2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Algebraic-Graphical Approach for Signed Dynamical Networks

Guodong Shi, Claudio Altafini, and John S. Baras

***Abstract*— A signed network is a network with each link associated with a positive or negative sign. Models for nodes interacting over such signed networks, where two types of interactions are defined along the positive and negative links, respectively, arise from various biological, social, political, and economical systems. Starting from standard consensus dynamics, there are two basic types of negative interactions along negative links, namely state flipping or relative-state flipping. In this paper, we provide an algebraic-graphical method serving as a systematic tool of studying these dynamics over signed networks. Utilizing generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions, we show this method is useful to establish a series of basic convergence results for dynamics over signed networks.**

I. Introduction and Preliminaries

In the past decades, the study of network dynamics has attracted various research attentions from a variety of scientific disciplines [1]. Particularly, with its root traced back to 1960s on products of stochastic matrices [2], to 1970s on DeGroot social interactions [3], and to 1980s on distributed optimization [4], consensus algorithms serve as a primary model for network dynamics as well as being a foundation for some prominent engineering applications of large-scale complex networks [5]–[8]. It has become a common understanding that cooperative node dynamics will lead to certain collective network behaviors.

On the other hand, in various biological, social, political, and economical systems, there are often two different, activating or inhibitive, trustful or mistrustful, cooperative or antagonistic, types of node interactions. Using a positive or negative sign to denote the type of a link, the structure of these systems can be modeled as signed graphs. After specifying node dynamical relations among each positive or negative links, the evolution of node states defines signed network dynamics. Consensus algorithms with positive and negative links have been recently investigated [9]–[19], where there exist two basic types of negative interactions along negative links, namely the state flipping negative dynamics introduced in [10] and the relative-state flipping negative dynamics introduced in [11].

G. Shi is with the Research School of Engineering, The Australian National University, ACT 0200, Canberra, Australia. Email: guodong.shi@anu.edu.au

C. Altafini is with the Division of Automatic Control, Department of Electrical Engineering, Linköping University, SE-58183 Linköping, Sweden. E-mail: claudio.altafini@liu.se

J. S. Baras is with the Institute for Systems Research and the Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742, USA. E-mail: baras@isr.umd.edu

A. Signed Graphs

Consider a network with n nodes indexed in the set $\mathrm{V} = \{1,\dots,n\}$. The structure of the network is represented as an undirected graph $\mathrm{G} = (\mathrm{V},\mathrm{E})$, where an edge (link) $\{i,j\} \in \mathrm{E}$ is an unordered pair of two distinct nodes in the set V. Each edge in E is associated with a sign, positive or negative, defining G as a signed graph. The positive and negative edges are collected in the sets E^+ and E^-, respectively. Then $\mathrm{G}^+ = (\mathrm{V},\mathrm{E}^+)$ and $\mathrm{G}^- = (\mathrm{V},\mathrm{E}^-)$ are respectively termed positive and negative subgraphs. Throughout the paper and without further specific mention we assume that G is connected and G^- contains at least one edge.

For a node $i \in \mathrm{V}$, its positive neighbors are the nodes that share a positive link with i, forming the set $\mathrm{N}_i^+ := \big\{j : \{i,j\} \in \mathrm{E}^+\big\}$. Similarly the negative neighbor set of node i is denoted as $\mathrm{N}_i^- := \big\{j : \{i,j\} \in \mathrm{E}^-\big\}$. The set $\mathrm{N}_i = \mathrm{N}_i^+ \bigcup \mathrm{N}_i^-$ then contains all nodes that interact with node i in the graph G. We use $\deg_i = \big|\mathrm{N}_i\big|$ to denote the degree of node i, i.e., the number of neighbors of node i. Similarly, $\deg_i^+ = \big|\mathrm{N}_i^+\big|$ and $\deg_i^- = \big|\mathrm{N}_i^-\big|$ represent the positive and negative degree of node i, respectively.

B. Signed Laplacian

The Laplacian of the positive graph G^+ is defined as [20]

$$L_{\mathrm{G}^+} := D_{\mathrm{G}^+} - A_{\mathrm{G}^+}$$

where A_{G^+} is the adjacency matrix of the graph G^+ with $[A_{\mathrm{G}^+}]_{ij} = 1$ if $\{i,j\} \in \mathrm{E}^+$ and $[A_{\mathrm{G}^+}]_{ij} = 0$ otherwise, and $D_{\mathrm{G}^+} = \mathrm{diag}(\deg_1^+,\dots,\deg_n^+)$ is the positive degree matrix.

Next, we denote $D_{\mathrm{G}^-} = \mathrm{diag}(\deg_1^-,\dots,\deg_n^-)$ as the negative degree matrix. Let A_{G^-} be the *signed* adjacency matrix of the graph G^-, where $[A_{\mathrm{G}^-}]_{ij} = -1$ if $\{i,j\} \in \mathrm{E}^-$ and $[A_{\mathrm{G}^+}]_{ij} = 0$ otherwise. Then the matrix

$$L_{\mathrm{G}^-} := D_{\mathrm{G}^-} - A_{\mathrm{G}^-}$$

is defined as the signed Laplacian of the negative graph G^-. The signed Laplacian of the signed graph G is then given by $L_{\mathrm{G}^+} + L_{\mathrm{G}^-}$.

Particularly, we can also neglect the sign of edges in G^- and let $A^*_{\mathrm{G}^-}$ be the adjacency matrix of G^- with the signs being neglected, i.e., $[A^*_{\mathrm{G}^-}]_{ij} = 1$ if $\{i,j\} \in \mathrm{E}^-$ and $[A_{\mathrm{G}^+}]_{ij} = 0$. Then $L^*_{\mathrm{G}^-} := D_{\mathrm{G}^-} - A^*_{\mathrm{G}^-}$, is defined as the Laplacian of G^- neglecting the sign of the links.

C. Structural Balance Theory

Introduced in the 1940s [21] and primarily motivated by social-interpersonal and economic networks, a fundamental notion in the study of signed graphs is the so-called structural balance. We recall the following definition (see [1] for a detailed introduction).

Definition 1: A signed graph G is *structurally balanced* if there is a partition of the node set into $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$, where V_1 and V_2 are nonempty and mutually disjoint, such that any edge between the two node subsets V_1 and V_2 is negative, and any edge within each V_i is positive.

The notion of structural balance can be weakened in the following definition [22].

Definition 2: A signed graph G is *weakly structurally balanced* if there is a partition V into $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2 \cdots \bigcup \mathrm{V}_m$ with $m \geq 2$, where $\mathrm{V}_1, \dots, \mathrm{V}_m$ are nonempty and mutually disjoint, such that any edge between different V_i's is negative, and any edge within each V_i is positive.

D. Positive/Negative Interactions

Time is slotted at $t = 0, 1, \dots$. Each node i holds a state $x_i(t) \in \Re$ at time t, and interacts with its neighbors at each time to revise its state. The interactions rule is specified by the sign of the links. Let $\alpha, \beta \geq 0$. We first focus on a particular link $\{i, j\} \in \mathrm{E}$ and specify for the moment the dynamics along this link isolating all other interactions.

- If the sign of $\{i, j\}$ is positive, then each node $s \in \{i, j\}$ updates its value by
$$x_s(t+1) = x_s(t) + \alpha\big(x_{-s}(t) - x_s(t)\big) = (1-\alpha)x_s(t) + \alpha x_{-s}(t), \quad (1)$$
where $-s \in \{i, j\} \setminus \{s\}$.
- If the sign of $\{i, j\}$ is negative, then each node $s \in \{i, j\}$ updates its value by either
  - State Flipping Rule:
$$x_s(t+1) = x_s(t) + \beta\big(-x_{-s}(t) - x_s(t)\big) = (1-\beta)x_s(t) - \beta x_{-s}(t); \quad (2)$$
  or
  - Relative-state Flipping Rule:
$$x_s(t+1) = x_s(t) - \beta\big(x_{-s}(t) - x_s(t)\big) = (1+\beta)x_s(t) - \beta x_{-s}(t). \quad (3)$$

The positive interaction is consistent with DeGroot's rule of social interactions, which indicates that the opinions of trustful social members are attractive to each other [3]. The state-flipping rule, introduced in [10], states that a node will be attracted by the opposite of its neighbor's state if they share a negative link. The relative-state-flipping rule, introduced in [11], on the other hand states that two nodes sharing a negative link take repulsive interactions rather than attraction. The two parameters α and β describe the strength of positive and negative links, respectively.

E. Contributions and Paper Organization

In this paper, we establish an algebraic-graphical method serving as a system-theoretic tool for studying consensus dynamics over signed networks. Combining generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions, we will show that this approach provides simple proofs to a series of basic convergence results for dynamics over signed networks, for both deterministic and random node interactions. We note that some of the presented results are essentially no longer new to the literature, however, new insights can be gained by putting the results and analysis in a uniform framework. For example, we prove that signed Laplacian leads to eventually positive matrices in both state-flipping and relative-state-flipping definitions, whenever convergence is guaranteed.

The remainder of the paper is organized as follows. Section II presents a series of basic results for dynamics over deterministic networks. Section III extends the discussions to random networks with convergence results established using similar algebraic-graphical analysis but with additional probabilistic ingredient. Finally Section IV concludes the paper with a few remarks and future work.

II. Deterministic Networks

In this section, we investigate the evolution of the node states with deterministic interactions. The pairwise interactions among the signed links are collected over a deterministic network. We are interested in characterizing the asymptotic limits of the node states.

A. Fundamental Convergence Results

1) State-Flipping Negative Dynamics: With the state flipping rule (2) along with the negative links, the update of $x_i(t)$ reads as

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha \sum_{j \in \mathrm{N}_i^+} \Big(x_j(t) - x_i(t)\Big) \\ &\quad - \beta \sum_{j \in \mathrm{N}_i^-} \Big(x_j(t) + x_i(t)\Big) \\ &= \Big(1 - \alpha \mathrm{deg}_i^+ - \beta \mathrm{deg}_i^-\Big) x_i(t) + \alpha \sum_{j \in \mathrm{N}_i^+} x_j(t) \\ &\quad - \beta \sum_{j \in \mathrm{N}_i^-} x_j(t). \end{aligned} \quad (4)$$

Denote $\mathbf{x}(t) = (x_1(t) \dots x_n(t))^{\mathrm{T}}$. We can now rewrite (4) into the following compact form:

$$\mathbf{x}(t+1) = W_{\mathrm{G}} \mathbf{x}(t) = \big(I - \alpha L_{\mathrm{G}^+} - \beta L_{\mathrm{G}^-}\big)\mathbf{x}(t) \quad (5)$$

where L_{G^+} and L_{G^-} are the signed Laplacian of the positive and negative graphs G^+ and G^-, respectively.

Recall that a real matrix (or vector) is called positive (non-negative) if all its entries are positive (non-negative); a stochastic matrix is a nonnegative matrix with row sum equal to one [23]. A key property of the matrix W_{G} lies in

$$\sum_{j=1}^{n} \big|[W_{\mathrm{G}}]_{ij}\big| = 1, \ i \in \mathrm{V} \quad (6)$$

which indicates that W_{G} will become a stochastic matrix if all its entries are taken into their absolute values. The following result holds.

Theorem 1: Assume that $0 < \alpha + \beta < 1/\max_{i\in \mathrm{V}} \deg_i$. Then under the state flipping rule (2), the following statements hold for any initial value $\mathbf{x}(0)$.

(i) If G is structurally balanced subject to partition $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$, then $\lim_{t\to\infty} x_i(t) = \big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in \mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_1$, and $\lim_{t\to\infty} x_i(t) = -\big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in \mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_2$.

(ii) If G is not structurally balanced, then $\lim_{t\to\infty} x_i(t) = 0,\ i \in \mathrm{V}$.

Proof. (i) Let G be structurally balanced with partition $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$. Consider a gauge transformation given by

$$z_i(t) = x_i(t), i \in \mathrm{V}_1; \quad z_i(t) = -x_i(t), i \in \mathrm{V}_2.$$

The evolution of the $z_i(t)$ follows standard consensus algorithm and the result holds from Theorem 2 in [7].

(ii) Applying Geršhgorin's Circle Theorem (see, e.g., Theorem 6.1.1 in [23]), it is easy to see that $-1 < \lambda_i(W_{\mathrm{G}}) \leq 1$ for all $\lambda_i \in \sigma(W_{\mathrm{G}})$ when $0 < \alpha + \beta < 1/\deg_i$ for all i. This immediately implies that for any initial value $\mathbf{x}(0)$, there exists $\mathbf{y}(\mathbf{x}(0)) = (y_1(\mathbf{x}(0)), \dots, y_n(\mathbf{x}(0)))^{\mathrm{T}}$ satisfying $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ such that

$$\lim_{t\to\infty} x_i(t) = y_i.$$

Claim. $|y_1| = \dots = |y_n|$ for any $\mathbf{x}(0)$.

Suppose there are two distinct nodes i and j with $|y_i| \neq |y_j|$. The fact that $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ gives

$$|y_i| \leq \sum_{j=1}^{n} \big|[W_{\mathrm{G}}]_{ij}\big| \cdot |y_j|,\ i \in \mathrm{V}. \tag{7}$$

This is impossible if G is connected noting the fact that $\sum_{j=1}^{n} \big|[W_{\mathrm{G}}]_{ij}\big| = 1,\ i \in \mathrm{V}$. This proves the above claim.

Now let $y_* = |y_1| = \dots = |y_n| \neq 0$ for some $\mathbf{x}(0)$. There must be a set V_* (which, of course, may be an empty set) with

$$y_i = y_*, i \in \mathrm{V}_*;\ y_i = -y_*, i \in \mathrm{V} \setminus \mathrm{V}_*.$$

It is straightforward to verify that in order for $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ to hold, all links (if any) in either V_* or $\mathrm{V} \setminus \mathrm{V}_*$ must be positive, and the links (if any) between V_* and $\mathrm{V} \setminus \mathrm{V}_*$ must be negative. This is to say, G must be structurally balanced since by our standing assumption G^- is nonempty.

We have now completed the proof. □

We remark that the condition $0 < \alpha+\beta < 1/\max_{i\in \mathrm{V}} \deg_i$ in Theorem 1 can be certainly relaxed, e.g., a straightforward one would be $0 < \alpha\deg_i^+ + \beta\deg_i^- < 1$ for all i. Further relaxations can be obtained making use of the structure of L_{G^+} and L_{G^-}, and the fact that the spectrum of W_{G} will be restricted within the unit cycle for sufficiently small α and β.

The essential message of Theorem 1 is that the structural balance of G determines whether one is within the spectrum of W_{G}. In fact, there holds

$$\big\|\mathbf{x}(t+1)\big\|^2 \leq \lambda_{\max}\big(W_{\mathrm{G}}^2\big)\big\|\mathbf{x}(t)\big\|^2 \leq \big\|\mathbf{x}(t)\big\|^2 \tag{8}$$

with sufficiently small α and β guaranteeing $\lambda_{\max}\big(W_{\mathrm{G}}^2\big) \leq 1$. Therefore, the algorithm (4) defines an overall contraction mapping, consistent with the standard consensus algorithms without negative links.

2) Relative-State-Flipping Negative Dynamics: Now consider the state flipping rule (3) for negative links. The update of $x_i(t)$ reads as

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha \sum_{j\in \mathrm{N}_i^+} \Big(x_j(t) - x_i(t)\Big) \\ &\quad - \beta \sum_{j\in \mathrm{N}_i^-} \Big(x_j(t) - x_i(t)\Big) \\ &= \Big(1 - \alpha\deg_i^+ + \beta\deg_i^-\Big)x_i(t) \\ &\quad + \alpha \sum_{j\in \mathrm{N}_i^+} x_j(t) - \beta \sum_{j\in \mathrm{N}_i^-} x_j(t). \end{aligned} \tag{9}$$

Recall that the Laplacian of the matrix G^- neglecting the sign of the links is defined as $L^*_{\mathrm{G}^-}$. The algorithm (9) is written into

$$\mathbf{x}(t+1) = M_{\mathrm{G}}\mathbf{x}(t) = \big(I - \alpha L_{\mathrm{G}^+} + \beta L^*_{\mathrm{G}^-}\big)\mathbf{x}(t). \tag{10}$$

From (10), $M_{\mathrm{G}}\mathbf{1} = \mathbf{1}$ always holds. We present the following result.

Theorem 2: Suppose G^+ is connected. Then for any $0 < \alpha < 1/\max_{i\in \mathrm{V}} \deg_i^+$, there exists a critical value $\beta_* > 0$ for β such that

(i) If $\beta < \beta_*$, then an average consensus is reached along (9), i.e.,

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j=1}^{n} x_i(0)}{n}$$

for all initial value $\mathbf{x}(0)$;

(ii) If $\beta > \beta_*$, then $\lim_{t\to\infty} \big\|\mathbf{x}(t)\big\| = \infty$ for almost all initial values w.r.t. Lebesgue measure.

Proof. Since $0 < \alpha < 1/\max_{i\in \mathrm{V}} \deg_i^+$, all eigenvalues of $I - \alpha L_{\mathrm{G}^+}$ are nonnegative again invoking the Geršhgorin's Circle Theorem. This in turn leads to the following fact noticing that $L^*_{\mathrm{G}^-}$ is positive semi-definite: $I - \alpha L_{\mathrm{G}^+} + \beta L^*_{\mathrm{G}^-}$ is positive semi-definite. Define $J = \mathbf{1}\mathbf{1}^{\mathrm{T}}/n$ and consider

$$f(\beta) := \lambda_{\max}\Big(I - \alpha L_{\mathrm{G}^+} + \beta L^*_{\mathrm{G}^-} - \mathbf{1}\mathbf{1}^{\mathrm{T}}/n\Big).$$

The Courant-Fischer Theorem (see Theorem 4.2.11 in [23]) implies $f(\cdot)$ is a continuous and non-decreasing function over $[0, \infty)$. Now that G^+ is connected, we have $f(0) < 1$ since the second smallest eigenvalue of L_{G^+} is positive. Apparently $f(\infty) > 1$. Therefore, there exists a critical value $\beta_* > 0$ satisfying $f(\beta_*) = 1$ such that

- There holds $f(\beta) < 1$ if $\beta < \beta_*$. In this case, along (10) $\mathbf{x}(t)$ converges to the eigenspace corresponding to the eigenvalue one of M_{G}, which leads to the average consensus statement in (i).
- There holds $f(\beta) > 1$ if $\beta > \beta_*$. In this case, along (10) $\mathbf{x}(t)$ diverges as long as the initial value $\mathbf{x}(0)$ has a nonzero projection onto the eigenspace corresponding to $\lambda_{\max}\big(M_{\mathrm{G}}\big)$ of M_{G}. This leads to the almost everywhere divergence statement in (ii).

The proof is now complete. □

The condition that G^+ is a connected graph is crucial for Theorem 2: If G^+ is not connected, it is easy to see that one single negative link and an arbitrarily small $\beta > 0$ will drive the network state to diverge for almost all initial values.

B. Directed Graphs

Directional links in a network can also be associated with signs [24]. We now present generalizations of the previous model and results to signed directed networks. For the ease of presentation, we keep the previous notation and simply adapt them to the directed graph case. Their usage is of course restricted to the current subsection.

Now let the graph $G = (V, E)$ be a directed graph (digraph), where a link $(i, j) \in E$ is directed starting from i and pointing to j. A diagraph is termed a signed digraph if each of its link has a positive or negative sign. By revising the definition of positive and negative neighbor sets of node i to

$$N_i^+ := \{j : (j, i) \in E^+\}; \quad N_i^- := \{j : (j, i) \in E^-\},$$

the network dynamics (4) and (9) are then readily defined for the digraph G. The set $N_i = N_i^+ \bigcup N_i^-$ continues to represent the overall neighbor set of node i. In this directed graph case we continue to define $\deg_i^+ = \deg_i^+$, $\deg_i^- = \deg_i^-$, and $\deg_i = |N_i|$ as the positive, negative, and overall degrees of node i.

Definition 1 can be generalized to digraphs by replacing the undirected edges with directional links:

Definition 3: A signed digraph G is *structurally balanced* if there is a partition of the node set into $V = V_1 \bigcup V_2$, where V_1 and V_2 are nonempty and mutually disjoint, such that any directional link between V_1 and V_2 is negative, and any link whose two end nodes belong to the same V_i is positive.

The following theorem corresponds to Theorem 1 for signed digraphs.

Theorem 3: Consider network dynamics (4) over a digraph G. Assume that $0 < \alpha+\beta < 1/\max_{i\in V} \deg_i$. Suppose G is strongly connected. The following holds for any initial value $\mathbf{x}(0)$.

(i) If G is structurally balanced subject to partition $V = V_1 \bigcup V_2$, then there are n positive numbers $w_1, \dots, w_n$ with $\sum_{i=1}^n w_i = 1$ such that $\lim_{t\to\infty} x_i(t) = \big(\sum_{j\in V_1} w_j x_j(0) - \sum_{j\in V_2} w_j x_j(0)\big)/n$, $i \in V_1$; and $\lim_{t\to\infty} x_i(t) = -\big(\sum_{j\in V_1} w_j x_j(0) - \sum_{j\in V_2} w_j x_j(0)\big)/n$, $i \in V_2$.

(ii) If G is not structurally balanced, then $\lim_{t\to\infty} x_i(t) = 0,\ i \in V$.

Likewise, the following theorem corresponds to Theorem 2 for signed digraphs.

Theorem 4: Consider network dynamics (9) over a digraph G. Suppose G^+ is strongly connected and fix $0 < \alpha < 1/\max_{i\in V} \deg_i^+$. There exists $\beta_* > 0$ such that for any $\beta < \beta_*$, there are $q_1(\beta), \dots, q_n(\beta) \in \mathbb{R}^+$ with $\sum_{i=1}^n q_i = 1$ satisfying that a consensus is reached at

$$\lim_{t\to\infty} x_i(t) = \sum_{j=1}^n q_i x_i(0),\ i \in V$$

for all initial value $\mathbf{x}(0)$.

In the statement of Theorem 4, for any $\beta < \beta_*$, $\big(q_1(\beta) \dots q_n(\beta)\big)$ is a left eigenvector related to eigenvalue 1 of M_{G}. It is worth emphasizing that the β_* in Theorem 4 is merely an upper bound for β under which the network can still reach a consensus in the presence of the negative links, and it is unclear whether such β_* would remain a critical value as the undirected case. The actual value of β_* can be estimated using standard matrix perturbation theory [25].

C. Weighted Signs, Continuous-time Dynamics, Switching Structures

More sophisticated signed networks can certainly be studied by generalizing the previous tools and analysis. This subsection presents a survey to related results in the literature.

1) Weighted Signs: The strength of positive and negative links, represented by α and β, can also be link dependent. This means that for the positive and negative dynamics (1), (2), and (3) along the edge $\{i, j\}$, α and β will be replaced by α_{ij} and β_{ij}, respectively. The results of Theorems 1–4 can be extended to networks with weighted signs straightforwardly [10].

2) Continuous-time Dynamics: The signed network dynamics considered above clearly have their continuous-time counter part. For the state flipping negative dynamics (5), the corresponding node state evolution in continuous time reads as

$$\frac{d}{dt}\mathbf{x}(t) = -\Big(\alpha L_{G^+} + \beta L_{G^-}\Big)\mathbf{x}(t). \tag{11}$$

On the other hand, the continuous-time counter part of the dynamics (10) is

$$\frac{d}{dt}\mathbf{x}(t) = -\Big(\alpha L_{G^+} - \beta L^*_{G^-}\Big)\mathbf{x}(t). \tag{12}$$

Theorems 1 and 2 can be translated to the these continuous time models straightforwardly, even for nonlinear node interactions [9], [15]. As illustrated in (8), under the state-flipping negative dynamics, both positive and negative links lead to non-expansive network state evolution[1]. The mathematical reason behind those non-linear generalizations is due to the fact that the non-expansive property can be preserved for suitable nonlinear interaction rules.

3) Switching Network Structures: In the study of standard consensus algorithms, one particular interest was to establish convergence conditions under time-varying network structures [5], [8], [26], [27], for which earlier work was dated to 1960s [2]. Such analysis can be challenging due to the lacking of a common convergence metric that works for all possible choices of the interaction graphs. Nevertheless, possibilities of generalizing the analysis of time-varying network structures have been shown in the literature [10], [14], [15], [18], [28], [29].

[1]With directed graphs, (8) in general no longer holds under the state-flipping negative dynamics. However, there still holds that $\max_{i\in V} \big|x_i(t+1)\big| \le \max_{i\in V} \big|x_i(t)\big|$ as shown in the proof of Theorem 3. Therefore, the network state evolution continues to be non-expansive.

Let $\mathrm{G}_t = (\mathrm{V}, \mathrm{E}_t), t = 0, 1, \dots$ be a sequence of graphs with each G_t being a (directed or undirected) signed graph. Then the positive and negative neighbor set of node i, are determined by connections in G_t and therefore become time-dependent, denoted $\mathrm{N}_i^+(t)$ and $\mathrm{N}_i^-(t)$, respectively. The network dynamics under the state-flipping rule (2) are then represented by

$$x_i(t+1) = x_i(t) + \alpha \sum_{j \in \mathrm{N}_i^+(t)} \Big(x_j(t) - x_i(t)\Big) - \beta \sum_{j \in \mathrm{N}_i^-(t)} \Big(x_j(t) + x_i(t)\Big). \quad (13)$$

The following result holds.

Proposition 1: Suppose there exists a constant $0 < \delta < 1$ such that $\alpha\big|\mathrm{N}_i^+(t)\big| + \beta\big|\mathrm{N}_i^-(t)\big| \le 1 - \delta$ for all $i \in \mathrm{V}$ and all $t \ge 0$.

(i) Let there exist $T \ge 0$ such that the graph $\mathrm{G}_{[s,s+T]} := \big(\mathrm{V}, \bigcup_{t=s}^{s+T} \mathrm{E}_t\big)$ is strongly connected for all $s \ge 0$. Then along (13), for any initial value $\mathbf{x}(0)$, there exists $y_*(\mathbf{x}(0)) \ge 0$ such that $\lim_{t\to\infty} \big|x_i(t)\big| = y_*(\mathbf{x}(0))$ for all $i \in \mathrm{V}$.

(ii) Suppose G_t is undirected for all $t \ge 0$. Let the graph $\mathrm{G}_{[s,\infty]} := \big(\mathrm{V}, \bigcup_{t=s}^{\infty} \mathrm{E}_t\big)$ be connected for all $s \ge 0$. Then along (13), for any initial value $\mathbf{x}(0)$, there exists $y_*(\mathbf{x}(0)) \ge 0$ such that $\lim_{t\to\infty} \big|x_i(t)\big| = y_*(\mathbf{x}(0))$ for all $i \in \mathrm{V}$.

Proof. The desired conclusions follow from Theorem 2.1 and Theorem 2.2 in [16], where the positive and negative weights are link dependent. □

Structural balance condition can be applied to the sequence of graphs $\mathrm{G}_t = (\mathrm{V}, \mathrm{E}_t)$, under which bipartite consensus result can be similarly established for state-flipping negative dynamics [14], [18], [28]. On the other hand, for relative-state flipping negative dynamics, analysis for switching network structures can be extremely challenging since the network state is no longer non-expansive in the presence of one single negative link. It turned out that in order to preserve convergence to consensus, it is important that at each time step, the influence of the negative links can be overcome by the positive links. We refer to [29] for such treatment under continuous-time node dynamics.

III. Random Networks

Node interactions happen randomly in many real-world networks, and how consensus can be reached over a random node interaction processes have been extensively studied [6], [30]–[35]. We proceed to discuss network dynamics over signed random graph processes.

We use the following gossiping model [31] to describe the random node interactions. The undirected, signed graph, $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, continue to define the world of the network where interactions take place. Each node initiates interactions at the instants of a rate-one Poisson process, and at each of these instants, picks a node at random to interact with. Under this model, at a given time, at most one node initiates an interaction. This allows us to order interaction events in time and to focus on modeling the node pair selection at interaction times. The node pair selection is then performed as follows.

Definition 4: Independently at each interaction event $t \ge 0$, (i) a node $i \in \mathrm{V}$ is drawn uniformly at random, i.e., with probability $1/n$; (ii) node i picks a neighbor j uniformly with probability $1/\mathrm{deg}_i$ for $j \in \mathrm{N}_i$. In this case, we say that the unordered node pair $\{i, j\}$ is selected.

The node pair selection process is assumed to be i.i.d., i.e., the nodes that initiate an interaction and the selected node pairs are identically distributed and independent over $t \ge 0$. Let $(\mathrm{E}, \mathscr{S}, \mu)$ be the probability space, where $\mathscr{S}$ is the discrete σ-algebra on E, and μ is the probability measure defined by $\mu(\{i,j\}) = (1/\mathrm{deg}_i + 1/\mathrm{deg}_j)/n$ for all $\{i,j\} \in \mathrm{E}$. The node selection process can then be seen as a random event in the product probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\Omega = \mathrm{E}^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \dots,) : \forall t, \omega_t \in \mathrm{E}\}$, $\mathscr{F} = \mathscr{S}^{\mathbb{N}}$, and $\mathbb{P}$ is the product probability measure (uniquely) defined by: for any finite subset $K \subset \mathbb{N}$, $\mathbb{P}((\omega_t)_{t\in K}) = \prod_{t\in K} \mu(\omega_t)$ for any $(\omega_t)_{t\in K} \in \mathrm{E}^{|K|}$. For any $t \in \mathbb{N}$, we define the coordinate mapping $\mathcal{G}_t : \Omega \to \mathrm{E}$ by $\mathcal{G}_t(\omega) = \omega_t$, for all $\omega \in \Omega$. Then formally $\mathcal{G}_t,\ t = 0, 1, \dots$ describe the node pair selection process. We denote $\mathscr{F}_t = \sigma(\mathcal{G}_0, \dots, \mathcal{G}_t)$ as the σ-algebra capturing the $t+1$ first interactions of the selection process.

After the pair of nodes $\{i, j\}$ have been selected at time t, they update their states $x_i(t)$ and $x_j(t)$ according to the sign of the link that they share: if the link is positive, they update their states by (1); if the link is negative, they update their states by either (2) or (3). The nodes that are not selected at time t will keep their states unchanged. In this way, $\mathbf{x}(t)$, $t = 0, 1, \dots$ specifies a random process over the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and we are interested in the mean, mean-square, and almost sure convergence of $\mathbf{x}(t)$.

For state-flipping and relative-state flipping models, we present the following results, respectively, for the mean-square and almost sure convergence of $\mathbf{x}(t)$.

Theorem 5: Let $0 < \alpha, \beta < 1$ and consider state flipping rule (2) for dynamics over negative links.

(i) If G is structurally balanced subject to partition $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$, then both in mean-square and almost surely,

$$x_i(t) \to \big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in\mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_1 \quad (14)$$

and

$$x_i(t) \to -\big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in\mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_2. \quad (15)$$

(ii) If G is not structurally balanced, then $x_i(t) \to 0$ both in mean-square and almost surely for all $i \in \mathrm{V}$.

Theorem 6: Suppose G^+ is connected and consider the relative-state flipping rule (3). For any $0 < \alpha < 1$, there exists $\beta^*(\alpha) > 0$ such that $x_i(t) \to \sum_{j=1}^n x_i(0)/n$ both in mean-square and almost surely for all initial value $\mathbf{x}(0)$ if $\beta < \beta^*$.

The following results characterize possible almost sure divergence of $\mathbf{x}(t)$ caused by large β related to the negative links.

Theorem 7: Fix $0 < \alpha < 1$ with $\alpha \neq 1/2$.

(i) Suppose both G^+ and G^- are connected. Then under the state flipping negative dynamics (2), there exists β_* such that whenever $\beta > \beta_*$, there holds

$$\mathbb{P}\Big(\limsup_{t\to\infty} \max_{i\in \mathrm{V}} \big|x_i(t)\big| = \infty\Big) = 1 \tag{16}$$

for almost all initial values w.r.t. Lebesgue measure.

(ii) Under the relative-state flipping negative dynamics (3), there exists β_* such that whenever $\beta > \beta_*$, there holds

$$\mathbb{P}\Big(\limsup_{t\to\infty} \max_{i,j\in \mathrm{V}} \big|x_i(t) - x_j(t)\big| = \infty\Big) = 1 \tag{17}$$

for almost all initial values w.r.t. Lebesgue measure.

These results can be similarly obtained by adapting the algebraic-graphical analysis for deterministic networks to the random setting. We refer to [12], [13], [17] for related treatments, and omit the details due to space limitations.

IV. Conclusions

We have established an algebraic-graphical method which can systematically investigate consensus dynamics over signed networks, in view of generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions. After basic convergence results have been clear, interesting future directions include inverse problems such as estimating characteristics of the annotations of links and nodes from observations of various network characteristics at a subset of nodes (called boundary nodes) like delay and throughput.

References

[1] E. David and J. Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. New York, NY, USA: Cambridge University Press, 2010.

[2] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proc. Amer. Math. Soc.*, vol. 14, no. 5, pp. 733–737, 1963.

[3] M. H. Degroot, "Reaching a Consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974. [Online]. Available: http://www.jstor.org/stable/2285509

[4] J. N. Tsitsiklis, *Problems in decentralized decision making and computation*. Ph.D. dissertation, Dept. Elect. Eng. Comput. Sci., Massachusetts Inst. Technol., Boston, MA, USA, 1984.

[5] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, pp. 988–1001, 2003.

[6] S. Kar and J. M. F. Moura, "Distributed consensus algorithms in sensor networks: Quantized data and random link failures," *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1383–1400, 2010.

[7] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[8] W. Ren and R. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, pp. 655–661, 2005.

[9] C. Altafini, "Dynamics of opinion forming in structurally balanced social networks," *Plos One*, vol. 7, no. 6, p. e38135, 2012.

[10] ——, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.

[11] G. Shi, M. Johansson, and K. H. Johansson, "How agreement and disagreement evolve over random dynamic networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 6, pp. 1061–1071, 2013.

[12] G. Shi, A. Proutiere, M. Johansson, J. S. Baras, and K. H. Johansson, "Emergent behaviors over signed random dynamical networks: state-flipping model," *IEEE Transactions on Control of Network Systems*, pp. 142–153, 2015.

[13] ——, "Emergent behaviors over signed random dynamical networks: relative-state-flipping model," *IEEE Transactions on Control of Network Systems*, in press, 2017.

[14] J. Liu, X. Chen, T. Basar, and M. A. Belabbas, "Exponential convergence of the discrete-time altafini model," *Preprint*, arXiv [math.OC]:1512.0715, 2015.

[15] Z. Meng, G. Shi, and K. H. Johansson, "Dynamics of opinion forming in structurally balanced social networks," *SIAM Journal on Control and Optimization*, vol. 53, no. 5, pp. 3057–3080, 2015.

[16] Z. Meng, G. Shi, K. H. Johansson, M. Cao, and Y. Hong, "Behaviors of networks with antagonistic interactions and switching topologies," *Automatica*, vol. 73, 2016.

[17] G. Shi, A. Proutiere, M. Johansson, J. S. Baras, and K. H. Johansson, "The evolution of beliefs over signed social networks," *Operations Research*, vol. 64, pp. 585–604, 2016.

[18] W. Xia, M. Cao, and K. H. Johansson, "Structural balance and opinion separation in trustmistrust social networks," *IEEE Transactions on Control of Network Systems*, 2015.

[19] J. M. Hendrickx, "A lifting approach to models of opinion dynamics with antagonisms," in *Proc. of the 53th IEEE Conference on Decision and Control*, 2014, pp. 2118–2123.

[20] M. Egerstedt and M. Mesbahi, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[21] F. Heider, "Attitudes and cognitive organization," *Journal of Psychology*, vol. 21, pp. 107–112, 1946.

[22] J. A. Davis, "Clustering and structural balance in graphs," *Human Relations*, vol. 20, pp. 181–187, 1967.

[23] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[24] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications*. Cambridge University Press, 1994.

[25] J. S. G. W. Stewart, *Matrix Perturbation Theory*. Academic Press, 1990.

[26] V. Blondel, J. M. Hendrickx, A. Olshevsky, and J. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," *IEEE Conf. Decision and Control*, pp. 2996–3000, 2005.

[27] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, pp. 169–182, 2005.

[28] A. V. Proskurnikov, A. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Transactions on Automatic Control*, 2015.

[29] B. D. O. Anderson, G. Shi, and J. Trumpf, "Convergence and state reconstruction of time-varying multi-agent systems from complete observability theory," *IEEE Transactions on Automatic Control*, 2017.

[30] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1867–1872, 2005.

[31] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2508–2530, 2006.

[32] F. Fagnani and S. Zampieri, "Randomized consensus algorithms over large scale networks," *IEEE Journal on Selected Areas in Communications*, vol. 26, no. 4, pp. 634–649, 2008.

[33] A. Tahbaz-Salehi and A. Jadbabaie, "A necessary and sufficient condition for consensus over random networks," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 791–795, 2008.

[34] ——, "Consensus over ergodic stationary graph processes," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 225–230, 2010.

[35] G. Shi, B. D. O. Anderson, and K. H. Johansson, "Consensus over random graph processes: Network borel-cantelli lemmas for almost sure convergence," *IEEE Transactions on Information Theory*, vol. 61, no. 10, pp. 5690–5707, 2015.