x) := \inf_{\mu}\{\mu \ : \ x \in \mu L, \ \mu \in \mathbb{R}_+\} \text{ for } x \in \mathbb{R}^n$$

is called the gauge (Minkowski) function of the set L. If L is a symmetric PC–set in $\mathbb{R}^n$, then $\mathrm{g}(L,\cdot)$ induces the vector norm $|x|_L := \mathrm{g}(L,x)$ whose unit norm ball is the set L. For $X \in \mathrm{Com}(\mathbb{R}^n)$ and $Y \in \mathrm{Com}(\mathbb{R}^n)$, the *Hausdorff distance (metric)* is given by

$$H(L,X,Y) := \min_{\alpha}\{\alpha : X \subseteq Y \oplus \alpha L, Y \subseteq X \oplus \alpha L, \alpha \geq 0\},$$

where L is a given, symmetric, proper C set in $\mathbb{R}^n$. The norm of a non–empty compact subset X of $\mathbb{R}^n$ (i.e., $X \in \mathrm{Com}(\mathbb{R}^n)$) is given by $\|X\|_L := H(L,X,\{0\})$.

Remark 1: [17] We recall that if A is a stable matrix ($\rho(A) < 1$), then there exists a symmetric, proper C-set L in $\mathbb{R}^n$ and a scalar $\lambda \in [0,1)$ such that $AL \subseteq \lambda L$. In particular $\lambda \in [\rho(A),1)$.

II. Problem Setup

We consider the following autonomous discrete–time, linear, time–varying (DLTV) system:

$$x_{k+1} = A_{\theta_k} x_k + w_k, \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the current state, x_{k+1} is the successor state, and θ_k is a random process taking values in a finite set $\mathcal{S} = \{1,\ldots,q\}$. Consequently, the matrix A_{θ_k} is a matrix valued random process taking values in the finite set $\mathcal{A} = \{A_1,\ldots,A_q\}$. We assume θ_k to be a homogeneous Markov chain with probability transition matrix $P = (p_{ij})$ such that $Pr(\theta_{k+1} = j|\theta_k = i) = p_{ij}$.

The linear dynamics is driven by an unknown but bounded disturbance $w_k \in \mathbb{R}^n$. The unknown disturbance variable is bounded in the sense that, for all $k \in \mathbb{N}$, it holds that:

$$w_k \in W_k, \tag{2.2}$$

where, for each $k \in \mathbb{N}$, the disturbance set W_k is a random compact set, as defined in Section III. The initial state of the system (2.1) belongs to a random compact set X_0:

$$x_0 \in X_0. \tag{2.3}$$

Inspired by the set–dynamics theoretical framework [16], similarly as in [17], we introduce the map $\mathcal{R}(\cdot,\cdot,\cdot)$, given by:

$$\mathcal{R}(X,W,\theta) := A_\theta X \oplus W. \tag{2.4}$$

Clearly, the function $\mathcal{R}(\cdot,\cdot,\cdot)$ maps $\mathrm{Com}(\mathbb{R}^n) \times \mathrm{Com}(\mathbb{R}^n) \times \mathcal{S}$ to $\mathrm{Com}(\mathbb{R}^n)$ as well as $\mathrm{ComConv}(\mathbb{R}^n) \times \mathrm{ComConv}(\mathbb{R}^n) \times \mathcal{S}$ to $\mathrm{ComConv}(\mathbb{R}^n)$. Reachability analysis reduces to the characterization of the reach sets $X_k, \ k \in \mathbb{N}$ which are generated by the stochastic set–dynamics:

$$X^+ = \mathcal{R}(X,W,\theta) \text{ so that}$$
$$\forall k \in \mathbb{N}, \ X_{k+1} = \mathcal{R}(X_k,W_k,\theta_k). \tag{2.5}$$

Thus, the reach set at time $k \in \mathbb{N}$ is the k^{th} iterate of the map $\mathcal{R}(\cdot,W,\theta) \ : \ \mathrm{Com}(\mathbb{R}^n) \to \mathrm{Com}(\mathbb{R}^n)$ evaluated at X_0, while the reachable tube is the trajectory of the system (2.5) with the initial condition equal to X_0. In this case, the stochastic reach sets admit an explicit representation given by:

$$\forall k \in \mathbb{N}_+, \ X_k := \Phi(k,0)X_0 \oplus \bigoplus_{i=0}^{k-1} \Phi(k,i+1)W_i. \tag{2.6}$$

where $\Phi(k,i) = A_{\theta_{k-1}} A_{\theta_{k-2}} \ldots A_{\theta_i}$ is the state transition matrix, with $\Phi(k,k) = I$.

Next, we recall a reachability analysis result in the deterministic case. Let $A_{\theta_k} = A$ for all k where A is a deterministic matrix. In addition, let $W_k = W$ for all k, and let $X_0 \in \mathrm{Com}(\mathbb{R}^n)$ and $W \in \mathrm{Com}(\mathbb{R}^n)$ be deterministic sets. Under these assumptions, the reach sets admit the explicit representation

$$X_k := A^k X_0 \oplus \bigoplus_{i=0}^{k-1} A^i W. \tag{2.7}$$

and the following result holds.

Proposition 1 ([16], [17]): If $\rho(A) < 1$ then the reach sets X_k converge to the unique solution of the fixed point set equation $X = AX \oplus W$, which can be explicitly written as

$$X_\infty := \bigoplus_{i=0}^{\infty} A^i W. \tag{2.8}$$

Paper Objective: Our main aims are to:

- derive the set-dynamics of the reach set expectations whose evolution is controlled by the Markovian random process θ_k;
- provide conditions under which the expectations of the reach sets converge.

III. Random Sets Technical Preliminary

It is well–established that $\mathrm{Com}(\mathbb{R}^n)$ endowed with the Hausdorff distance $H(L,\cdot,\cdot)$ is a complete metric space [11], [13]. In fact, with the use of the Hausdorff distance $H(L,\cdot,\cdot)$ the space $\mathrm{Com}(\mathbb{R}^n)$ can be made into a separable, locally compact metric space [10], [11], [13]. It is also known that $\mathrm{ComConv}(\mathbb{R}^n)$ is a closed subset of $\mathrm{Com}(\mathbb{R}^n)$ and that the convex hull is a map $\mathrm{co}(\cdot) \ : \ \mathrm{Com}(\mathbb{R}^n) \to \mathrm{ComConv}(\mathbb{R}^n)$ which is continuous w.r.t. the Hausdorff distance $H(L,\cdot,\cdot)$ [10], [11], [13]. Additionally, as shown in [10], [11], [13], $\mathrm{ComConv}(\mathbb{R}^n)$ is an abstract, locally compact, convex cone which can be embedded isometrically into the Banach space $\mathcal{C}(L^*)$ of continuous functions on the dual unit ball $L^* := \{y \in \mathbb{R}^n \ : \ \forall x \in L, \ y^T x \leq 1\}$ (w.r.t. the unit ball L) of $\mathbb{R}^n$ by identifying a set $X \in \mathrm{ComConv}(\mathbb{R}^n)$ with its support function:

$$\mathrm{s}(X,y) = \sup_x\{y^T x \ : \ x \in X\}, \tag{3.1}$$

for all $y \in L^*$. Most often, L is chosen as the Euclidean norm ball $L = \mathcal{B}_2 := \{x \in \mathbb{R}^n \ : \ x^T x \leq 1\}$ so that its dual L^* satisfies $L^* = L$. This mapping preserves both the metric and linear structure [10], [13]. A random set X can be taken in the Borel sense [8], [9], [11], and thus a random set X *can be regarded as a measurable map defined on a probability space* (Ω,Σ,P)*, and taking values in the collection* $\mathrm{Com}(\mathbb{R}^n)$ *of non–empty compact subsets of* $\mathbb{R}^n$ [10], [12].

A selection of the random set X is a random vector x such that $x(\omega) \in X(\omega)$ holds almost surely. By adapting the definition of the expectation of a random set as introduced by Artstein and Vitale [10] and using $E(\cdot)$ to denote expectation, we have the following definition.

Definition 1: Let X be a random set such that each selection x has finite expectation $\mathrm{E}(x)$. The expectation of a random set X, denoted by $\mathrm{E}(X)$, is given by:

$$E[X] := \{E[x] \ : \ x \text{ is a selection of } X\}. \tag{3.2}$$

A necessary and sufficient condition for $\mathrm{E}(X)$ to be well–defined, (i.e., that $E[X] \in \mathrm{Com}(\mathbb{R}^n)$), is that $E[\|X\|_L] < \infty$ [10]. If the underlying probability space is nonatomic, then it is known [7], [9], [10] that $E[\mathrm{co}(X)] = E[X]$. Using the strong law of large numbers, the set expectation definition has been specialized for different assumptions on the countability of the realizations of X.

Proposition 2 (finite realizations [18]): Suppose X is a random set so that $X = K_j \in \mathrm{Com}(\mathbb{R}^n)$ with probability p_j, for $j = 1 \dots, N$. The expectation of X is given by:

$$E[X] = \oplus_{j=1}^N p_j \, \mathrm{co}(K_j).$$

Proposition 3 (countable realizations [18]): Suppose X is a random set so that $\mathrm{E}(\|X\|_L) < \infty$, $X = K_j \in \mathrm{Com}(\mathbb{R}^n)$ with probability p_j, for $j = 1, 2 \dots$. It follows that if $\sum_j \|K_j\|_L < \infty$ then the expectation of X is given by:

$$E[X] = \lim_{N\to\infty} \oplus_{j=1}^N p_j \, \mathrm{co}(K_j) \triangleq \oplus_{j=1}^\infty p_j \, \mathrm{co}(K_j).$$

The expectation of a random set can be extended also to the case where X is a continuous map and is expressed in terms of a Stieltjes-Minkowski integral.

In light of the previous definitions we have the following assumption on the sets X_0 and W_k.

Assumption 1: (i) The sets $W_k, \ k \in \mathbb{N}$ are i.i.d. RCCSs, such that $E[\|W_k\|_L^2] = E[\|W\|_L^2] < \infty$ (which implies that $\mathrm{E}(\|W\|_L) < \infty$). Furthermore, the expectation $E[W_k] = \overline{W} \in \mathrm{ComConv}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.

(ii) The set X_0 is a RCCS such that $\mathrm{E}(\|X_0\|_L^2) < \infty$ (which implies that $E[\|X_0\|_L] < \infty$). Moreover, its expectation $E[X_0] = \overline{X}_0 \in \mathrm{ComConv}(\mathbb{R}^n)$.

(iii) The random process θ_k, the random sets X_0 and $W_k, \ k \in \mathbb{N}$ are independent.

IV. Reach Sets Expectations Dynamics

In [6] the linear set–dynamics driven by random convex compact sets in the case where the parameter matrix A is deterministic was studied. In addition, set-dynamics for the expectations of the reach sets was derived. In this section, we pursue a similar objective, with the difference that the matrix of the system parameters is a matrix valued random process. The dynamics of the expected reach sets associated to (2.5) is given by

$$E[X_{k+1}] = E[A_{\theta_k} X_k] \oplus E[W_k]. \tag{4.1}$$

The following remark paves the way for a more explicit characterization of (4.1).

Remark 2: Let $\{X_i\}_{i=1}^N$ be sets in $\mathrm{Com}(\mathbb{R}^n)$ and let $\{A_{ij}\}_{i,j=1}^N$ be matrices in $\mathbb{R}^{n\times n}$. The sets $Y_i = \oplus_{j=1}^N A_{ij} X_i$, $i = 1, \dots, N$ can be equivalently expressed as

$$Y_i = \{y_i : \mathbf{y}^T = (y_1^T, \dots, y_N^T), \mathbf{y} \in \mathbf{Y}\}, \tag{4.2}$$

where $\mathbf{Y} = \mathbf{A}\mathbf{X}$, with $\mathbf{A} = (A_{ij})$ and $\mathbf{X} = \otimes_{i=1}^N X_i$. Basically, Y_i is the set induced by i^{th} n–dimensional elements of $\mathbf{Y}$. The set $\mathbf{Y}$ can be equivalently expressed as $\mathbf{Y} = \oplus_{i=1}^N \mathbf{A}\mathbf{X}_i$, where $\mathbf{X}_i = \{\mathbf{x} : \mathbf{x}^T = (0, \dots, 0, x_i^T, 0, \dots, 0), x_i \in X_i\}$.

We recall that θ_k evolves according to a homogeneous Markov chain with probability matrix $P = (p_{ij})$ and with time dependent probability distribution $\pi_{k,i} = Pr(\theta_k = i)$. We first note that a direct computation of $E[A_{\theta_k} X_k]$ is not immediate since θ_k and X_k are not independent (X_k encodes information about the past trajectory of θ_k). To deal with this, we apply an approach used in the theory of Markovian jump linear systems (MJLSs) [19]. Namely, we define the sets $X_{k,i} = \mathbb{1}_{\{\theta_k = i\}} X_k$, where $\mathbb{1}_{\{\omega\}}$ is the indicator function of the event ω. It follows that with probability one $X_k = \oplus_{i=1}^q X_{k,i}$ and the random set dynamics becomes $X_{k+1} = A_{\theta_k}\left(\oplus_{j=1}^q X_{k,j}\right) \oplus W_k = \oplus_{j=1}^q \left(A_j X_{k,j}\right) \oplus W_k$. Multiplying both sides by $\mathbb{1}_{\{\theta_{k+1}=i\}}$ we get $X_{k+1,i} = \oplus_{j=1}^q \left(\mathbb{1}_{\{\theta_{k+1}=i\}} A_j X_{k,j}\right) \oplus \mathbb{1}_{\{\theta_{k+1}=i\}} W_k$. By noting that $E[\oplus_{j=1}^q \left(\mathbb{1}_{\{\theta_{k+1}=i\}} A_j X_{k,j}\right)] = \oplus_{j=1}^q \left(p_{ji} A_j E[X_{k,j}]\right)$ we obtain the following coupled linear set–dynamics

$$\bar{X}_{k+1,i} = \oplus_{j=1}^q p_{ji} A_j \bar{X}_{k,j} \oplus \bar{W}_{k,i}, \ i = 1, \dots, q, \tag{4.3}$$

where $E[X_{k,i}] = \bar{X}_{k,i}$, $\bar{W}_{k,i} = \pi_{k+1,i} E[W_k] = \pi_{k+1,i}\bar{W}$ and $E[X_k] = \bar{X}_k = \oplus_{i=1}^q \bar{X}_{k,i}$. Recalling Remark 2, a compact representation of (4.3) is given by

$$\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k \oplus \mathbf{W}_k, \tag{4.4}$$

where $\mathbf{X}_0 = \otimes_{i=1}^q \bar{X}_{0,i} = \otimes_{i=1}^q \pi_{0,i} E[X_0]$, $\mathbf{W}_k = \otimes_{i=1}^q \bar{W}_{k,i} = \otimes_{i=1}^q \pi_{k+1,i}\bar{W}$, and

$$\mathbf{A} = \begin{pmatrix} p_{11}A_1 & \dots & p_{q1}A_q \\ \vdots & \dots & \vdots \\ p_{1q}A_1 & \dots & p_{qq}A_q \end{pmatrix}.$$

Note that the disturbance set is time varying and therefore we cannot directly apply Proposition 1 do derive a convergence result. Based on Assumption 1–(i), $\|\bar{W}_{k,i}\|_L = \|\pi_{k+1,i}\bar{W}\|_L \leq \|\bar{W}\|_L \leq E[\|W_k\|_L]| < \infty$, for all k, i. But since $\{W_k\}_{k\geq 0}$ is a sequence of i.i.d. RCCs ($E[\|W_k\|_L]|$ are equal for all k), this means that there exists $\mu \geq 0$ such that $\|\mathbf{W}_k\|_L \leq \mu$, for all k. To study the convergence properties of (4.4), we use a similar approach as in [16], [17], and define the sets

$$\tilde{\mathbf{S}}_k = \oplus_{i=0}^k \mathbf{R}_i, \tag{4.5}$$

with $\mathbf{R}_i = \mathbf{A}^i \tilde{\mathbf{W}}$, where $\tilde{\boldsymbol{W}} = \otimes_{i=1}^q \bar{W}$. We have the following result.

Proposition 4: Let $\rho(\mathbf{A}) < 1$ and let Assumption 1 hold. Then the sequence $\{\tilde{\mathbf{S}}_k\}_{k\geq 0}$ converges to a set $\mathbf{S}_\infty$ denoted by

$$\tilde{\mathbf{S}}_\infty = \oplus_{k=0}^\infty \mathbf{A}^k \tilde{\boldsymbol{W}}, \tag{4.6}$$

Proof: The proof relies on showing that $\{\tilde{\mathbf{S}}_k\}_{k\geq 0}$ is Cauchy. This follows by observing that $H(L,\tilde{\mathbf{S}}_{k+j},\tilde{\mathbf{S}}_k) \leq \sum_{i=k+1}^{k+j} \boldsymbol{\lambda}^i \|\tilde{\boldsymbol{W}}\|_L \leq \frac{\boldsymbol{\lambda}^{k+1}}{1-\boldsymbol{\lambda}}\|\tilde{\boldsymbol{W}}\|_L$, where $\boldsymbol{\lambda}$ is as in Remark 1. It follows that $\{H(L,\tilde{\mathbf{S}}_{k+j},\tilde{\mathbf{S}}_k)\}_{k\geq 0}$ converges to zero. ■

Consider now the set dynamics

$$\tilde{\mathbf{X}}_{k+1} = \mathbf{A}\tilde{\mathbf{X}}_k \oplus \tilde{\boldsymbol{W}},\ \tilde{\mathbf{X}}_0 = \mathbf{X}_0. \tag{4.7}$$

Corollary 1: Let $\rho(\mathbf{A}) < 1$ and let Assumption 1 hold. Then the sequence $\{\tilde{\mathbf{X}}_k\}_{k\geq 0}$ converges to the set $\tilde{\mathbf{S}}_\infty$.

Proof: Follows from Proposition 4 together with the observations that $\tilde{\mathbf{X}}_k = \mathbf{A}^k\tilde{\mathbf{X}}_0 \oplus \tilde{\mathbf{S}}_{k-1}$ and that $H(L,\tilde{\mathbf{X}}_k,\tilde{\mathbf{S}}_{k-1}) \leq \boldsymbol{\lambda}^k\tilde{\mathbf{X}}_0$, which in turn implies that $\{H(L,\tilde{\mathbf{X}}_k,\tilde{\mathbf{S}}_{k-1})\}_{k\geq 0}$ converges to zero. ■

Corollary 2: Let $\rho(\mathbf{A}) < 1$ and let Assumption 1 hold. Then $E[X_k] \subseteq \tilde{X}_k$ for all k, where $\tilde{X}_k = \oplus_{i=1}^q \tilde{X}_{k,i}$, with $\tilde{X}_{k,i} = \{x_i \in \mathbb{R}^n : \mathbf{x}^T = [x_1^T,\dots,x_q^T], \mathbf{x} \in \tilde{\mathbf{X}}_k\}$. In addition $\limsup_{k\to\infty} \|E[X_k]\|_L \leq \|\tilde{S}_\infty\|_L$, where $\tilde{S}_\infty = \oplus_{i=1}^q \tilde{S}_{\infty,i}$ with $\tilde{S}_{\infty,i} = \{x_i \in \mathbb{R}^n : \mathbf{x}^T = [x_1^T,\dots,x_q^T], \mathbf{x} \in \tilde{\mathbf{S}}_\infty\}$.

Proof: First note that by construction $\mathbf{W}_k \subseteq \tilde{\boldsymbol{W}}$ for all k. Since $\mathbf{X}_0 = \tilde{\mathbf{X}}_0$, by an induction argument we get that $\mathbf{X}_k \subseteq \tilde{\mathbf{X}}_k$ which further implies that $\bar{X}_{k,i} \subseteq \tilde{X}_{k,i}$, where we recall that $\bar{X}_{k,i} = \{x_i : \mathbf{x}^T = (x_1^T,\dots,x_q^T), \mathbf{x} \in \mathbf{X}_k\}$. Consequently $E[X_k] = \oplus_{i=1}^q \bar{X}_{k,i} \subseteq \oplus_{i=1}^q \tilde{X}_{k,i} = \tilde{X}_k$ for all k. The last claim of this corollary follows from Corollary 1. ■

Remark 3: The previous result established that if the matrix $\mathbf{A}$ is strictly stable, then the limiting behavior of expectations of the reach sets are such that they are subsets of $\tilde{S}_\infty$. It should not come as a surprise that, as shown in [19], matrix $\mathbf{A}$ is the same matrix that appears in the convergence analysis of $E[x_k]$, where x_k evolves according to a MJLS driven by disturbance $x_{k+1} = A_{\theta_k}x_k + w_k$, with similar independency assumptions on θ_k, w_k and x_0 as in the set–dynamics setup described in Section II.

A. Special cases

In the previous section we established set–dynamics for evaluating the expectations of the reach sets. For such evaluations to be tractable we need the initial expectations for X_0 and W_0. In the case where they admit finite realizations, Proposition 3 gives us the way to compute the expectations. If however the set–valued random variables are continuous maps, obtaining the initial expectations becomes more challenging. One approach would be to approximate the Stieltjes-Minkowski integral, by partitioning the sample space and computing $E[X] \approx \oplus_i p_i \text{co}(X_i(\omega_i))$, where p_i is the measure of ω_i. Naturally, the quality of the approximation depends on the granularity of the partition. A special case of a RCCS with continuous map is the Gaussian RCCS defined as $X = \hat{X} \oplus x$, where $\hat{X}$ is a deterministic nonempty convex compact subset of $\mathbb{R}^n$ and x is a Gaussian random vector in $\mathbb{R}^n$. If we assume that X_0 and W_k are Gaussian RCCSs given by $X_0 = \hat{X}_0 \oplus x_0$ and $W_k = \hat{W} \oplus w_k$, with $x_0 \sim \mathcal{N}(\mu_0, Q_0)$ and $w_k \sim \mathcal{N}(0,R)$, then the random set–dynamics corresponding to X_k can be decomposed in two parts: $X_k = \hat{X}_k \oplus x_k$, where $\hat{X}_{k+1} = A_{\theta_k}\hat{X}_k \oplus \hat{W}$ and $x_{k+1} = A_{\theta_k}x_k + w_k$. Note that unlike the case where A_{θ_k} is deterministic [6], $\hat{X}_k$ are in fact RCCSs. However, since the initial set and disturbance set are deterministic, (4.3) can be iterated more efficiently.

In Section IV we derived results for convergence in expectation for the sequence of sets $\{X_k\}_{k\geq 0}$. Showing convergence with probability one is more challenging due to the time varying nature of the matrix of parameters. However, under stronger assumptions on A_{θ_k}, W_k and X_0, we can show that $\{X_k\}_{k\geq 0}$ remain bounded with probability one.

Proposition 5: Let Assumption 1 hold and assume there exists $0 \leq \lambda < 1$ such that the linear matrix inequalities $A_i^TPA_i - \lambda^2P \preceq 0$, $i = 1,\dots,q$ are feasible, where P is a symmetric positive definite matrix. Let $L \triangleq \{x : x^TPx \leq 1\}$ and assume there exist positive scalars $\alpha_w \geq 0$ and $\alpha_x \geq 0$ such that $\|W_k\|_L \leq \alpha_w$ and $\|X_0\|_L \leq \alpha_x$ with probability one. Then there exists a sequence of sets $\{\tilde{X}_k\}_{k\geq 0}$ such that with probability one $X_k \subseteq \tilde{X}_k$ for all k, where

$$\tilde{X}_{k+1} = \lambda\tilde{X}_k \oplus \tilde{W}, \tag{4.8}$$

with $\tilde{X}_0 = \alpha_xL$ and $\tilde{W} = \alpha_wL$ deterministic compact sets, and L the proper, compact set used in the Hausdorff distance. In addition $\{\tilde{X}_k\}$ converges to a set $\tilde{X}_\infty$ denoted by

$$\tilde{X}_\infty = \oplus_{i=0}^\infty \lambda^i\alpha_wL,$$

Proof: Unless stated otherwise, all the following statements hold with probability one. First note that, as discussed in [17], the feasibility of the linear matrix inequalities means that $A_iL \subseteq \lambda L$ for all i. Recalling that X_k can be explicitly represented as $X_k := \Phi(k,0)X_0 \oplus \bigoplus_{i=0}^{k-1}\Phi(k,i+1)W_i$, we define the sets $S_k = \bigoplus_{i=0}^{k-1}\Phi(k,i+1)W_i$. According to the definition of the Hausdorff distance, the assumption $\|W_k\|_L \leq \alpha_w$ implies that $W_k \subseteq \alpha_wL$. This means that each term in the sum of S_k satisfies $\Phi(k-1,i+1)W_i \subseteq \alpha_w\Phi(k,i+1)L \subseteq \lambda^{k-i-1}\alpha_wL$. Introducing $\tilde{S}_k = \bigoplus_{i=0}^{k-1}\lambda^{k-i-1}\alpha_wL$, it holds that $S_k \subseteq \tilde{S}_k$ for all k. But since $\Phi(k,0)X_0 \subseteq \lambda^k\alpha_xL$ and $X_k = \Phi(k,0)X_0 \oplus S_k$, we have that $X_k \subseteq \tilde{X}_k$ where $\tilde{X}_k = \lambda^k\alpha_xL \oplus \tilde{S}_k$. Similar to Proposition 4, we can show that $\{\tilde{S}_k\}$ is Cauchy, and therefore by completeness $\{\tilde{S}_k\}$ converges to a set $\tilde{S}_\infty \triangleq \oplus_{i=0}^\infty\lambda^i\alpha_wL$. Observing that $H(L,\tilde{X}_k,\tilde{S}_k) \leq \lambda^k\alpha_x$ it follows that the sequence $\{H(L,\tilde{X}_k,\tilde{S}_k)\}_{k\geq 0}$ converges to zero and consequently $\{\tilde{X}_k\}_{k\geq 0}$ converges to the set $\tilde{S}_\infty$. ■

The previous result shows that under certain conditions we can guarantee that the sets X_k do not grow unbounded for all possible realizations. In addition, it provides a superset for the limiting behavior of X_k which implies that $\limsup_{k\to\infty}\|X_k\|_L \leq \|\tilde{X}_\infty\|_L$ with probability one. The search for a λ as described in Proposition 5 can be done by solving a sequence of semi–definite programs, as part of a linear search algorithm.

V. Illustrative Example

In this section we study the dynamics of the expectations of the reach sets induced by the position of a quadrotor. We use a simplification of the model studied in [20], to which a small–angle approximation was applied to obtain the following linear system

$$\begin{aligned} \ddot{\phi} &= a\dot{\theta} + b\alpha_2 U2 \\ \ddot{\theta} &= -a\dot{\phi} + b\alpha_3 U3 \\ \ddot{x} &= g\theta \\ \ddot{y} &= -g\phi. \end{aligned}$$

The meaning of the symbols shown in the model are: (ϕ, θ) are the pitch and yaw angles, (x, y) are the position coordinates, $a = 8.667\text{e}{-04}$ and $b = 30.667$ are nominal parameters that depend on the moments of inertia and residual propeller angular speed, $g = 9.81m/s^2$ is the gravitational acceleration, and U_2 and U_3 are thrust dependent inputs.

The state–space dimension of the original model is 12. We assume that that the quadrotor remains at a constant altitude (the velocity on the z direction is zero) and no roll takes place, and hence the lower dimensional model shown above. For consistency with the original description of the model we kept the notations used in [20]. Note however, that the symbols have no connection with the symbols used in the previous sections.

Scalars α_2 and α_3 are drag coefficients that affect the thrusts generated by motors 2 and 4, and 1 and 3, respectively. Their nominal values are $\alpha_2 = \alpha_3 = 1$. We will randomly vary their values and study their effect on the quadrotor position. The evolution of α_i's is determined by the evolution of a random process $\theta_k \in \{1, 2, 3, 4\}$ that maps into the set of pairs $\{(1, 1), (0.4, 1), (1, 0.3), (0.25, 0.25)\}$ from where (α_2, α_3) take values.

Using a sampling period of $h = 0.1$s, we obtain a 8–dimensional discrete state–space representation for the nominal case $\mathbf{x}_{k+1} = A\mathbf{x}_k + B_1\mathbf{u}_k + \mathbf{w}_k$, where we introduced also the state disturbance $\mathbf{w}_k$. For the nominal case, we construct a stabilizing controller that brings the quadrotor (when no disturbance is present) to the origin of the (x, y) plane, resulting in the close loop system $\mathbf{x}_{k+1} = \bar{A}_1\mathbf{x}_k + \mathbf{w}_k$, where $\bar{A}_1 = A + B_1K$. We now consider the random evolution of the parameters, and define the matrix $\bar{A}_{\theta_k} = A + B_{\theta_k}K$. Consequently we have the stochastic dynamics $\mathbf{x}_{k+1} = \bar{A}_{\theta_k}\mathbf{x}_k + \mathbf{w}_k$. According to the set where parameters (α_2, α_3) take values, modes 3 and 4 are unstable (that is, the matrix $A + B_iK$, $i = 3, 4$ is unstable). We assume that the initial state belong to a Gaussian RCCS $X_0 = \hat{X}_0 \oplus \hat{x}_0$, where $\hat{x}_0 \sim \mathcal{N}(\mu, Q)$. Similarly, for all $k \in \mathbb{N}$, $\mathbf{w}_k \in W_k$, where $W_k = \hat{W}$. The compact set X_0 is the lifting of the rectangle defined by $(\bar{x}_0 \pm \delta, \bar{y}_0 \pm \delta)$, with $0 < \delta \leq \delta_{xy}$ to the 8–dimensional Euclidean space. The rectangle reflects the uncertainty in the initial position: $\bar{x}_0 - \delta_{xy} \leq x_0 \leq \bar{x}_0 + \delta_{xy}$, $\bar{y}_0 - \delta_{xy} \leq y_0 \leq \bar{y}_0 + \delta_{xy}$. All other components in the initial state are affected by $\hat{x}_0$ only. Similarity, W_0 is the lifting of the rectangle $(\bar{w}(7) \pm \delta, \bar{w}(8) \pm \delta)$, with $0 < \delta \leq \delta_w$ to the 8–dimensional Euclidean space, where indices 7 and 8 correspond to the accelerations on the x and y–directions, respectively. This way we can model additional forces acting on the quadrotor such as wind forces. The linear set-dynamics (4.3) corresponding to the expectations of the reach sets were simulated using the MPT Matlab toolbox [21]. The compact sets representing the initial state and the disturbance were represented through a set of vertices and rays. It is well known that the number of vertices representing a set obtained by repeated Minkowski sums increases exponentially. Consequently, after each time iteration we limit the number of vertices describing the expectations of the reach sets.

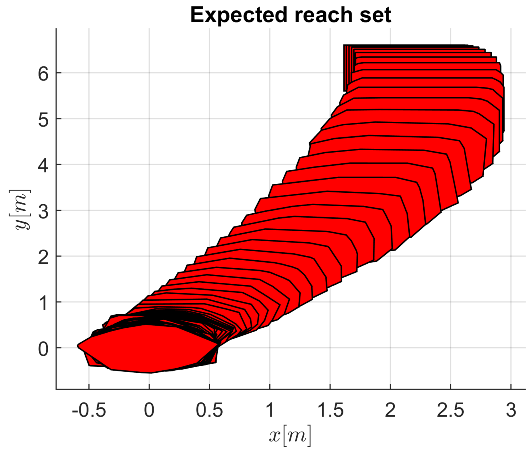

Fig. 1: i.d.d. case: $p_1 = 0.9$, $p_2 = 1/3$, $p_3 = 1/3$, $p_4 = 1/3$, $\bar{x}_0 = 2$, $\bar{y}_0 = 6$, $\delta_{xy} = 0.5$, $\delta_w = 0.05$, $\mu = 0.1$, $\rho(\mathbf{A}) = 0.9145$.

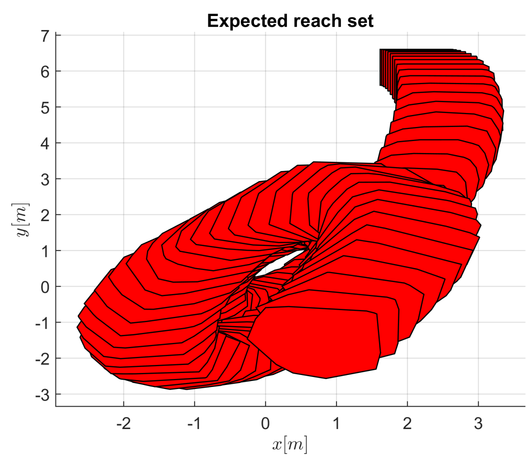

Fig. 2: i.d.d. case: $p_1 = 0.1/3$, $p_2 = 0.1/3$, $p_3 = 0.1/3$, $p_4 = 0.9$, $\bar{x}_0 = 2$, $\bar{y}_0 = 6$, $\delta_{xy} = 0.5$, $\delta_w = 0.05$, $\mu = 0.1$, $\rho(\mathbf{A}) = 1.082$.

The set dynamics numerical simulations were executed for 100 iterations. We depict the projections of the expectations of the reach sets on the dimensions corresponding to the position. We consider first a particular case for the Markovian process θ_k, namely we assume it to be an i.i.d. random process. Figures 1–2 shows simulation results for the i.i.d. case under different choices of probability distribution. As expected, as the probabilities associated to unstable modes increase, the system becomes unstable, in the mean sense ($E[X_k]$ is not bounded); behavior that can be observed in the

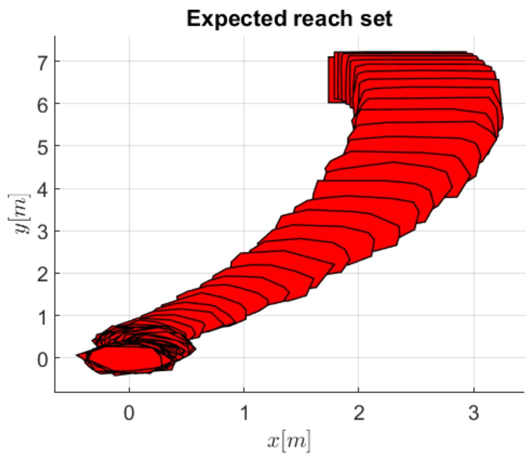

Fig. 3: Markovian case: P=[0.7,0.1,0.1,0.1; 0.2,0.5,0.2,0.1;0.1,0.1,0.6,0.2;0.2,0.2,0.2,0.4], π_0=[0.25,0.25,0.25,0.25], $\bar{x}_0 = 2$, $\bar{y}_0 = 6$, $\delta_{xy} = 0.5$, $\delta_w = 0.05$, $\mu = 0.1$, $\rho(\mathbf{A}) = 0.9595$.

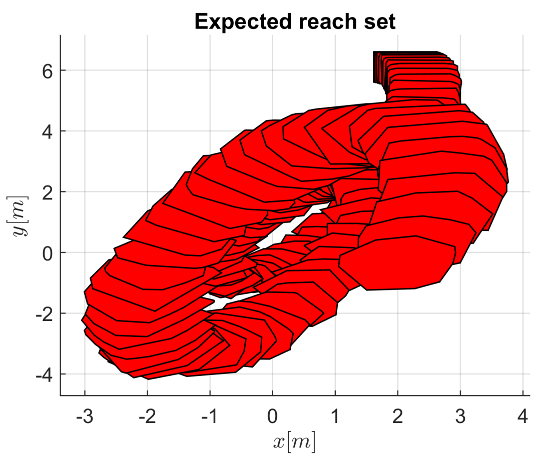

Fig. 4: Markovian case: P=[0.7,0.1,0.1,0.1; 0.1,0.7,0.1,0.1; 0.1/3,0.1/3,0.1/3,0.9;0.01/3,0.01/3,0.01/3,0.99], π_0=[0.25,0.25,0.25,0.25], $\bar{x}_0 = 2$, $\bar{y}_0 = 6$, $\delta_{xy} = 0.5$, $\delta_w = 0.05$, $\mu = 0.1$, $\rho(\mathbf{A}) = 1.0132$.

evolution of the reach set expectations. Figures 3-4 depict the evolution of the expected reach sets in the Markovian case. The first figure shows a stable behavior. By appropriately selecting the probability transition matrix, we can generate an unstable behavior as shown in Figure 4. Note that the lack of smooth transitions between the shapes of the reach sets is due to reducing the number of vertices representing the sets in order to keep the Minkowski additions scalable with time.

VI. Concluding Remarks

We have studied linear set–dynamics driven by RCCS, with random matrix of parameters. The random process governing the evolution of the matrix of parameters was assumed to be a homogeneous Markov chain. We derived the set–dynamics for the expectations of the reach sets. In addition, we gave sufficient conditions for convergence in terms of the spectral radius of a particular matrix. As an illustrative example, we used the results derived in this paper to quantify the expectations of the reach sets for the position of a quadrotor

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